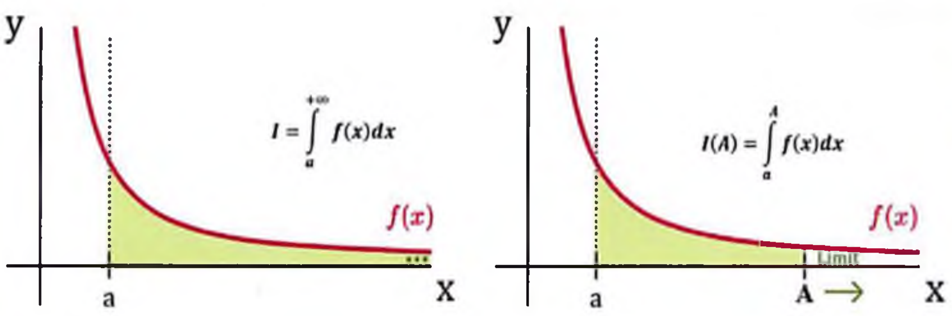
YAXSHIBAYEV D.S.

INTEGRALLARNI HISOBLASH

$$\int_a^{+\infty} f(x)dx = \lim_{A\to\infty}\int_a^A f(x)dx$$



O‘QUV QO‘LLANMA

O‘ZBEKISTON RESPUBLIKASI OLIY TA’LIM, FAN VA INNOVATSIYALAR VAZIRLIGI
MUHAMMAD AL-XORAZMIY NOMIDAGI TOSHKENT AXBOROT TEXNOLOGIYALARI UNIVERSITETI

D.S. Yaxshibayev

INTEGRALLARNI HISOBLASH

o‘quv qo‘llanma

Muhammad al-Xorazmiy nomidagi Toshkent axborot texnologiyalari universiteti bakalavriyatining barcha ta’lim yo‘nalishlari talabalari uchun tavsiya etilgan

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O‘quv qo‘llanma “Hisob (Calculus)” fanining integrallar moduliga to‘g‘ri keladi. Unda aniqmas integral, aniq integral va ularning xossalari, xosmas integrallar, ikki va uch karrali integrallar va ularning xossalari yetarli darajada o‘rganilgan. Bundan tashqari har bir mavzuga doir misollar yechib ko‘rsatilgan. Amaliy mashg‘ulotlarda yechiladigan misollar hamda ushbu darsda mustaqil yechish uchun misollari bilan berilgan.

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Kirish

Muhtaram Prezidentimiz Sh.M.Mirziyoyev ta'kidlaganidek, "Matematika hamma fanlarga asos. Bu fanni yaxshi bilgan bola aqlli, keng tafakkurli bo'lib o'sadi, istalgan sohada muvaffaqiyatli ishlab ketadi". Haqiqatdan ham matematika fani inson aqlini charxlaydi, diqqatini rivojlantiradi, ko'zlangan maqsadga erishish uchun qat'iyat va irodani tarbiyalaydi, algoritmik tarzda tartib-intizomlilikka o'rgatadi va eng muhimi mulohaza yuritishga chorlaydi hamda tafakkurni kengaytiradi. Hozirgi kunda respublikamizda ta'lim sohasida olib borilayotgan islohotlar Milliy o'quv dasturi talablari asosida o'qitishni tashkil etish, talabalar uchun zamon talabiga javob beradigan darslik, darsliklar yaratish dolzarb masalaligicha qolmoqda. Ushbu darslik yuqoridagi talablarni hisobga olib yaratildi. O'quv qo'llanmada aniqmas integral, aniq integral va ularning xossalari, xosmas integrallar, ikki va uch karrali integrallar va ularning xossarari yetarli darajada o'rganilgan. Bundan tashqari har bir mavzuga doir misollar yechib ko'satilgan hamda ushbu darsda mustaqil yechish uchun misollar ham keltiriladi

O'quv qo'llanmada keltirilgan mavzular iloji boricha qat'iy va tushunarli bo'lishiga harakat qilindi hamda ko'p miqdordagi misollar bilan ta'minlandi, bu esa nazariy mazmunning ma'nosini ochishga yordam beradi.

O'quv qo'llanma kamchiliklardan xoli emas albatta, shu sababli muallif uni takomillashtirishga qaratilgan fikr va mulohazalarni mamnuniyat bilan qabul qiladi va oldindan o'z minnatdorchiligini bildiradi.

I BOB. ANIQMAS INTEGRAL

1.1.Boshlang'ich funksiya va aniqmas integral

Berilgan funksiyaning hosilasini topish differensial hisobning asosiy masalalaridan biridir. Matematik analizning geometriya, mexanika, fizika va texnikadagi masalalarga keng miqyosdagi tatbiqi teskari masalani yechishga, ya'ni berilgan $f(x)$ funksiya uchun hosilasi shu funksiyaga teng bo'lgan $F(x)$ funksiyani topishga olib keladi.

Funksiyaning berilgan hosilasiga ko'ra uning o'zini topish masalasi integral hisobning asosiy masalalaridan biri hisoblanadi.

$y = f(x)$ funksiya $(a; b)$ intervalda aniqlangan bo'lsin.

1-ta'rif. Agar $(a; b)$ intervalda differensiallanuvchi $F(x)$ funksiyanig hosilasi berilgan $f(x)$ funksiyaga teng, ya'ni

$$F'(x) = f(x) \qquad (\text{yoki } dF(x) = f(x)dx)$$

bo'lsa, $F(x)$ funksiyaga $(a; b)$ intervalda $f(x)$ funksiyaning boshlang'ich funksiyasi deyiladi

Masalan: $F(x) = x^6$ funksiya butun sonlar oqida $f(x) = 6x^5$ funksiyaning boshlang'ich funksiyasi bo'ladi, chunki $x \in R$da $(x^6)' = 6x^5$; $F(x) = \sqrt{1-x^2}$ funksiya $(-1; 1)$ intervalda $f(x) = -\frac{x}{\sqrt{1-x^2}}$ funksiyaning boshlang'ich funksiyasi bo'ladi, chunki $x \in (-1; 1)$ da $(\sqrt{1-x^2})' = -\frac{x}{\sqrt{1-x^2}}$.

Lemma. Agar $F(x)$ va $\Phi(x)$ funksiyalar $(a; b)$ intervalda $f(x)$ funksiyaning boshlang'ich funksiyalari bo'lsa, u holda $F(x)$ va $\Phi(x)$ bir-biridan o'zgarmas songa farq qiladi.

Isboti. $F(x)$ va $\Phi(x)$ funksiyalar $(a; b)$ intervalda $f(x)$ funksiyaning boshlang'ich funksiya bo'lsin: $F'(x) = f(x)$, $\Phi'(x) = f(x)$.

U holda istalgan $x \in (a; b)$ da

$$(\Phi(x) - F(x))' = \Phi'(x) - F'(x) = f(x) - f(x) = 0$$

bo'ladi. Bundan $\Phi(x) - F(x) = C$ yoki $\Phi(x) = F(x) + C$ kelib chiqadi, bu yerda C –ixtiyoriy o'zgarmas son.

Shunday qilib, $f(x)$ funksiya $(a; b)$ intervalda biror $F(x)$ boshlang'ich funksiyaga ega bo'lsa, uning qolgan barcha boshlang'ich funksiyalari $\{F(x) + C\}$ to'plamni tashkil qiladi.

2-ta'rif. $f(x)$ funksiyaning $(a; b)$ intervaldagi boshlang'ich funksiyalari to'plamiga $f(x)$ funksiyaning aniqmas integrali deyiladi va $\int f(x)dx$ kabi belgilanadi.

Shunday qilib, ta'rifga ko'ra

$$\int f(x)dx = F(x) + C, \qquad (1.1)$$

bu yerda $f(x)$ - integral ostidagi funksiya, $f(x)dx$ -integral ostidagi ifoda; x - integrallash o'zgaruvchisi, $\int$ - integrallash belgisi deb ataladi.

Aniqmas integralni topish, ya'ni berilgan funksiyaning boshlang'ich funksiyalari to'plamini aniqlash masalasi funksiyani integrallash deyiladi.

Demak, funksiyani integrallash amali funksiyani differensiallashga teskari amal bo'ladi.

Berilgan $f(x)$ funksiya qachon boshlang'ich funksiyaga ega bo'ladi degan savolga quyidagi teorema javob beradi (teoremani isbotsiz keltiramiz).

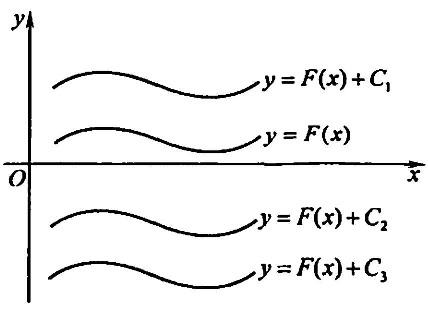
1-teorema. Agar $f(x)$ funksiya $[a;b]$ kesmada uzluksiz bo'lsa, u holda u bu kesmada uzluksiz bo'lgan boshlang'ich funksiyaga ega bo'ladi.

Ko'p hollarda $F(x)$ funksiya $f(x)$ funksiyaning boshlang'ich funksiyasi bo'ladigan $(a;b)$ interval ko'rsatilmaydi. Bunday holda $(a;b)$ interval sifatida $f(x)$ funksiyaning aniqlanish sohasi tushuniladi. Shu sababli bundan keyin integral ostidagi funksiyalar uzluksiz va (1.1) formula ma'noga ega deb hisoblaymiz. Masalan, $f(x) = \frac{1}{x}$ funksiya $(-\infty;0)$ va $(0;\infty)$ intervalda uzluksiz.

Shu sababli uning aniqmas integrali deb

$$\int \frac{dx}{x} = \begin{cases} \ln x + C, & x > 0, \\ \ln(-x) + C, & x < 0 \end{cases} = \ln|x| + C \ (x \neq 0)$$

funksiya tushuniladi.



1-shakl

Boshlang'ich funksiyaning grafigi integral egri chiziq deb ataladi.

Aniqmas integral geometrik jihatdan ixtiyoriy C o'zgarmasga bog'liq bo'lgan barcha integral egri chiziqlar to'plamini ifodalaydi. Agar $F(x)$ funksiyaning grafigi integral egri chiziq bo'lsa, boshqa integral egri chiziqlar uni Oy o'qi bo'yicha parallel ko'chirish yordamida hosil qilinadi.

1.2. Aniqmas integralning xossalari

Aniqmas integral quyidagi xossalarga ega.

1^o. Aniqmas integralning hosilasi integral ostidagi $f(x)$ funksiyaga teng:

$$\left(\int f(x)dx\right)' = f(x);$$

Isboti. $F(x)$ funksiya $f(x)$ funksiyaning boshlang‘ich funksiyasi, ya’ni $F'(x) = f(x)$ bo‘lsin. U holda

$$\left(\int f(x)dx\right)' = (F(x) + C)' = F'(x) + 0 = f(x).$$

Bu xossa integrallashning to‘g‘riligini differensiallash orqali tekshirish imkonini beradi.

Masalan, $\int (3x^2 + 5)\,dx = x^3 + 5x + C$ to‘g‘ri, chunki

$$(x^3 + 5x + C)' = 3x^2 + 5.$$

2^o. Funksiya differensialining aniqmas integrali shu funksiya bilan o‘zgarmas sonning yig‘indisiga teng:

$$\int dF(x) = F(x) + C.$$

Isboti. $F'(x) = f(x)$ bo‘lsin. U holda

$$\int dF(x) = \int F'(x)dx = \int f(x)dx = F(x) + C.$$

3^o. O‘zgarmas ko‘paytuvchini aniqmas integral belgisidan tashqariga chiqarish mumkin:

$$\int kf(x)dx = k\int f(x)dx, \qquad k = const, k \neq 0.$$

Isboti. $F'(x) = f(x)$ bo‘lsin. Bundan

$$\int kf(x)dx = \int kF'(x)dx = \int (kF(x))'dx = kF(x) + C_1 =$$
$$k(F(x) + C) = k\int f(x)dx \; (C_1 = kC \text{ deb olindi}).$$

4^o. Chekli sondagi funksiyalar algebraik yig‘indisining aniqmas integrali shu funksiyalar aniqmas integrallarining algebraik yig‘indisiga teng:

$$\int (f(x) \pm g(x))\,dx = \int f(x)dx \pm \int g(x)dx.$$

Isboti. $F'(x) = f(x)$, $G'(x) = g(x)$ bo‘lsin. U holda

$$\int (f(x) \pm g(x))\,dx = \int (F'(x) \pm G'(x))\,dx = \int (F(x) \pm G(x))'\,dx =$$
$$= \int d(F(x) \pm G(x)) = F(x) \pm G(x) + C =$$
$$= (F(x) + C_1) \pm (G(x) + C_2) =$$
$$= \int f(x)dx \pm \int g(x)dx\,, C_1 \pm C_2 = C.$$

5^o. Agar $\int f(x)dx = F(x) + C$ bo'lsa, u holda x ning istalgan differensiallanuvchi funksiyasi $u = u(x)$ uchun $\int f(u)du = F(u) + C$ bo'ladi.

Isboti. x erkli o'zgaruvchi, $f(x)$ uzluksiz funksiya, $F(x)$ funksiya $f(x)$ funksiyaning boshlang'ich funksiyasi bo'lsin. U holda $\int f(x)dx = F(x) + C$ bo'ladi.

$u = \phi(x)$ bo'lsin, bu yerda $\phi(x)$ - uzluksiz hosilaga ega bo'lgan funksiya.

Birinchi differensialning invariantlik xossasiga ko'ra $dF(u) = F'(u)du = f(u)du$ bo'ladi.

Bundan

$$\int f(u)du = \int d(F(u) = F(u) + C.$$

Bu xossa integrallash formulasining invariantligi xossasi deyiladi. Demak, aniqmas integral integrallash o'zgaruvchisi erkli o'zgaruvchi yoki erkli o'zgaruvchining uzluksiz hosilaga ega bo'lgan ixtiyoriy funksiyasi bo'lishidan qat'iy nazar bir xil formula bilan topiladi.

1.3. Asosiy elementar funksiyalarning integrallar jadvali

Integrallash differensiallash amaliga teskari amal bo'lgani uchun asosiy integrallar jadvalini differensial hisobning mos formulalarini qo'llash va aniqmas integralning xossalaridan foydalanish orqali hosil qilish mumkin.

Masalan, $d(\sin u) = \cos u\, du$ ekanidan $\int \cos u\, du = \int d(\sin u) = \sin u + C$.

Quyida keltiriladigan integrallar asosiy integrallar jadvali deyiladi.

Asosiy integrallar jadvalida integrallash o'zgaruvchisi u erkli o'zgaruvchi yoki erkli o'zgaruvchining funksiyasi (5^o xossaga ko'ra) bo'lishi mumkin.

Jadvalda keltirilgan formulalarning to'g'riligiga uning o'ng tomonini differensiallash va bu differensialning formula chap tomonidagi integral ostidafi ifodaga teng bo'lishini tekshirish orqali ishonch hosil qilish mumkin.

Asosiy integrallar jadvali

1. $\int u^\alpha\, du = \frac{u^{\alpha+1}}{\alpha+1} + C,\ (\alpha \neq -1)$;
2. $\int \frac{du}{u} = ln\,|u| = C$;
3. $\int a^u du = \frac{a^u}{ln\, a} + C,\ (0 < a \neq 1)$;
4. $\int e^u\, du = e^u + C$;

5. $\int sin\,u\,du = -cos\,u + C;$

6. $\int cos\,u\,du = sin\,u + C;$

7. $\int tgudu = -ln\,|\,cos\,u\,| = C;$

8. $\int ctgudu = ln\,|\,sin\,u\,| = C;$

9. $\int \frac{du}{cos^2 u} = tgu + C;$

10. $\int \frac{du}{sin^2 u} = -ctgu + C;$

11. $\int \frac{du}{sin\,u} = ln\left|tg\frac{u}{2}\right| + C;$

12. $\int \frac{du}{cos\,u} = ln\left|tg\left(\frac{u}{2} + \frac{\pi}{4}\right)\right| + C;$

13. $\int \frac{du}{\sqrt{a^2-u^2}} = arcsin\frac{u}{a} + C;$

14. $\int \frac{du}{\sqrt{u^2 \pm a^2}} = ln\left|u + \sqrt{u^2 \pm a^2}\right| + C.$

15. $\int \frac{du}{a^2+u^2} = \frac{1}{a} arctg\frac{u}{a} + C;$

16. $\int \frac{du}{u^2-a^2} = \frac{1}{2a} ln\left|\frac{u-a}{u+a}\right| + C;$

17. $\int \sqrt{u^2 \pm a^2}\,du = \frac{u}{2}\sqrt{u^2 \pm a^2} \pm \frac{a^2}{2} ln\left|u + \sqrt{u^2 \pm a^2}\right| + C.$

18. $\int \sqrt{a^2 - u^2}du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2} arcsin\frac{u}{a} + C;$

19. $\int shudu = chu + C;\ \int chudu = shu + C;$

20. $\int \frac{du}{ch^2 u} = thu + C;$

21. $\int \frac{du}{sh^2 u} = -cthu + C.$

Bu integrallardan birining, masalan 13- formulaning to‘g‘riligini ko‘rsatamiz:

$$d\left(arcsin\frac{u}{a} + C\right) = \frac{1}{\sqrt{1 - \left(\frac{u}{a}\right)^2}} \cdot \frac{1}{a}du = \frac{du}{\sqrt{a^2 - u^2}}.$$

Amaliy mashg‘ulotda yechiladigan misollar.

Berilgan aniqmas integrallarni toping.

1. $\int \left(x^3 + 5x + \frac{5}{x}\right)dx$

2. $\int (3 - x^2)^2 dx$

3. $\int \left(3 - \frac{1}{x^3}\right)\sqrt{x^2\sqrt{x}}dx$

4. $\int a^{-x}\left(1 - \frac{a^x}{\sqrt{x^3}}\right)dx$

5. $\int \frac{cos\,2\,xdx}{sin^2 x\,cos^2 x}$

6. $\int \frac{5 - 2ctg^3 x}{cos^2 x}dx$

7. $\int tg3x\,dx$

8. $\int (\sin 7x - \frac{1}{4x+4} + \frac{1}{\cos^2 4x})dx$

9. $\int \frac{x-4}{1-x^2}dx.$

10. $\int \frac{x-x^3}{\sqrt{9+x^4}}dx.$

11. $\int \frac{3^x}{\sqrt{9-9x}}dx$

Mustaqil yechish uchun misollar.

Berilgan aniqmas integrallarni toping.

1. $\int \frac{e^{2x}dx}{e^{2x}+3}$

2. $\int (3x - \sqrt{7x^4-1} + \frac{1}{\sin^2 4x})dx$

3. $\int \frac{dx}{2x^2+5}$

4. $\int \frac{dx}{\sqrt{5-x^2}}$

5. $\int \frac{5x-2}{x^2+4}dx$

6. $\int \frac{x+1}{\sqrt{x^2+2}}dx$

1.4. Aniqmas integralda integrallash usullari

Bevosita integrallash usuli

Integral ostidafi funksiyada (yoki ifodada) almashtirishlar bajarish va aniqmas integralning xossalarini qo‘llash orqali berilgan integralni bir yoki bir nechta jadval integraliga keltirib integrallash usuliga bevosita integrallash usuli deyiladi.

Misollar

1. $\int \frac{\cos 2x}{\cos^2 x \sin^2 x}dx = \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x \sin^2 x}dx = \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x}\right)dx =$

$$= \int \frac{dx}{\sin^2 x} - \int \frac{dx}{\cos^2 x} = -ctgx - tgx + C = -\frac{2}{\sin 2x} + C;$$

2. $\int \left(5\sin x - \frac{2}{x^2+1} + x^3\right)dx = 5\int \sin x\,dx - 2\int \frac{dx}{x^2+1} + \int x^3\,dx =$

$$= -5\cos x - 2arctgx + \frac{x^4}{4} + C;$$

3. $\int \frac{x^4}{1+x^2}dx = -\int \frac{1-x^4-1}{1+x^2}dx = -\int (1-x^2)dx + \int \frac{dx}{1+x^2} =$

$$= -\int dx + \int x^2\,dx + \int \frac{dx}{1+x^2} = -x + \frac{x^3}{3} + arctgx + C.$$

4. $\int \frac{dx}{\sqrt{x-3}-\sqrt{x-7}} = \int \frac{\sqrt{x-3}+\sqrt{x-7}}{\sqrt{x-3}+\sqrt{x-7}} \cdot \frac{dx}{\sqrt{x-3}-\sqrt{x-7}} =$

$$= \frac{1}{4}\int (\sqrt{x-3} + \sqrt{x-7})dx = \frac{1}{6}\sqrt{(x-3)^3} + \frac{1}{6}\sqrt{(x-7)^3} + C.$$

5. $\int \frac{dx}{\sqrt{3+x+x^2}} = \int \frac{dx}{\sqrt{\frac{11}{4}+\left(\frac{1}{4}+x+x^2\right)}} = \int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}} =$

$$= \left(u = x + \frac{1}{2}, \quad a = \left(\frac{\sqrt{11}}{2}\right) \right) =$$

$$= ln\left| x + \frac{1}{2} + \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{11}}{2}\right)^2} \right| + C =$$

$$= ln\left| x + \frac{1}{2} + \sqrt{3+x+x^2} \right| + C.$$

Berilgan integralni jadval integrallariga keltirishda differensialning quyidagi almashtirishlari («differensial amali ostiga kiritish» jarayoni) qo'llaniladi:

$du = d(u+a)$, a –son; $du = \frac{1}{a}d(au)$; $udu = \frac{1}{2}d(u^2)$; $cos\,u\,du = d(sin\,u)$; $sin\,u\,du = -d(cos\,u)$; $\frac{1}{u}du = d(ln\,u)$; $\frac{1}{cos^2 u}du = d(tgu)$.

Umuman olganda, $f'(u)du = d(f(u))$. Bu formuladan integrallarni topishda ko'p foydalaniladi.

Misollar

1. $\int \frac{cos\,x+sin\,x}{sin\,x-cos\,x}dx = \int \frac{d(sin\,x-cos\,x)}{sin\,x-cos\,x} = ln\,|\,sin\,x - cos\,x\,| + C.$

2. $\int \frac{dx}{16+9x^2} = \frac{1}{3}\int \frac{d(3x)}{16+(3x)^2} = \frac{1}{3}\cdot\frac{1}{4}arctg\frac{3x}{4} + C = \frac{1}{12}arctg\frac{3x}{4} + C;$

O'rniga qo'yish (o'zgaruvchini almashtirish) usuli

Ko'pchilik hollarda integralda o'zgaruvchini almashtirish uni bevosita integrallashga olib keladi. Integrallashning bu usuli o'rniga qo'yish (o'zgaruvchini almashtirish) usuli deb yuritiladi. Bu usul quyidagi teoremaga asoslanadi.

1-teorema. Biror T oraliqda aniqlangan va differensiallanuvchi $x = \phi(t)$ funksiyaning qiymatlar sohasi Xdan iborat bo'lib, Xda $f(x)$ funksiya aniqlangan bo'lsin, y'ani Toraliqda $f(\phi(t))$ murakkab funksiya aniqlangan bo'lsin. Agar $F(x)$ funksiya X oraliqda $f(x)$ funksiyaning boshlang'ich funksiyasi bo'lsa, u holda

$$\int f(x)dx = \int f(\phi(t))\phi'(t)dt \qquad (1.4.1)$$

bo'ladi.

Isboti. X oraliqda $f(x)$ va $F(x)$ funksiyalar aniqlangan.

Shu sababli $f(\phi(t))$ va $F(\phi(t))$ murakkab funksiyalar T oraliqda aniqlangan, differensiallanuvchi hamda

$$(F(\phi(t)))' = F'(\phi(t))\phi'(t) = f(\phi(t))\phi'(t)$$

bo'ladi.

Bundan

$$\int f(\phi(t))\phi'(t)dt = \int (F(\phi(t)))'dt = F(\phi(t)) + C$$

$$= (F(x) + C)\,|x = \phi(t) = \int f(x)dx\bigg|_{x=\phi(t)}$$

hisobga olinsa,

$$\int f(x)dx = \int f(\phi(t))\phi'(t)dt$$

(1.4.1) formula aniqmas integralda o'zgaruvchini almashtirish formulasi deb yuritiladi.

Ayrim hollarda $t = \phi(x)$ o'rniga qo'yish bajarishga to'g'ri keladi.

U holda $\int f(\phi(x))\phi'(x)dx = \int f(t)dt$ bo'ladi. Demak, (1.4.1) formula o'ngdan chapga qo'llanishi ham mumkin.

Misollar.

1.$\int \frac{\sqrt{1+ln\,x}}{x\,ln\,x}dx$ integralni topamiz. $1 + ln\,x = t^2$ bo'lsin.

Bundan $ln\,x = t^2 - 1,\quad \frac{dx}{x} = 2tdt.$

(1.1) formulaga ko'ra

$$\int \frac{\sqrt{1+ln\,x}}{x\,ln\,x}dx = \int \frac{t\cdot 2tdt}{t^2-1} = 2\int \frac{t^2dt}{t^2-1} = 2\int \left(1 + \frac{1}{t^2-1}\right)dt$$

$$= 2\left(t + \frac{1}{2}ln\left|\frac{t-1}{t+1}\right|\right) + C =$$

$$= 2t + ln\left|\frac{(t-1)^2}{t^2-1}\right| + C = 2\sqrt{1+ln\,x} + ln\left|\frac{(\sqrt{1+ln\,x}-1)^2}{1+ln\,x-1}\right| + C =$$

$$= 2\sqrt{1+ln\,x} + 2\,ln\left|\sqrt{1+ln\,x} - 1\right| - ln|ln\,x| + C.$$

2.$\int x\sqrt{x-3}\,dx$ integralni topamiz. Bunibg uchun $\sqrt{x-3} = t$ o'rniga qo'yish bajaramiz. U holda $x = t^2 + 3,\ dx = 2tdt.$

Shu sababli

$$\int x\sqrt{x-3}dx = \int (t^2+3)\cdot t\cdot 2tdt = 2\int (t^4 + 3t^2)\,dt =$$

$$= 2\int t^4 dt + 6\int t^2\,dt = 2\cdot\frac{t^5}{5} + 6\cdot\frac{t^3}{3} + C$$
$$= \frac{2}{5}\sqrt{(x-3)^5} + 2\sqrt{(x-3)^3} + C.$$

3. $\int\sqrt{1+\cos^2 x}\sin 2x dx$integralni topamiz. Bunda $1+\cos^2 x = t^2$ deymiz.

Bundan $-2\cos x\sin x\,dx = 2tdt$ yoki $\sin 2x dx = -2tdt.$

U holda
$$\int\sqrt{1+\cos^2 x}\sin 2x dx = \int t(-2t)dt = -2\cdot\frac{t^3}{3} + C =$$
$$= -\frac{2}{3}\sqrt{(1+\cos^2 x)^3} + C.$$

4. $\int\frac{\sqrt{9-x^2}}{x^2}dx$ integralda $x = 3\sin t,\ dx = 3\cos t\,dt,\ \sqrt{9-x^2} = 3\cos t$ deymiz. Bunda $t = \arcsin\frac{x}{3}$. U holda
$$\int\frac{\sqrt{9-x^2}}{x^2}dx = \int\frac{\cos^2 t}{\sin^2 t}dt = \int\frac{1-\sin^2 t}{\sin^2 t}dt = \int\frac{dt}{\sin^2 t} - \int dt =$$
$$= -ctgt - t + C = -\frac{\cos t}{\sin t} - t + C = -\frac{\sqrt{1-\sin^2 t}}{\sin t} - t + C =$$
$$= -\frac{\sqrt{1-\left(\frac{x}{3}\right)^2}}{\frac{x}{3}} - arctg\frac{x}{3} + C = -\frac{\sqrt{9-x^2}}{x} - arctg\frac{x}{3} + C$$

Izoh. Ayrim hollarda integrallashning o‘zgaruvchini almashtirish usuli takroran qo‘llaniladi, ya’ni bunda bajarilgan o‘rniga qo‘yishdan so‘ng shunday integral hosil bo‘ladiki, bu integralni boshqa o‘rniga qo‘yish orqali soddalashtirish yoki jadval integraliga keltirish mumkiin bo‘ladi.

Bo‘laklab integrallash usuli

Bo‘laklab integrallash usuli ikki funktsiya ko‘paytmasining differensiali formulasiga asoslanadi.

2-teorema. $u(x)$ va $v(x)$ funksiyalar qandaydir X oraliqda aniqlangan va differentsiallanuvchi bo‘lib, $u'(x)v(x)$ funksiya bu oraliqda boshlang‘ich funksiyaga ega, y’ani $\int u'(x)v(x)dx$ integral mavjud bo‘lsin. U holda X oraliqda $u(x)v'(x)$ funksiya boshlang‘ich funksiyaga ega va
$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx \qquad (1.2)$$
bo‘ladi.

Isboti. $(u(x)v(x))' = u'(x)v(x) + v'(x)u(x)$ tenglikdan

$$u(x)v'(x) = (u(x)v(x))' - v(x)u'(x).$$

$(u(x)v(x))'$ va $u'(x)v(x)$ funksiyalar X intervalda boshlang'ich funksiyaga ega bo'lgani uchun $v'(x)u(x)$ ham X intervalda boshlang'ich funksiyaga ega bo'ladi. Oxirgi tenglikning chap va o'ng tomonini integrallasak, formula kelib chiqadi.

(1.2) formulaga aniqmas integralni bo'laklab integrallash formulasi deyiladi.

Ma'lumki, $v'(x)dx = dv, u'(x)dx = du$. Bundan (1.2) formula

$$\int udv = uv - \int vdu \qquad (1.3)$$

ko'rinishga keltiriladi.

Bo'laklab integrallash usulining mohiyati berilgan integralda integral ostidagi $f(x)dx$ ifodani udv ko'paytma shaklida tasvirlash va (1.3) formulani qo'llagan holda berilgan $\int udv$ integralni oson integrallanadigan $\int vdu$ integral bilan almashtirib topishdan iborat.

Bo'laklab integrallash orqali topiladigan integrallarni asosan uch guruhga ajratish mumkin:

1) $\int P(x)arctgx\,dx$, $\int P(x)\,arcctgxdx$, $\int P(x)\ln x\,dx$, $\int P(x)\arcsin x\,dx$,

$\int P(x)\arccos x\,dx$ (bu yerda $P(x)$ - ko'phad) ko'rinishdagi 1-guruh integrallar.

Bunda $dv = P(x)dx$, qolgan ko'paytuvchilar u bilan belgilanadi;

2) $\int P(x)e^{kx}dx$, $\int P(x)\sin kxdx$, $\int P(x)\cos kxdx$ ko'rinishdagi 2-guruh integrallar. Bunday bo'laklashda $u = P(x)$, qolgan ko'paytuvchilar dv deb olinadi;

3) $\int e^{kx}\sin kxdx$, $\int e^{kx}\cos kxdx$ ko'rinishdagi 3-guruh integrallar bo'laklab integrallash formulasini takroran qo'llash orqali topiladi.

Misollar

1. $$I = \int \sin x e^{2x} dx = \left| \begin{matrix} u = e^{2x}, & du = 2e^{2x}dx \\ dv = \sin xdx, & v = -\cos x \end{matrix} \right| = -e^{2x}\cos x +$$

$$+2\int e^{2x}\cos x\,dx = \left| \begin{matrix} u = e^{2x}, du = 2e^{2x}dx \\ dv = \cos x\,dx, v = \sin x \end{matrix} \right| =$$

$$= -e^{2x}\cos x + 2(e^{2x}\sin x - 2\int e^{2x}\sin x\,dx) =$$

$$= e^{2x}(2\sin x - \cos x) - 4I.$$

Bundan

$$I = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C.$$

$$2.\int arctgx\,dx = \begin{vmatrix} u = arctgx, & du = \frac{dx}{1+x^2} \\ dv = dx, v = x \end{vmatrix} =$$

$$= xarctgx - \int \frac{x}{1+x^2}dx = xarctgx - \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2}dx =$$

$$= xarctgx - \frac{1}{2}ln|1+x^2| + C.$$

$$3.\ \int xe^x dx = \begin{vmatrix} u = x, & du = dx \\ dv = e^x dx, & v = e^x \end{vmatrix} = xe^x - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C.$$

$$4.\int ln^2 x\,dx = \begin{vmatrix} ln^2 x = u, & du = 2\ln x \cdot \frac{dx}{x}, \\ dx = dv, & v = x \end{vmatrix} = x\,ln^2 x - 2\int ln\,x\,dx =$$

$$= \begin{vmatrix} ln\,x = u, du = \frac{dx}{x}, \\ dx = dv, v = x \end{vmatrix} = x\,ln^2 x - 2x\,ln\,x + 2\int dx =$$

$$= x\,ln^2 x - 2x\,ln\,x + 2x + C.$$

$$5.\ \int x^2 \sin 2x dx = \begin{vmatrix} x^2 = u, du = 2xdx, \\ \sin 2xdx = dv, \ v = -\frac{\cos 2x}{2} \end{vmatrix} = -\frac{1}{2}x^2\cos 2x +$$

$$+ \int x\cos 2xdx =$$

$$= \begin{vmatrix} x = u, du = dx, \\ \cos 2xdx = dv, v = \frac{\sin 2x}{2} \end{vmatrix}$$

$$= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2xdx =$$

$$= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C.$$

Ko'rsatilgan uch guruh bo'laklab integrallanadigan barcha integrallarni o'z ichiga olmaydi.

Masalan,

$$6.\ \int \sin x\,lncos\,x\,dx = \begin{vmatrix} lncos\,x = u, du = -\frac{\sin x}{\cos x}dx \\ \sin x\,dx = dv, v = -\cos x \end{vmatrix} =$$

$$-\cos x\,lncos\,x - \int \sin x\,dx = -\cos x\,lncos\,x + \cos x + C =$$

$$= \cos x\,(1 - lncos\,x) + C.$$

7. $\int \frac{xdx}{\cos^2 x} = \left| \begin{matrix} u = x, \ du = dx \\ dv = \frac{dx}{\cos^2 x}, v = tgx \end{matrix} \right| = xtgx -$

$$- \int tgxdx = xtgx - \ln|\cos x| + C.$$

8. $\int \frac{\ln \sin x}{\cos^2 x} dx = \left| \begin{matrix} \ln \sin x = u, \ du = \frac{\cos x}{\sin x} dx \\ \frac{dx}{\cos^2 x} = dv, \ v = tgx \end{matrix} \right| = tgx \ln \sin x - \int dx =$

$$tgx \ln \sin x - x + C;$$

Amaliy **mashg'ulot**da yechiladigan misollar

Berilgan aniqmas integrallarni toping

1. $\int \arccos x \, dx \left(Javob. x \arccos x - \sqrt{1 - x^2} + c \right)$

2. $\int xe^x dx (Javob. xe^x - e^x + C)$

3. $\int e^x \cos x \, dx \left(Javob. \frac{e^x}{2} (\cos x + \sin x) + c \right).$

4. $\int \frac{x \cos x dx}{\sin^2 x} (Javob. -\frac{x}{\sin x} + \ln \left| tg \frac{x}{2} \right| + c)$

5. $\int \frac{1}{\cos x} dx (Javob. \ln \left| tg \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + c)$

6. $\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}} dx \left(Javob. \frac{3}{2} \sqrt[3]{1 - \sin 2x} + c \right)$

7. $\int \frac{dx}{(\arcsin x)^2 \sqrt{1 - x^2}} \left(Javob. -\frac{1}{\arcsin x} + c \right)$

8. $\int \frac{x^2 + 1}{x^4 + 1} dx \left(Javob. \frac{1}{\sqrt{2}} arctg \frac{x^2 - 1}{x\sqrt{2}} + c \right)$

9. $\int \frac{dx}{\sin^2 x \sqrt[4]{ctgx}} \left(Javob. -\frac{4}{3} \sqrt[4]{ctg^3 x} + c \right)$

10. $\int \frac{dx}{\sin^2 x + 2\cos^2 x} \left(Javob. \frac{1}{\sqrt{2}} arctg \left(\frac{tgx}{\sqrt{2}} \right) + c \right)$

11. $\int \frac{dx}{\sin x} \left(Javob. \ln \left| tg \frac{x}{2} \right| + c \right)$

12. $\int \sin^2 x \, dx \, (Javob. \frac{x}{2} - \frac{1}{4} \sin 2x + c)$

Mustaqil yechish uchun misollar.

Berilgan aniqmas integrallarni toping

1. $\int \left(3x - \sqrt[7]{x^5} + 2 \sin x - 3 \right) dx$

2. $\int \frac{\ln x}{x} dx$

3. $\int \left(\sin 3x + x\sqrt{1+x^2}\right) dx$

4. $\int (x^7 - \frac{1}{\sqrt[3]{x}} + 2^x) dx$

5. $\int \frac{5x-2}{x^2+4} dx$

6. $\int \cos(3x-4)\, dx$

1.5. Sodda ratsional kasrlarni integrallash

I va II turdagi sodda kasrlar jadval integrallari orqali topiladi:

$$\int \frac{Adx}{x-\alpha} = A\int \frac{d(x-\alpha)}{x-\alpha} = A\ln|x-\alpha| + C; \qquad (1.1)$$

$$\int \frac{Adx}{(x-\alpha)^k} = A\int (x-\alpha)^{-k} d(x-\alpha) =$$

$$= A\frac{(x-\alpha)^{-k+1}}{-k+1} + C = \frac{A}{(1-k)(x-\alpha)^{k-1}} + C. \qquad (1.2)$$

III turdagi sodda kasrni qaraymiz. $\int \frac{Mx+N}{x^2+px+q} dx$ integralining suratida kasrning maxrajidan olingan hosila $(x^2+px+q)' = 2x+p$ ni ajratamiz va natijani integrallaymiz:

$$\int \frac{Mx+N}{x^2+px+q} dx = \int \frac{\frac{M}{2}(2x+p) + N - \frac{Mp}{2}}{x^2+px+q} dx$$

$$= \frac{M}{2}\int \frac{2x+p}{x^2+px+q} dx +$$

$$+\left(N - \frac{Mp}{2}\right)\int \frac{dx}{x^2+px+q} = \frac{M}{2}J_1 + \left(N - \frac{Mp}{2}\right)J_2.$$

Integrallardan birinchisi $J_1 = \ln|x^2+px+q|$.

Ikkinchi integral maxrajida to'liq kvadrat ajratamiz va uni integrallaymiz:

$$J_2 = \int \frac{dx}{x^2+px+q} = \int \frac{d\left(x+\frac{p}{2}\right)}{\left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4}}$$

$$= \frac{2}{\sqrt{4q-p^2}} arctg \frac{2x+p}{\sqrt{4q-p^2}},$$

bunda $4q - p^2 > 0$, chunki $D < 0$.

Natijada quyidagiga ega bo'lamiz:

$$\int \frac{Mx+N}{x^2+px+q}dx = \frac{M}{2}ln\,|x^2+px+q| + \frac{2N-Mp}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}} + C. \quad (1.3)$$

Misol

$\int \frac{5x+11}{x^2+6x+13}dx$ integralni topamiz.

$$\int \frac{5x+11}{x^2+6x+13}dx = \int \frac{\frac{5}{2}(2x+6)+11-\frac{5}{2}\cdot 6}{x^2+6x+13}dx =$$

$$= \frac{5}{2}\int \frac{(2x+6)dx}{x^2+6x+13} - 4\int \frac{dx}{x^2+6x+13} = \frac{5}{2}ln\,|x^2+6x+13| - 4J_3.$$

J_3 integralni topamiz:

$$J_3 = \int \frac{dx}{(x+3)^2+4} = \int \frac{d(x+3)}{(x+3)^2+2^2} = \frac{1}{2}arctg\frac{x+3}{2}.$$

Bundan

$$\int \frac{5x+11}{x^2+6x+13}dx = \frac{5}{2}ln\,|x^2+6x+13| - 2arctg\frac{x+3}{2} + C.$$

IV turdagi sodda kasrning integralni topamiz:

$$\int \frac{Mx+N}{(x^2+px+q)^s}dx = \frac{M}{2}\int \frac{(2x+p)dx}{(x^2+px+q)^s} +$$

$$+\left(N-\frac{Mp}{2}\right)\int \frac{d\left(x+\frac{p}{2}\right)}{\left(\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}\right)^s}. \quad (1.4)$$

Bunda birinchi integral jadvaldagi integralga keltirib, topiladi:

$$\int \frac{(2x+p)dx}{(x^2+px+q)^s}$$

$$= \int (x^2+px+q)^{-s}d(x^2+px+q)$$

$$= \frac{1}{(1-s)(x^2+px+q)^{s-1}}.$$

Ikkinchi integralga (uni I_s bilan belgilaymiz) $\left(x+\frac{p}{2}\right) = t$ belgilash kiritamiz va $0 < q - \frac{p^2}{4} = a^2$ almashtirish bajaramiz.

U holda

$$I_s = \int \frac{d\left(x+\frac{p}{2}\right)}{\left(\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}\right)^s} = \int \frac{dt}{(t^2+a^2)^s} =$$

$$\frac{1}{a^2}\int \frac{(t^2+a^2)-t^2}{(t^2+a^2)^s}dt = \frac{1}{a^2}\int \frac{dt}{(t^2+a^2)^{s-1}} - \frac{1}{a^2}\int \frac{t^2dt}{(t^2+a^2)^s}.$$

Bunda birinchi integral I_s ga o'xshash bo'lib, unda maxrajning darajasi bir birlikka kichik. Shu sababli uni I_{s-1} bilan belgilaymiz. Ikkinchi integralni bo'laklab integrallaymiz:

$$\int \frac{t^2dt}{(t^2+a^2)^s} = \frac{1}{2}\int \frac{t\cdot 2tdt}{(t^2+a^2)^s} =$$
$$= \frac{1}{2}\left(\frac{-t}{(s-1)(t^2+a^2)^{s-1}} + \frac{1}{s-1}\int \frac{dt}{(t^2+a^2)^{s-1}}\right) =$$
$$= -\frac{t}{2(s-1)(t^2+a^2)^{s-1}} + \frac{1}{2(s-1)} I_{s-1}.$$

Demak, I_s integralni hisoblash uchun s darajani pasaytirish formulasini hosil qilamiz:

$$I_s = \frac{1}{a^2} I_{s-1} + \frac{t}{2a^2(s-1)(t^2+a^2)^{s-1}} - \frac{1}{2a^2(s-1)} I_{s-1}$$

yoki

$$I_s = \frac{t}{2a^2(s-1)(t^2+a^2)^{s-1}} + \frac{2s-3}{2a^2(s-1)} I_{s-1}. \qquad (1.5)$$

Shunday qilib, (1.5) formula bo'yicha I_s integralni topamiz, I_s dagi barcha $t$ni $x+\frac{p}{2}$ bilan almashtiramiz va birinchi va ikkinchi integralni (1.4) tenglikka qo'yib IV turdagi sodda kasr integralini topish uchun ifoda hosil qilamiz.

(1.5) formula bo'yicha I_s integralni topish indeksi bittaga kichik bolgan I_{s-1} integralni topishga, I_{s-1} integralni topish esa o'z navbatida I_{s-2} integralni topishga keltiriladi va bu jarayon quyidagi jadval integralni topishgacha davom ettiriladi:

$$I_1 = \int \frac{dt}{t^2+a^2} = \frac{1}{a} arctg \frac{t}{a} + C.$$

Demak, (1.5) formula orqali $I_s$dan I_{s-1}ga , so'ngra I_{s-2} qaytiladi va hokazo. Shu sababli bunday formulalar keltirish yoki rekurrent (qaytuvchan) formulalar deyiladi.

Misol

$\int \frac{2x+5}{(x^2+4x+8)^2} dx$ integralni topamiz.

$$\int \frac{2x+4+1}{(x^2+4x+8)^2} dx = \int \frac{2x+4}{(x^2+4x+8)^2} dx + \int \frac{dx}{(x^2+4x+8)^2} =$$
$$= -\frac{1}{x^2+4x+8} + \int \frac{d(x+2)}{[(x+2)^2+4]^2} = -\frac{1}{x^2+4x+8} + \int \frac{dt}{(t^2+a^2)^2}$$

bu yerda $t = x+2,\ a = 2.$

Bundan

$$I_2 = \frac{t}{2a^2(t^2+a^2)} + \frac{1}{2a^3} arctg\frac{t}{a} = \frac{x+2}{8(x^2+4x+8)} + \frac{1}{16} arctg\frac{x+2}{2}.$$

Demak,

$$\int \frac{2x+5}{(x^2+4x+8)^2}dx$$

$$= -\frac{1}{x^2+4x+8} + \frac{x+2}{8(x^2+4x+8)} + \frac{1}{16} arctg\frac{x+2}{2} + C =$$

$$= \frac{x-6}{8(x^2+4x+8)} + \frac{1}{16} arctg\frac{x+2}{2} + C.$$

1.6. Ratsional kasr funksiyalarni integrallash

Yuqorida aytilganlardan kelib chiqadiki, ushbu $R(x) = \frac{Q_m(x)}{P_n(x)}$ ratsional kasr funksiyani integrallash quyidagi tartibda amalga oshiriladi:

1)berilgan ratsional kasrning to'g'ri yoki noto'g'ri kasr ekanini tekshirish; agar kasr noto'g'ri bo'lsa, kasrdan butun qismini ajratish;

2)to'g'ri kasrning maxrajini ko'paytuvchilarga ajratish;

3)to'g'ri kasrni sodda kasrlar yig'indisiga yoyish va yoyilmaning koeffitsiyentlarni topish;

4)hosil bo'lgan ko'phad va sodda kasrlar yig'indisini integrallash.

Misol

$\int \frac{x^4+6}{x^3-2x^2+2x}dx$ integralni topamiz.

$R(x) = \frac{x^4+6}{x^3-2x^2+2x}$ noto'g'ri kasr, chunki $m = 4, n = 3 (m > n)$.

Suratni maxrajga bo'lish orqali kasrdan butun qismini ajratamiz:

$$\begin{array}{r|l} -\;x^4+6 & x^3-2x^2+2x \\ x^4-2x^3+2x^2 & \overline{x+2} \\ \hline -\;2x^3-2x^2+6 & \\ 2x^3-4x^2+4x & \\ \hline 2x^2-4x+6 & \end{array}$$

Bundan

$$R(x) = x + 2 + \frac{2x^2-4x+6}{x^3-2x^2+2x}.$$

To'g'ri kasrning maxrajini ko'paytuvchilarga ajratamiz:

$$x^3 - 2x^2 + 2x = x(x^2 - 2x + 2).$$

To'g'ri kasrni sodda kasrlar ga yoyamiz:

$$\frac{2x^2-4x+6}{x(x^2-2x+2)}=\frac{A}{x}+\frac{Mx+N}{x^2-2x+2}.$$

Yoyilmaning koeffitsiyentlarini topamiz:

$$2x^2-4x+6=A(x^2-2x+2)+Mx^2+Nx.$$

Bundan

$$\begin{cases} x^2: A+M=2, \\ x^1: -2A+N=-4, \\ x^0: 2A=6. \end{cases}$$

yoki $A=3, M=-1, N=2.$

Shunday qilib,

$$R(x)=x+2+\frac{3}{x}+\frac{-x+2}{x^2-2x+2}.$$

Ko'phad va sodda kasrlar yig'indisini integrallaymiz:

$$\int\frac{(x^4+6)dx}{x^3-2x^2+2x}=\int(x+2)dx+\int\frac{3dx}{x}+\int\frac{-x+2}{x^2-2x+2}dx$$

$$=\frac{x^2}{2}+2x+3\,ln\,|x|-\int\frac{\frac{1}{2}(2x-2)+1-2}{x^2-2x+2}dx=$$

$$=\frac{x^2}{2}+2x+3\,ln\,|x|-\frac{1}{2}\int\frac{2x-2}{x^2-2x+2}dx+\int\frac{dx}{(x-1)^2+1}=$$

$$=\frac{x^2}{2}+2x+3\ln|x|-\frac{1}{2}ln\,|x^2-2x+2|+arctg(x-1)+C.$$

Amaliy **mashg'ulot**da yechiladigan misollar.

Berilgan aniqmas integrallarni toping

1. $\int\frac{2x-3}{x(x-1)(x-2}dx$ $(Javob.-\frac{3}{2}ln|x|+ln|x-1|+\frac{1}{2}ln|x-2|+c)$

2. $\int\frac{x}{(x-1)(x+1)^2}dx$ $(Javob.\frac{1}{4}ln\left|\frac{x-1}{x+1}\right|+\frac{1}{2}\frac{1}{x+1}+c)$

3. $\int\frac{x}{(x-1)(x^2+1)}dx$ $(Javob.\frac{1}{2}en|x-1|-\frac{1}{4}ln|x^2+1|+\frac{1}{2}arctgx+c)$

4. $\int\frac{x^5+x^4-8}{x^3-4x}dx$ $(Javob.\frac{x^3}{3}+\frac{x^2}{2}+4x+ln\left|\frac{x^2(x-2)^5}{(x+2)^3}\right|+c)$

5. $\int\frac{x^3+1}{x^3-x^2}dx$ $(Javob.x+\frac{1}{x}+ln\frac{(x-1)^2}{|x|}+c)$

6. $\int\frac{x^2-2x+3}{(x-1)(x^3-4x^2+3x)}dx$ $(Javob.\frac{1}{x-1}+ln\frac{\sqrt{(x-1)(x-3)}}{|x|}+c)$

7. $\int\frac{x^2}{x^4-1}dx$ $(Javob.\frac{1}{2}arctgx+\frac{1}{4}ln\left|\frac{1-x}{1+x}\right|+c)$

Mustaqil yechish uchun misollar.

Berilgan aniqmas integrallarni toping.

1. $\int \frac{x-4}{x^2-5x+6}dx$ (Javob. $\ln\frac{(x-2)^2}{|x-3|}+c$)

2. $\int \frac{1}{(x-1)(x+2)(x+3)}dx$ (Javob. $\frac{1}{12}\ln\left|\frac{(x-1)(x+3)^3}{(x+2)^4}\right|+c$)

3. $\int \frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}dx$ (Javob. $\ln\frac{\sqrt{(x^2-2x+5)^3}}{|x-1|}+\frac{1}{2}arctg\frac{x-1}{2}+c$)

4. $\int \frac{13}{x(x^2+6x+13}dx$ (Javob. $\ln\frac{x}{\sqrt{x^2+6x+13}}+5arctg\frac{x+3}{2}+c$)

5. $\int \frac{1}{x(x^2-1)}dx$ (Javob. $\ln\frac{\sqrt{x^2-1}}{|x|}+c$)

1.7. Trigonometrik funksiyalarni integrallash

Trigonometrik funksiyalarni integrallas usullaridan ayrimlari bilan tanishamiz. Faqat trigonometrik o‘zgaruvchlar ustida ratsional amallar (qo‘shish, ayirish, ko‘paytirish va bo‘lish) bajarilgan ifoda berilgan bo‘lsin. Bunday ifodani barcha trigonometrik funksiyalarni $\sin x$ va $\cos x$ funksiyalar orqali ratsional ravishda ifodalash va $R(\sin x, \cos x)$ ko‘rinishga keltirish mumkin.

1. $\int R(\sin x, \cos x)dx$ ko‘rinishidagi integrallar

$\int R(\sin x, \cos x)dx$ ko‘rinishidagi integralni $tg\frac{x}{2}=t$ o‘rniga qo‘yish orgali hamma vaqt t o‘zgaruvchili ratsional funksiyaning integraliga almashtirish, ya’ni ratsionallashtirish mumkin.

Haqiqatan ham, $\int R(\sin x, \cos x)dx$ ifodadan $\sin x=\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{2t}{1+t^2}$, $\cos x=\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$, $x=arctgt$, $dx=\frac{2dt}{1+t^2}$ o‘rniga qo‘yishlar yordamida t o‘zgaruvchili $\int R\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\cdot\frac{2dt}{1+t^2}=\int R_1(t)dt$ ratsional funksiya kelib chiqadi.

$tg\frac{x}{2}=t$ o‘rniga qo‘yish orgali $\int R(\sin x, \cos x)dx$ ko‘rinishidagi har qanday integralni topish mumkin. Shu sababli bu o‘rniga qo‘yish universal trigonometrik o‘rniga qo‘yish deb ataladi.

Misol. $\int\frac{dx}{3\sin x+2\cos x+3}$ integralni topamiz. Bunda $tg\frac{x}{2}=t$ o‘rniga qo‘yish bajaramiz. U holda

$$\int \frac{dx}{3\sin x + 2\cos x + 3} = \int \frac{\frac{2dt}{1+t^2}}{3\cdot\frac{2t}{1+t^2} + 2\cdot\frac{1-t^2}{1+t^2} + 3}$$
$$= 2\int \frac{dt}{t^2+6t+5} = 2\int \frac{dt}{(t+1)(t+5)} =$$
$$= \int \left(\frac{A}{t+1} + \frac{B}{t+5}\right) dt = A\ln|t+1| + B\ln|t+5| + C.$$

No‘malum koeffitsiyentlarni aniqlaymiz: $A = \frac{1}{2}$, $B = -\frac{1}{2}$.

Demak,

$$\int \frac{dx}{3\sin x + 2\cos x + 3} = \frac{1}{2}(\ln|t+1| - \ln|t+5|) + C = \frac{1}{2}\ln\left|\frac{t+1}{t+5}\right|$$
$$= \frac{1}{2}\ln\left|\frac{tg\frac{x}{2}+1}{tg\frac{x}{2}+5}\right| + C.$$

Universal trigonometrik o‘rniga qo‘yish $R(\sin x, \cos x)$ ko‘rinishidagi har qanday funksiyani ratsionallashtirish imkonini beradi, ammo amalda ko‘pincha ancha murakkab ratsional funksiyalar hosil bo‘lishi mumkin. Shu sababli ba’zan yuqorida keltirilgan integralni topishda quyidagi sodda o‘rniga qo‘yishlardan foydalaniladi:

a) agar $R(\sin x, \cos x)$ ifoda $\sin x$ ga nisbatan toq, ya’ni $R(-\sin x, \cos x) = -R(\sin x, \cos x)$ bo‘lsa, u holda $\cos x = t$ o‘rniga qo‘yish bu funksiyani ratsionallashtiradi;

b) agar $R(\sin x, \cos x)$ ifoda $\cos x$ ga nisbatan toq, ya’ni $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ bo‘lsa, u holda $\sin x = t$ o‘rniga qo‘yish orqali bu funksiya ratsionallashtiriladi;

c) agar $R(\sin x, \cos x)$ ifoda $\sin x$ va $\cos x$ larga nisbatan juft, ya’ni $R(-\sin x, -\cos x) = R(\sin x, \cos x)$ bo‘lsa, u holda $tgx = t$ o‘rniga qo‘yish bu funksiyani ratsionallashtiradi.

Bunda quyidagi almashtirishlardan foydalaniladi:

$\sin^2 x = \frac{tg^2 x}{1+tg^2 x} = \frac{t^2}{1+t^2}$, $\cos^2 x = \frac{1}{1+tg^2 x} = \frac{1}{1+t^2}$, $x = arctgt$, $dx = \frac{dt}{1+t^2}$.

Misollar

1. $\int \frac{\cos x dx}{\sin^2 x - 4\sin x + 5}$ integralni topami. Integral ostidagi funksiya $\cos x$ ga nisbatan toq funksiya. Shu sababli $\sin x = t, \cos x\, dx = dt$ deb olamiz.

U holda

$$\int \frac{\cos x\,dx}{\sin^2 x - 4\sin x + 5} = \int \frac{dt}{t^2 - 4t + 5} = \int \frac{d(t-2)}{(t-2)^2 + 1} =$$
$$= arctg(t-2) + C = arctg(\sin x - 2) + C.$$

2. $\int \frac{dx}{1 - 2\sin^2 x}$ integralni topamiz. Integral ostidagi funksiya $\sin x$ ga nisbatan juft funksiya, shu sababli $tgx = t$ o‘rniga qo‘yishdan foydalanamiz:

$$\int \frac{dx}{1 - 2\sin^2 x} = \int \frac{\frac{dt}{1+t^2}}{1 - \frac{2t^2}{1+t^2}} = \int \frac{dt}{1 - t^2} = \frac{1}{2} \ln \left| \frac{t+1}{t-1} \right|$$
$$= \frac{1}{2} \ln \left| \frac{tgx + 1}{tgx - 1} \right| + C.$$

2. $\int \sin^n x \cos^m x\,dx$ ko‘rinishidagi integrallar

$\int \sin^n x \cos^m x\,dx$ ko‘rinishidagi integrallar m va n butun sonlarga bog‘liq holda quyidagicha topiladi:

a) $n > 0$ va toq bo‘lganida $\cos x = t$ o‘rniga qo‘yish integralni ratsionallashtiradi;

a) $m > 0$ va toq bo‘lganida $\sin x = t$ o‘rniga qo‘yish orqali integral ratsionallashtiriladi;

c) ikkala m va n daraja ko‘rsatkichlar juft va nomanfiy bo‘lganida

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

formulalaridan foydalanib, darajalar pasaytiriladi;

d) $m + n < 0$ va juft bo‘lganida $tgx = t$ yoki $ctgx = t$ o‘rniga qo‘yishdan foydalaniladi. Bunda $m < 0$ va $n < 0$ bo‘lsa, u holda suratda $1 = (\sin^2 x + \cos^2 x)^k$, $\left(k = \frac{|m+n|}{2} - 1\right)$ almashtirishdan iborat sun’iy usul qo‘llab, ratsional funksiyalarni integrallashga keltiriladi;

e) $m, n \leq 0$ va ulardan biri toq bo‘lganida $\sin x$ va $\cos x$lardan qaysi birining darajasi toqligiga qarab, surat va maxrajni shu funksiyaga qo‘shimcha ko‘paytirishdan foydalaniladi.

Misollar

1. $\int \sin^5 x \cos^2 x dx$ $(n > 0$ va toq, $\cos x = t) = \int \sin^4 x \cos^2 x \sin x\,dx =$

$$= -\int (1 - t^2)^2 t^2 dt = -\int t^2\,dt + 2\int t^4\,dt - \int t^6\,dt$$
$$= -\frac{t^3}{3} + \frac{2t^5}{5} - \frac{t^7}{7} + C =$$
$$= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C.$$

2. $\int sin^4 x\, cos^2 x dx (n, m \geq 0\ va\ juft) = \int (sin\, x\, cos\, x)^2 sin^2 x dx =$

$$= \int \left(\frac{sin^2 2x}{4}\right) \cdot \left(\frac{1 - cos\, 2\, x}{2}\right) dx = \frac{1}{8} \int (sin^2 2\, x - sin^2 2\, x\, cos\, 2\, x) dx$$

$$=$$

$$= \frac{1}{8} \int \frac{1 - \cos 4x}{2} dx - \frac{1}{16} \int sin^2 2\, x d(sin\, 2\, x) =$$

$$= \frac{1}{16}\left(x - \frac{sin\, 4\, x}{4}\right) - \frac{sin^3 2\, x}{48} + C = \frac{1}{16}\left(x - \frac{sin\, 4\, x}{4} - \frac{sin^3 2\, x}{3}\right) + C.$$

3. $\int \frac{dx}{sin^4 x\, cos^2 x}$ integralda $n = -4,\ m = -2,\ n + m = -6 < 0,\ k = \frac{|m+n|}{2} - 1 = 2.$

Demak,

$$\int \frac{dx}{sin^4 x\, cos^2 x} = \int \frac{(sin^2 x + cos^2 x)^2}{sin^4 x\, cos^2 x} dx =$$

$$= \int \frac{sin^4 x + 2\, sin^2 x\, cos^2 x + cos^4 x}{sin^4 x\, cos^2 x} dx =$$

$$= \int \frac{dx}{cos^2 x} + 2 \int \frac{dx}{sin^2 x} + \int \frac{cos^2 x}{sin^4 x} d\, x$$

$$= tgx - 2ctgx - \int ctg^2 x d(ctgx) =$$

$$= tgx - 2ctgx - \frac{1}{3} ctg^3 x + C.$$

3. $\int tg^n x dx$ va $\int ctg^n x dx$ ko'rinishidagi integrallar

$\int tg^n x dx$ va $\int ctg^n x dx$ (bu yerda $n > 0$ butun son) ko'rinishidagi integrallar mos rasvishda $tgx = t$ va $ctgx = t$ o'rniga qo'yish orqali topiladi.

Bunday integrallarni orniga qo'yishllardan foydalanmasdan, bevosita

$$tg^2 x = \frac{1}{cos^2 x} - 1, ctg^2 x = \frac{1}{sin^2 x} - 1$$

formulalar yordamida hisoblash ham mumkin.

Misol.

$\int tg^5\, x dx$integralni ikki usul bilan topamiz.

1-usul. $\int tg^5\, x dx = \left| tgx = t, dx = \frac{dt}{1+t^2} \right| = \int \frac{t^5 dt}{1+t^2} = \int t^3\, dt -$

$$- \int t dt + \int \frac{tdt}{1 + t^2} = \frac{t^4}{4} - \frac{t^2}{2} + \frac{1}{2} \int \frac{d(1 + t^2)}{1 + t^2}$$

$$= \frac{t^4}{4} - \frac{t^2}{2} + \frac{1}{2} ln\, |\, 1 + t^2 | + C =$$

$$= \frac{1}{4}tg^4x - \frac{1}{2}tg^2x - \frac{1}{2}ln\,|\cos^2 x| + C =$$

$$= \frac{1}{4}tg^4x - \frac{1}{2}tg^2x - ln\,|\cos x| + C.$$

2-usul. $\int tg^5\, xdx = \int tg^3x \cdot tg^2xdx = \int tg^3\, x \cdot \left(\frac{1}{\cos^2 x} - 1\right)dx$

$$= \int tg^3x \cdot \frac{dx}{\cos^2 x} - \int tg^3xdx = \int tg^3xd(tgx) - \int tgx \cdot \left(\frac{1}{\cos^2 x} - 1\right)dx =$$

$$= \frac{1}{4}tg^4x - \int tgxd\,(tgx) - \int tgxdx = \frac{1}{4}tg^4x - \frac{1}{2}tg^2x - ln\,|\cos x| + C.$$

4. $\int \sin m\,x \cos n\,xdx,$ $\int \sin m\,x \sin n\,xdx,$ $\int \cos m\,x \cos n\,xdx$ ko'rinishidagi integrallar

Bu ko'rinishdagi integrallar

$$\sin m\,x \cos n\,x = \frac{1}{2}(\sin(m+n)x + \sin(m-n)x),$$

$$\sin m\,x \sin n\,x = \frac{1}{2}(\cos(m-n)x - \cos(m+n)x),$$

$$\cos m\,x \cos n\,x = \frac{1}{2}(\cos(m+n)x + \cos(m-n)x)$$

trigonometrik formulalar yordamida topiladi.

Misol

$$\int \cos 3\,x \cdot \cos 5\,xdx = \frac{1}{2}\int (\cos 8\,x + \cos 2\,x)dx =$$

$$= \frac{1}{2}\left(\frac{1}{8}\sin 8\,x + \frac{1}{2}\sin 2\,x\right) + C = \frac{1}{16}(\sin 8\,x + 4\sin 2\,x) + C.$$

Amaliy mashg'ulotda yechiladigan misollar.

Berilgan aniqmas integrallarni toping

1. $\int \frac{1}{3\sin^2 x + 5\cos^2 x}dx$ $(Javob. \frac{1}{\sqrt{15}} arctg\frac{\sqrt{3}tgx}{\sqrt{5}} + c)$

2. $\int \frac{1}{3+5\cos x}dx$ $(Javob. \frac{1}{4}ln\left|\frac{2+tg\frac{x}{2}}{2-tg\frac{x}{2}}\right| + c)$

3. $\int \frac{1}{3\sin^2 x + 3\sin x \cos x + \cos^2 x}dx$ $(Jabob. \frac{1}{\sqrt{13}} ln\left|\frac{\frac{1}{2}tgx+3-\sqrt{3}}{2tgx+3+\sqrt{3}}\right| + c)$

4. $\int \sin^4 3\,x\,dx$ $(Javob. \frac{3}{8}x - \frac{1}{2}\sin 6\,x + \frac{1}{96}\sin 1\,2x + c)$

5. $\int \frac{\cos^4 x + \sin^4 x}{\cos^2 x - \sin^2 x}dx$ $(Javob. \frac{1}{4}ln\left|\frac{1+tgx}{1-tgx}\right| + \frac{1}{2}\sin x \cos x + c)$

6. $\int \frac{1}{\cos x \sin^3 x}dx$ $(Javob. ln|tgx| - \frac{1}{2\sin^2 x} + c)$

7. $\int \cos^3 x \sin^{10} x\,dx$ $(Javob. \frac{\cos^{11} x}{11} - \frac{\cos^{13} x}{13} + c)$

$8.\int \frac{\sin^2 x}{\cos^2 x+1}dx$ (Javob. $\sqrt{2}arctg\left(\frac{tgx}{\sqrt{2}}\right) - x + c$)

Mustaqil yechish uchun misollar.

Berilgan aniqmas integrallarni toping.

$1.\int \frac{1}{4-5\sin x}dx$ (Javob. $\frac{1}{3}\ln\left|\frac{tg\frac{x}{2}-2}{2tg\frac{x}{2}-1}\right| + c$)

$2.\int \frac{\sin x}{\sin x+1}dx$ (Javob. $\frac{2}{1+tg\frac{x}{2}} + x + c$)

$3.\int \frac{\cos 2x}{\sqrt{3+4\sin 2x}}dx$ (Javob. $\frac{1}{2}\sqrt{3+4\sin 2x} + c$)

$4.\int \frac{1}{8-4\sin x+5\cos^2 x}dx$ $\left(\text{Javob.} \ln\left|\frac{tg\frac{x}{2}-5}{tg\frac{x}{2}-3}\right| + c\right)$

1.8. Giperbolik funksiyalarni integrallash

Giberbolik funksiyalarni integrallash trigonometrik funksiyalarni integrallash kabi amalga oshiriladi. Bunda giperbolik funksiyalar uchun o‘rinli bo‘ladigan quyidagi formulalardan foydalaniladi:

$ch^2x - sh^2x = 1, 2shx \cdot chx = sh2x, ch^2x = \frac{ch2x+1}{2}, ch2x =$

$sh^2x = \frac{ch2x-1}{2}, 1 - th^2x = \frac{1}{ch^2x}, cth^2x - 1 = \frac{1}{sh^2x}, = ch^2x + shx$

$shx = \frac{2th\frac{x}{2}}{1-th^2\frac{x}{2}}, chx = \frac{1+th^2\frac{x}{2}}{1-th^2\frac{x}{2}}.$

Misollar

1. $\int \frac{dx}{shx} = \int \frac{dx}{2sh\frac{x}{2}ch\frac{x}{2}} = \int \frac{1}{th\frac{x}{2}} \cdot \frac{\frac{dx}{2}}{ch^2\frac{x}{2}} = \int \frac{d\left(th\frac{x}{2}\right)}{th\frac{x}{2}} = \ln\left|th\frac{x}{2}\right| + C.$

2. $\int \frac{dx}{ch^4x} = \int \frac{1}{ch^2x} \cdot \frac{dx}{ch^2x} = \int (1 - th^2x)d(thx) = thx - \frac{1}{3}th^3x + C.$

3. $\int th^3xdx = \int thx \cdot th^2xdx = \int thx\left(1 - \frac{1}{ch^2x}\right)dx = \int thxdx -$

$$-\int thx\, d(thx) = \int \frac{shxdx}{chx} - \frac{1}{2}th^2x = \int \frac{d(chx)}{chx} - \frac{1}{2}th^2x =$$

$$= \ln|chx| - \frac{1}{2}th^2x + C.$$

4. $\int \frac{dx}{3chx+2shx}$ integralni hisoblashda $th\frac{x}{2} = t$ belgilash kiritamiz. $dx = \frac{2dt}{1-t^2}, shx = \frac{2t}{1-t^2}, chx = \frac{1+t^2}{1-t^2}$ o‘rniga o‘yishlar yordamida topamiz:

$$\int \frac{dx}{3chx + 2shx} = \int \frac{\frac{2dt}{1-t^2}}{3\cdot\frac{1+t^2}{1-t^2} + 2\cdot\frac{2t}{1-t^2}} = \frac{2}{3}\int \frac{dt}{t^2 + \frac{4}{3}t + 1} =$$

$$= \frac{2}{3}\int \frac{d\left(t + \frac{2}{3}\right)}{\left(t + \frac{2}{3}\right)^2 + \left(\frac{\sqrt{5}}{3}\right)^2} = \frac{2}{\sqrt{5}} arctg\left(\frac{3t+2}{\sqrt{5}}\right) + C =$$

$$= \frac{2}{\sqrt{5}} arctg\left(\frac{3th\frac{x}{2} + 2}{\sqrt{5}}\right) + C.$$

Giberbolik funksiyalarni o‘z ichiga olgan integrallarni $R(e^x)$ ratsional funksiyaning integraliga keltirib topish mumkin. Bunda $\int R\,(e^x)dx$ ko‘rinishdagi integrallar

$e^x = t$ o‘rniga qo‘yish yordamida ratsionallashtiriladi.

Misollar

1. $\int \frac{dx}{chx} = \int \frac{2dx}{e^x + e^{-x}} = 2\int \frac{e^x dx}{e^{2x}+1} = (e^x = t, e^x dx = dt) = 2\int \frac{dt}{t^2+1} =$

$$= 2\,arg\,t\,gt + C = 2arctge^x + C.$$

2. $\int \frac{2e^x - 1}{e^{2x} - e^x - 2}dx = \left(e^x = t, dx = \frac{dt}{t}\right) = \int \frac{2t-1}{t(t^2-t-2)}dt = \int \frac{(2t-1)dt}{t(t+1)(t-2)}$

Ratsional kasrni sodda kasrlarga yoyamiz:

$$\frac{2t-1}{t(t+1)(t-2)} = \frac{A}{t} + \frac{B}{t+1} + \frac{C}{t-2}.$$

Yoyilmaning koeffitsiyentlarini topamiz:

$$2t - 1 = A(t^2 - t - 2) + B(t^2 - 2t) + C(t^2 + t).$$

Bundan

$$\begin{cases} t^2: A + B + C = 0, \\ t^1: -A - 2B + C = 2, \\ x^0: -2A = -1. \end{cases}$$

yoki $A = \frac{1}{2}, B = -1, C = \frac{1}{2}$.

Shunday qilib,

$$\int \frac{2e^x - 1}{e^{2x} - e^x - 2}dx = \int \frac{2t-1}{t(t+1)(t-2)}dt = \frac{1}{2}\int \frac{dt}{t} - \int \frac{dt}{t+1} + \frac{1}{2}\int \frac{dt}{t-2} =$$

$$= \frac{1}{2}ln|t| - ln|t+1| + \frac{1}{2}ln|t-2| + C =$$

$$= \frac{1}{2}ln\frac{|t(t-2)|}{(t+1)^2} + C = \frac{1}{2}ln\frac{|e^x(e^x-2)|}{(e^x+1)^2} + C.$$

1.9. Irratsional ifodalarni integrallash

Irratsional ifodalarni o'z ichiga olgan ayrim integrallarni ko'rib chiqamiz.

1. $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_1}{n_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_2}{n_2}}, \dots, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_k}{n_k}}\right)dx$ ko'rinishidagi integrallar

$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_1}{n_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_2}{n_2}}, \dots, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_k}{n_k}}\right)dx$ (R -ratsional funksiya, $m_1, n_1, m_2, n_2, \dots, m_k, n_k$,- butun sonlar) ko'rinishdagi integrallar $\frac{ax+b}{cx+d} = t^s$ o'rniga qo'ish yordamida ratsional funksiyaning integraliga keltiriladi, bunda $s = EKUK(n_1, n_2, \dots, n_k)$.

Misollar

1. $\int \frac{x^2+\sqrt[3]{1+x}}{\sqrt{1+x}}dx$ integralni topamiz. Bunda $EKUK(2,3) = 6$.

$1 + x = t^6$ deymiz.

U holda

$$\sqrt{1+x} = t^3, \sqrt[3]{1+x} = t^2, \quad dx = 6t^5dt.$$

Demak,

$$\int \frac{x^2 + \sqrt[3]{1+x}}{\sqrt{1+x}}dx = \int \frac{(t^6-1)^2 + t^2}{t^3} \cdot 6t^5dt$$

$$= 6\int t^2(t^{12} - 2t^6 + t^2 + 1)dt =$$

$$= 6\left(\frac{t^{15}}{15} - 2\frac{t^9}{9} + \frac{t^5}{5} + \frac{t^3}{3}\right) + C = \frac{2t^3}{15}(3t^{12} - 10t^6 + 9t^2 + 15) + C =$$

$$= \frac{2\sqrt{1+x}}{15}\left(3(1+x)^2 - 10(1+x) + 9\sqrt[3]{1+x} + 15\right) + C.$$

2. $\int \frac{4x^2+\sqrt[3]{2x+1}}{\sqrt{2x+1}}dx$ integralni topamiz. Bunda $EKUK(2,3) = 6$.

$2x + 1 = t^6$ deymiz.

U holda

$$\sqrt{2x+1} = t^3, \sqrt[3]{2x+1} = t^2, dx = 3t^5dt.$$

Demak,

$$\int \frac{4x^2 + \sqrt[3]{2x+1}}{\sqrt{2x+1}}dx = \int \frac{(t^6-1)^2 + t^2}{t^3} \cdot 3t^5 dt$$

$$= 3\int t^2\,(t^{12} - 2t^6 + t^2 + 1)dt =$$

$$= 3\left(\frac{t^{15}}{15} - 2\frac{t^9}{9} + \frac{t^5}{5} + \frac{t^3}{3}\right) + C = \frac{t^3}{15}(3t^{12} - 10t^6 + 9t^2 + 15) + C =$$

$$= \frac{\sqrt{2x+1}}{15} \cdot (12x^2 - 8x + 9\sqrt[3]{2x+1} + 8) + C.$$

2. $\int R(x, \sqrt{ax^2+bx+c})\,dx$ ko'rinishidagi integrallar

$\int R(x, \sqrt{ax^2+bx+c})\,dx$ ko'rinishidagi integrallar Eylerning uchta o'rniga qo'yichi orqali ratsional funksiyalardan olingan integrallarga keltiriladi:

a) $a > 0$ bo'lganida $\sqrt{ax^2+bx+c} = t \pm \sqrt{a}x$ almashtirish orqali integral ostidagi funksiya ratsionallashtiriladi (Eylerning birinchi o'rniga qo'yishi);

b) $c > 0$ bo'lganida $\sqrt{ax^2+bx+c} = tx \pm \sqrt{c}$ almashtirish yordamida integral ostidagi funksiya ratsionallashtiriladi (Eylerning ikkinchi o'rniga qo'yishi);

c) $ax^2 + bx + c$ kvadrat uchhad $a(x-x_1)(x-x_2)$ ko'rinishda ko'paytuvchilarga ajralganida integral ostidagi funksiya $\sqrt{ax^2+bx+c} = t(x-x_1)$ almashtirish bilan ratsionallashtiriladi (Eylerning uchinchi o'rniga qo'yishi).

Misollar

1. $\int \frac{dx}{1+\sqrt{x^2+2x+2}}$ integralni topamiz. Bunda $a > 0$. Shu sababli $\sqrt{x^2+2x+2} = t - x$ o'rniga qo'yish bajaramiz.

U holda

$$x^2 + 2x + 2 = t^2 - 2tx + x^2, 2x + 2tx = t^2 - 2.$$

Bundan

$$x = \frac{t^2-2}{2(1+t)}, dx = \frac{t^2+2t+2}{2(1+t)^2}, 1 + \sqrt{x^2+2x+2} =$$

$$= 1 + t - \frac{t^2-2}{2(1+t)} = \frac{t^2+4t+4}{2(1+t)}.$$

Topilganlarni berilgan integralga qo'yamiz:

$$\int \frac{dx}{1+\sqrt{x^2+2x+2}} = \int \frac{2(1+t)(t^2+2t+2)}{(t^2+4t+4)2(1+t)^2}dt =$$
$$= \int \frac{t^2+2t+2}{(1+t)(2+t)^2}dt.$$
Integral ostidagi to‘g‘ri kasrni sodda kasrlarga yoyamiz:
$$\frac{t^2+2t+2}{(1+t)(2+t)^2} = \frac{A}{1+t} + \frac{B}{2+t} + \frac{C}{(2+t)^2}.$$
Koeffitsiyentlarni tenglashtirish usulini qo‘llaymiz: $A = 1,$
$$B = 0, C = -2.$$
Bundan
$$\int \frac{dx}{1+\sqrt{x^2+2x+2}} = \int \frac{dt}{1+t} - 2\int \frac{dt}{(2+t)^2} = ln\,|1+t| + \frac{2}{2+t} + C.$$
x o‘zgaruvchiga qaytamiz:
$$\int \frac{dx}{1+\sqrt{x^2+2x+2}} =$$
$$= ln\left|1+x+\sqrt{x^2+2x+2}\right| + \frac{2}{x+2+\sqrt{x^2+2x+2}} + C.$$
2. $\int \frac{dx}{x\sqrt{x^2+x+1}}$ integralda $c > 0$. Shu sababli $\sqrt{x^2+x+1} = tx+1$ deymiz.

U holda
$$t = \frac{\sqrt{x^2+x+1}-1}{x} \text{ va } x^2+x+1 = t^2x^2+2xt+1, x - xt^2 = 2t-1.$$
Bundan
$$x = \frac{2t-1}{1-t^2}, dx = 2\frac{t^2-t+1}{(1-t^2)^2}dt, \sqrt{x^2+x+1} = \frac{t^2-t+1}{1-t^2}.$$
Topilganlarni berilgan integralga qo‘yamiz:
$$\int \frac{dx}{x\sqrt{x^2+x+1}} = \int \left(\frac{1-t^2}{2t-1}\right)\cdot\left(\frac{1-t^2}{t^2-t+1}\right)\cdot\left(2\frac{t^2-t+1}{(1-t^2)^2}dt\right)$$
$$= \int \frac{2dt}{2t-1}.$$
Bundan
$$\int \frac{dx}{x\sqrt{x^2+x+1}} = \int \frac{2dt}{2t-1} = ln\,|2t-1| + C =$$
$$= ln\left|\frac{2\sqrt{x^2+x+1}-2-x}{x}\right| + C.$$

4. $\int \frac{dx}{\sqrt{x^2-3x+2}}$ integralni topamiz. Bunda

$x^2 - 3x + 2 = (x-1)(x-2)$ bo‘lgani uchun $\sqrt{(x-1)(x-2)} = (x-1)t$ o‘rniga qo‘yish bajaramiz.

U holda

$$(x-1)(x-2) = (x-1)^2t^2, t = \sqrt{\frac{x-2}{x-1}}.$$

Bundan

$$x = \frac{t^2-2}{t^2-1}, dx = \frac{2tdt}{(t^2-1)^2}, \sqrt{x^2-3x+2} = \left(\frac{t^2-2}{t^2-1} - 1\right)t = = -\frac{t}{t^2-1}.$$

Topilganlarni berilgan integralga qo‘yamiz:

$$\int \frac{dx}{\sqrt{x^2-3x+2}} = \int \frac{-(t^2-1)2tdt}{(t^2-1)^2t} = -2\int \frac{dt}{t^2-1} = $$
$$= -\ln\left|\frac{t-1}{t+1}\right| + C = -\ln\left|\frac{1-2t+t^2}{t^2-1}\right| + C$$

Eski o‘zgaruvchiga qaytami:

$$\int \frac{dx}{\sqrt{x^2-3x+2}} = -\ln\left|\frac{1-2\sqrt{\frac{x-2}{x-1}} + \frac{x-2}{x-1}}{\frac{x-2}{x-1} - 1}\right| + C$$
$$= -\ln|3 - 2x + 2\sqrt{x^2-3x+2}| + C.$$

Eyler o‘rniga qo‘yishlari ko‘p integrallarda murakkab hisoblashlarga olib kelishi mumkin. Bunday hollarda integrallashning quyidagi usullaridan foydalaniladi.

$\int \boldsymbol{R(x, \sqrt{ax^2+bx+c})\, dx}$ ko‘rinishidagi integrallarni hisoblashning boshqa bir usuli kvadrat uchhaddan to‘la kvadrat ajratish usulidir. Bu usulda $\int \boldsymbol{R(x, \sqrt{ax^2+bx+c})\, dx}$ ko‘rinishidagi integrallar kvadrat uchhaddan to‘la kvadrat ajratish yo‘li bilan ushbu integrallardan biriga keltiriladi:

a) agar $a > 0$ va $b^2 - 4ac < 0$ bo‘lsa, u holda $\int R(t, \sqrt{m^2 + n^2t^2})\, dt,$ bu yerda

$n^2 = a, m^2 = -\frac{b^2-4ac}{4a}, t = x + \frac{b}{2a};$

b) agar $a > 0$ va $b^2 - 4ac > 0$ bo'lsa, u holda $\int R(t, \sqrt{n^2t^2 - m^2})\, dt$, bu yerda $n^2 = a, m^2 = \frac{b^2-4ac}{4a}$, $t = x + \frac{b}{2a}$;

c) agar $a < 0$ va $b^2 - 4ac > 0$ bo'lsa, u holda $\int R(t, \sqrt{m^2 - n^2t^2})\, dt$, bu yerda

$$n^2 = -a, m^2 = -\frac{b^2-4ac}{4a}, t = x + \frac{b}{2a}.$$

Mos ravishda $t = \frac{m}{n} tgz$, $t = \frac{m}{n \sin z}$, $t = \frac{m}{n} \sin z$ o'rniga qo'yishlar orqali oxirgi integrallar $\int R(\sin z, \cos z) dz$ ko'rinishga keltiriladi.

Misol.

$\int \sqrt{5 + 4x - x^2} dx$ integralni topamiz. Buning uchun kvadrat uchhaddan to'la kvadrat ajratamiz, yangi t o'zgaruvchi kiritamiz va trigonometrik o'rniga qo'yishdan foydalanib, topamiz:

$$\int \sqrt{5 + 4x - x^2}\, dx = \int \sqrt{9 - (x-2)^2}\, dx = \begin{vmatrix} x - 2 = t, \\ dx = dt \end{vmatrix}$$

$$= \int \sqrt{9 - t^2}\, dt =$$

$$= \begin{vmatrix} t = 3\sin z, \\ dt = \cos z dz \end{vmatrix} = \int \sqrt{9 - 9\sin^2 z}\, 3\cos z\, dz = \int 9\cos^2 z dz =$$

$$= \frac{9}{2}\int (1 + \cos 2z) dz = \frac{9}{2}\left(z + \frac{\sin 2z}{2}\right) + C = \quad \frac{9}{2}(z + \sin z\sqrt{1 - \sin^2 z}) + C =$$

$$= \left|z = \arcsin\frac{t}{3}\right| = \frac{9}{2}\left(\arcsin\frac{t}{3} + \frac{t}{3}\sqrt{1 - \frac{t^2}{9}}\right) + C = \frac{9}{2}\arcsin\frac{t}{3} + \frac{t}{2}\sqrt{9 - t^2} + C =$$

$$\frac{9}{2}\arcsin\frac{x-2}{3} + \frac{1}{2}(x-2)\sqrt{5 + 4x - x^2} + C.$$

Bundan tashqari $\int \boldsymbol{R(x, \sqrt{ax^2 + bx + c})\, dx}$ ko'rinishidagi integrallarni hisoblashda quyidagi usullarni qo'llash mumkin:

a) $\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}}$ ko'rinishidagi integrallar, bu yerda $P_n(x)$ n - darajali ko'phad:

1) $n = 0$ da $\int \frac{Adx}{\sqrt{ax^2+bx+c}}$ bo'ladi; bu integrallar $a > 0$ bo'lganda jadvaldagi 14 - integralga, $a < 0$ bo'lganda jadvaldagi 13 - integralga keltiriladi;

2) $n = 1$ da $\int \frac{(Ax+B)dx}{\sqrt{ax^2+bx+c}}$ bo'ladi; bu integrallar suratda kvadrat uchhadning hosilasini ajratish natijasida ikkita, biri jadvaldagi 1- integralga va ikkinchisi 1) banddagi integralga keltiriladi;

3) $n \geq 2$ bo'lganda berilgan integraldan keltirish formulalari yordamida quyidagi ko'rinishdagi ifoda hosil qilinadi:

$$\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}} = Q_{n-1}(x)\sqrt{ax^2+bx+c} + M\int \frac{dx}{\sqrt{ax^2+bx+c}},$$

bu yerda $Q_{n-1}(x)$ - koeffitsiyentlari noma'lum bo'lgan $n-1$- darajali ko'phad,

M - qandaydir o'zgarmas son. Bunda ko'phadning noma'lum koeffitsientlari va M soni oxirgi tenglikni differensiallash hamda x ning chap va o'ng tomondagi bir xil darajalari oldidagi koeffitsientlarni tenglashtirish orqali topiladi.

b) $\int \frac{dx}{(\alpha x+\beta)\sqrt{ax^2+bx+c}}$ ko'rinishidagi integral $\alpha x + \beta = \frac{1}{t}$ almashtirish yordamida 1) banddagi integralga keltiriladi;

c) $\int \frac{dx}{(\alpha x+\beta)^n\sqrt{ax^2+bx+c}} (n \in Z, n > 1)$ ko'rinishidagi integrallar $\alpha x + \beta = \frac{1}{t}$ o'rniga qo'yish orqali 3) banddagi integralga keltiriladi.

Misol

$\int \frac{dx}{(x-2)^3\sqrt{x^2-4x+5}}$ integralni topamiz. Bunda $x-2=\frac{1}{t}$ deymiz. U holda $dx = -\frac{dt}{t^2}, x^2-4x+5 = \frac{1}{t^2}+1.$

Bundan

$$\int \frac{dx}{(x-2)^3\sqrt{x^2-4x+5}} = -\int \frac{\frac{dt}{t^2}}{\frac{1}{t^3}\sqrt{\frac{1}{t^2}+1}} = -\int \frac{t^2dt}{\sqrt{t^2+1}}.$$

b) banddagi integral hosil qilindi. $n=2$ bo'lgani uchun

$$\int \frac{t^2dt}{\sqrt{t^2+1}} = (At+B)\sqrt{t^2+1} + M\int \frac{dt}{\sqrt{t^2+1}}.$$

Tenglikning har ikkala tomonini differensiallaymiz:

$$\frac{t^2}{\sqrt{t^2+1}} = A\sqrt{1+t^2} + \frac{(At+B)t}{\sqrt{t^2+1}} + \frac{M}{\sqrt{t^2+1}}$$

yoki

$$t^2 = A(1+t^2) + (At+B)t + M.$$

x ning bir xil darajalari oldidagi koeffitsientlarni tenglab, topamiz:

$$A = \frac{1}{2}, B = 0, M = -\frac{1}{2}.$$

U holda

$$\int \frac{t^2dt}{\sqrt{1+t^2}} = \frac{t\sqrt{1+t^2}}{2} - \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}} = \frac{t\sqrt{1+t^2}}{2} - \frac{1}{2}ln\left|t+\sqrt{1+t^2}\right| + C.$$

Eski o'zgaruvchiga qaytamiz:

$$\int \frac{dx}{(x-2)^3\sqrt{x^2-4x+5}} = -\frac{\sqrt{x^2-4x+5}}{2(x-2)^2} + \frac{1}{2}ln\left|\frac{1+\sqrt{x^2-4x+5}}{x-2}\right| + C.$$

3. $\int x^m(a+bx^n)^p dx$ binominal differensial integrali $\int x^m(a+bx^n)^p dx$ ko'rinishidagi integral binominal differensial integrali deyiladi. Bunda integral ostidagi ifoda $x^m(a+bx^n)^p$ ga binominal differensial deyiladi, bu yerda m, n, p – ratsional sonlar

Binominal differensial integrali uchta holdagina ratsional funksiyalarni integrallashga keltiriladi:

a) p butun son bo'lganida integral $x = t^s$ (bu yerda $s = EKUK(m,n)$) o'rniga qo'yish orqali ratsionallashtiriladi;

b) $\frac{m+1}{n}$ butun son bo'lganida integral $a + bx^n = t^s$ (bu yerda $s - p$ sonning maxraji) o'rniga qo'yish yordamida ratsionallashtiriladi;

c) $\frac{m+1}{n} + p$ butun son bo'lganida integralda $a + bx^n = t^s x^n$ (bu yerda $s - p$ sonning maxraji) almashtirish bajariladi.

Agar yuqorida keltirilgan shartlar bajarilmasa binominal differensial elementar funksiyalar orqali ifodalanmaydi, ya'ni integrallanmaydi.

Masalan, $\int \sqrt{1+x^3}\, dx$ integralning integral osti funksiyasi binominal differensial: $m = 0, n = 3, p = \frac{1}{2}$. Bunda $p = \frac{1}{2}$, $\frac{m+1}{n} = \frac{1}{3}$ $\frac{m+1}{n} + p = \frac{5}{6}$ sonlardan birortasi butun son emas. Shu sababli bu integral elementar funksiyalar orqali ifodalanmaydi.

Misol

$\int \frac{dx}{x^3\sqrt{1+x^4}}$ integralni topamiz. Shartga ko'ra $m = -3, n = 4,\ p = -\frac{1}{2}$,

$$\frac{m+1}{n} + p = \frac{-3+1}{4} - \frac{1}{2} = -1.$$

U holda

$$1 + x^4 = t^2x^4,\ x = (t^2-1)^{-\frac{1}{4}},\ dx = -\frac{1}{2}t(t^2-1)^{-\frac{5}{4}},\ t = \frac{\sqrt{1+x^4}}{x^2}.$$

Demak,

$$\int \frac{dx}{x^3\sqrt{1+x^4}} = \int x^{-3}(1+x^4)^{-\frac{1}{2}}dx =$$

$$= -\frac{1}{2}\int (t^2-1)^{-\frac{1}{4}\cdot(-3)}(t^2)^{-\frac{1}{2}}\left((t^2-1)^{-\frac{1}{4}\cdot 4}\right)^{-\frac{1}{2}} t(t^2-1)^{-\frac{5}{4}}dt =$$

$$= -\frac{1}{2}\int (t^2-1)^{\frac{3}{4}+\frac{1}{2}-\frac{5}{4}}\cdot t^{-1+1}dt = -\frac{1}{2}\int dt = -\frac{1}{2}t + C = -\frac{\sqrt{1+x^4}}{2x^2} + C.$$

Amaliy **mashg'ulot**da yechiladigan misollar.

Berilgan aniqmas integrallarni toping

1. $\int \frac{\sqrt{x}}{\sqrt[4]{x^3+4}}dx$ ($Javob. \frac{4}{3}\sqrt[4]{x^3} - \frac{16}{3}ln|\sqrt[4]{x^3}+4| + c$)

2. $\int \frac{\sqrt{x}}{\sqrt[3]{x^2}-\sqrt[4]{x}}dx$ ($Javob. \frac{6}{5}\sqrt[6]{x^5} + \frac{12}{5}\sqrt[12]{x^5} + \frac{12}{5}ln|\sqrt[12]{x^5} - 1| + c$)

3. $\int \frac{1}{3x-4\sqrt{x}}dx$ ($Javob. \frac{2}{3}ln|\sqrt{x}+4| + c$)

4. $\int \frac{1}{\sqrt{3x+4}+2\sqrt[4]{3x+4}}dx$($Javob. \frac{4}{3}(\frac{1}{2}\sqrt{3x+4} - 2\sqrt[4]{3x+4} +$

$4\,ln(\sqrt[4]{3x+4} + 21) + c$)

5. $\int \frac{1}{\sqrt{x}-7\sqrt[4]{x}}dx$($Javob. 4(\frac{1}{2}\sqrt{x} + 7\sqrt[4]{x} + 49\,ln|\sqrt[4]{x} - 7| + c$)

6. $\int \sqrt{\frac{1-x}{1+x}}\frac{dx}{x}$ ($Javob. \frac{2}{3}ln\left|\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right| + 2arctg\sqrt{\frac{1-x}{1+x}} + c$)

Mustaqil yechish uchun misollar.

Berilgan aniqmas integrallarni toping.

1. $\int x^5\sqrt[3]{(1+x^3)^2}\,dx$ ($Javob. \frac{1}{8}\sqrt[3]{(1+x^3)^8} - \frac{1}{5}\sqrt[3]{(1+x^3)^5} + c$)

2. $\int \frac{\sqrt{x}}{\sqrt[4]{x^3}+1}dx$ ($Javob. \frac{4}{3}(\sqrt[4]{x^3} - ln|\sqrt[4]{x^3}+1| + c$)

3. $\int \frac{x^3}{\sqrt{x^2+2}}dx$ ($Javob. (\frac{(x^2-4)\sqrt{x^2+2}}{3} + c$)

4. $\int \frac{\sqrt{x^3}-\sqrt[3]{x}}{\sqrt[4]{x}}dx$ ($Javob. (\frac{2}{9}\sqrt[4]{x^9} - \frac{12}{13}\sqrt[12]{x^{13}} + c$)

5. $\int \frac{1}{\sqrt[3]{x}+\sqrt{x}}dx$ ($Javob. 6(\sqrt[6]{x} - \frac{\sqrt[3]{x}}{2} + \frac{\sqrt{x}}{3} - ln(1+\sqrt[6]{x}) + c$)

6. $\int \frac{4x}{\sqrt[3]{(3x-8)^2}-2\sqrt[3]{3x-8}+4}dx$($Javob. \frac{1}{3}(\sqrt[3]{(3x-8)^4} + \frac{8}{9}(3x-8) + c$)

1.10. Boshlang‘ich funksiyasi elementar funksiyalarning chekli yigindisi ko‘rinishida tasvirlanmaydigan funksiylar

Biz integrallashning elementar funksiyalarning keng sinfini qamrab olgan muhim usullarini ko‘rib chiqdik. Bu usullar ko‘pchilik hollarda aniqmas integralni topish, ya’ni boshlang‘ich funksiyalarni aniqlash imkonini beradi.

Ma’lumki, har qanday uzluksiz funksiya boshlang‘ich funksiyaga ega bo‘ladi. Agar biror $f(x)$ elementar funksiyaning boshlang‘ich funksiyasi ham elementar funksiya bo‘lsa, u holda $\int f(x)dx$ integral elementar funksiyalarda olinadi deyiladi. Bunda integral elementar funksiyalar orqali ifodalanadi (yoki integralni topsa bo‘ladi). Agar integral elementar funksiyalar orqali ifodalanmasa, u holda $\int f(x)dx$ integral elementar funksiyalarda olinmaydi (yoki integralni topib bo‘lmaydi) deyiladi.

Masalan, $\int \sqrt{x} \cdot \cos x\, dx$ integral elementar funksiyalarda olinmaydi, chunki hosilasi $\sqrt{x} \cdot \cos x$ ga teng bo‘lgan elementar funksiya mavjud emas. Amaliy tatbiqda muhim ahamiyatga ega bo‘lgan elementar funksiyalarda olinmaydigan integrallarga misollar keltiramiz:

$\int e^{-x^2} dx$ – Puasson integrali (ehtimollar nazariyasi);

$\int \frac{dx}{\ln x}$ –integralli logarifm (sonlar nazariyasi);

$\int \cos x^2\, dx, \int \sin x^2\, dx$ –Frenel integrallari (fizika);

$\int \frac{\sin x}{x} dx, \int \frac{\cos x}{x} dx$ –integralli sinus va kosinus;

$\int \frac{e^x}{x} dx$ –integralli ko‘rsatkichli funksiya.

Elementar funksiyalarda ifolanmasada, $e^{-x^2}, \frac{1}{\ln x}, \cos x^2, \sin x^2, \frac{\sin x}{x}, \frac{\cos x}{x}, \frac{e^x}{x}$ funksiyalarning boshlang‘ich funksiyalari yetarlicha o‘rganilgan, ularning qiymatlari uchun x argumentning turli qiymatlarida mufassal jadvallar tuzilgan.

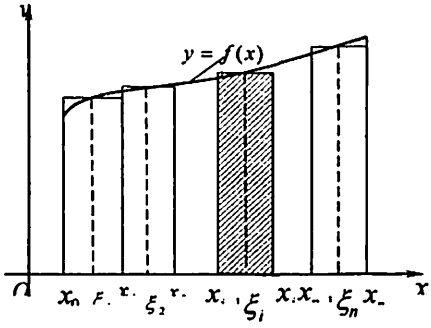
II BOB. ANIQ INTEGRAL

2.1.Aniq integral tushunchasiga olib keluvchi masalalar. Egri chiziqli trapetsiyaning yuzasni hisoblash masalasi

Aniq integral iqtisodiyotda va texnikaning bir qancha masalalarini yechishda, xususan, har xil geometrik va fizik kattaliklarni hisoblashda keng qo‘llaniladi.

Tekislikda Oxy to‘g‘ri burchakli dekart koordinatalar sistemasi kiritilgan va $[a; b]$,$b > a$ kesmada uzluksiz va manfiy bo‘lmafan $y = f(x)$, ya’ni $f(x) \geq 0$ funksiya aniqlangan bo‘lsin.

Yuqoridan $y = f(x)$ funksiya grafigining yoyi bilan, quyidan Ox o‘qning $[a; b]$ kesmasi bilan, yon tomonlaridan $x = a, 0 \leq y \leq f(a)$ va $x = b, 0 \leq y \leq f(b)$ to‘g‘ri chiziqlar bilan chegaralangan $aABb$ figuraga egri chiziqli trapetsiya deyiladi.



2-shakl

$aABb$ egri chiziqli trapetsiyaning S yuzasiga ta’rif beramiz. $[a; b]$ kesmani n ta kichik kesmalarga bo‘lamiz: bo‘linish nuqtalarining abssissalarini $a = x_0 < x_1 <\ldots< x_{i-1} < x_i <\ldots< x_{n-1} < x_n = b$ bilan belgilaymiz. $\{x_i\} = \{x_0, x_1, \ldots, x_n\}$ bo‘lish nuqtalari to‘plamini $[a; b]$ kesmaning bo‘linishi deymiz. x_i bo‘linish nuqtalari orqali Oy o‘qqa parallel $x = x_i$ to‘g‘ri chiziq o‘tkazamiz. Bu to‘g‘ri chiziqlar $aABb$ trapetsiyani asoslari $[x_{i-1}; x_i]$ bo‘lgan n ta bo‘lakka bo‘ladi. $aABb$ trapet-siyaning S yuzasi n ta tasma yuzalarining yig‘indisiga teng bo‘ladi. n yetarlicha katta va barcha $[x_{i-1}; x_i]$ kesmalar kichik bo‘lganida har bir n ta tasmaning yuzasini hisoblash oson bo‘lgan mos to‘g‘ri to‘trburchakning yuzasi bilan almashtirish mumkin bo‘ladi. Har bir $[x_{i-1}; x_i]$ kesmada biror ξ_i nuqtani tanlaymiz, $f(x)$ funksiyaning bu nuqtadagi qiymati $f(\xi_i)$ ni hisoblaymiz va uni to‘g‘ri to‘rtburchakning balandligi deb qabul qilamiz. $[x_{i-1}; x_i]$ kesma kichik bo‘lganida $f(x)$ uzluksiz funksiya bu kesmada kichik o‘zgarishga ega bo‘ladi. Shu sababli bu kesmalarda funksiyani o‘zgarmas va taqriban $f(\xi_i)$ ga teng deyish mumkin. Bitta tasmaning yuzasi $f(\xi_i)(x_i - x_{i-1})$ ga teng bo‘lganidan $aABb$ egri chiziqli trapetsiyaning S yuzasi taqriban S_n ga teng bo‘ladi:

$$S \approx S_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i, \Delta x_i = x_i - x_{i-1} \qquad (2.1)$$

(2.1) taqribiy qiymat $d = \max\{x_i - x_{i-1}\}, (i = \overline{1,n})$ kattalik qancha kichik bo'lsa shuncha aniq bo'ladi. d kattalikka $\{x_i\}$ bo'linishning diametri deyiladi. Bunda $n \to \infty$ da $d \to 0$.

Shunday qilib, egri chiziqli trapetsiyning S yuzasi deb, S_n to'g'ri to'rtburchaklar yuzasining bo'linish diametri nolga intilgandagi limitiga aytiladi, ya'ni

$$S = \lim_{d \to 0} S_n = \lim_{d \to 0} \sum_{i=1}^{n} f(\xi_i) \Delta x_i. \qquad (2.2)$$

Demak, egri chiziqli trapetsiyaning yuzasini hisoblash masalasi (2.2) ko'rinishdagi limitni hisoblashga keltiriladi.

Bosib o'tilgan yo'l masalasi

Agar nuqtaning harakat qonuni $s = f(t)$ (bunda t - vaqt, s - bosib o'tilgan yo'l) tenglama bilan berilgan bo'lsa $f(t)$ funksiyaning $f'(t)$ hosilasi material nuqtaning berilgan vaqtdagi harakat tezligi v ga teng, ya'ni $v = f'(t)$ bo'ladi Fizikada quyidagi teskari masalani tez-tez yechishga to'g'ri keladi. Material nuqta to'g'ri chiziq bo'ylab v tezlik bilan harakat qilayotgan bo'lsin. v tezlik $t$vaqtning uzluksiz funksiyasi bo'lsin deymiz. Material nuqta vaqtning $t = a$ dan $t = b$ gacha bo'lgan biror $[a; b]$ oralig'ida bosib o'tgan yo'l s ni topamiz. $[a; b]$ kesmani $a = t_0 < t_1 < \ldots < t_{i-1} < t_i < \ldots < t_{n-1} < t_n = b$ nuqtalar bilan vaqtning n ta yetarlicha kichik oraliqlariga bo'lamiz. Vaqtning kichik $[t_{i-1}; t_i]$ oralig'ida $v(t)$ tezlik deyarli o'zgarmaydi va uni bu vaqt oralig'ida o'zgarmas va $v(\xi_i)$ ($\xi_i \in [t_{i-1}; t_i]$) ga taqriban teng deyish mumkin. Bunda harakat $[t_{i-1}; t_i]$ kesmada tekis bo'ladi. U holda bosib o'tilgan yo'l bu vaqt oralig'ida $v(\xi_i)(t_i - t_{i-1})$ ga, $[a; b]$ vaqt oralig'ida $s \approx s_n = \sum_{i=1}^{n} v(\xi_i) \Delta t_i$, $\Delta t_i = t_i - t_{i-1}$ ga teng bo'ladi. Bu taqribiy qiymat $d = \max_i \Delta t_i (i = \overline{1,n})$ kattalik qancha kichik bo'lsa shuncha aniq bo'ladi.

Shunday qilib, s bosib o'tilgan yo'l deb, s_n yig'indining $d \to 0$dagi limitiga aytiladi, ya'ni

$$s = \lim_{d \to 0} s_n = \lim_{d \to 0} \sum_{i=1}^{n} v(\xi_i) \Delta t_i. \qquad (2.3)$$

Demak, bosib o'tilgan yo'lni hisoblash masalasi (2.3) ko'rinishdagi limitni hisoblashga keltiriladi.

Qaralgan har ikki masalada biror ko'rinishdagi yig'indining limitini topishga olib keluvchi bir xil usul qo'llanildi. Tabiatda bilim va texnikaning bir qancha masalalari yuqoridagi kabi yig'indining limitini topishga keltiriladi. Shu sababli (2.2) va (2.3) ifodalarni, aniq talqiniga qiziqmasdan, o'rganib chiqamiz.

2.2. Integral yig'indi va aniq integral

$y = f(x)$ funksiya $[a; b]$ kesmada aniqlangan va uzluksiz bo'lsin. $[a; b]$ kesmani ixtiyoriy ravishda $a = x_0 < x_1 < \ldots < x_{i-1} < x_i < \ldots < x_{n-1} < x_n = b$ nuqtalar bilan n ta qismga bo'lamiz, bunda $\{x_i\}$ ga $[a; b]$ kesmaning bo'linishi, $d = \max_{1 \le i \le n}(x_i - x_{i-1}), (i = \overline{1, n})$ kattalikka bo'linish diametri deymiz.

Har bir $[x_{i-1}; x_i]$ qismiy kesmadan ixtiyoriy ξ_i nuqtani tanlaymiz va $f(x)$ funksiyaning bu nuqtadagi qiymati $f(\xi_i)$ ni hisoblaymiz, bunda ξ_i nuqtalarga belgilangan nuqtalar deymiz.

$f(\xi_i)$ qiymatni mos $\Delta x_i = x_i - x_{i-1}$ uzunlikka ko'paytiramiz va barcha ko'paytmalarni qo'shamiz, ya'ni

$$w_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i \qquad (2.4)$$

yig'indini tuzamiz. (2.4) yig'indiga $f(x)$ funksiya uchun $[a; b]$ kesmaning $\{x_i\}$ bo'linishidagi Riman integral yig'indisi deyiladi.

$\{x_i\}$ bo'linishni maydalaymiz, ya'ni yangi bo'linish nuqtalarini qo'shamiz va yig'indining $d \to 0$ dagi limitini (agar u mavjud bo'lsa) topamiz.

w_n yig'indining $d \to 0$dagi limiti tuzunchasini kiritamiz.

1-ta'rif. Agar $\forall\ \varepsilon > 0$ son uchun shunday $\delta > 0$ son topilsaki, $|I - w_n| < \varepsilon$ tengsizlik $[a; b]$ kesmaning diametri $d < \delta$ bo'lgan istalgan $\{x_i\}$ bo'linishida ξ_i belgilangan nuqtalarning tanlanishiga bog'liq bo'lmagan holda bajarilsa, I soniga w_n Riman integral yig'indisining limiti deyiladi $I = \lim_{d \to 0} w_n$ deb yoziladi.

2-ta'rif. Agar Riman integral yig'indisi $d \to 0$da chekli limitga ega bo'lsa, u holda bu limitga $[a; b]$ kesmada $f(x)$ funksiyadan olingan aniq (bir karrali) integral (Riman integrali) deyiladi va $\int_a^b f(x)dx$ kabi belgilanadi.

Shunday qilib, ta'rifga ko'ra,

$$\int_a^b f(x)dx = \lim_{d \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i, \qquad (2.5)$$

bu yerda $f(x)$ - integral ostidagi funksiya, x - integrallash o'zgaruvchisi, a, b - integralning quyi va yuqori chegarasi, $[a; b]$ – integrallash sohasi (kesmasi) deyiladi.

$[a; b]$ kesmada $\int_a^b f(x)dx$ anig integral mavjud bo'lsa, $y = f(x)$ funksiya shu kesmada integrallanuvchi (Riman bo'yicha integrallanuvchi) deyiladi.

Izoh. Oliy matematika kursida bosha aniq integrallar qaralmagani sababli bundan keyin «Riman integrali» va «Riman bo'yicha integrallanuvchi» iboralarini mos ravishda «integral» va «integrallanuvchi» deb yozamiz.

Keltirilgan ta'riflarda $a < b$ bo'lsin deb faraz qilindi. Aniq integral tushunchasini $a = b$ va $a > b$ bo'lgan hollar uchun umumlashtiramiz.

$a > b$ bo'lganida 2-ta'rifga ko'ra

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \qquad (2.6)$$

bo'ladi. (2.6) tenglik integrallash chegaralari almashtirilganida aniq integralning ishorasi teskarisiga o'zgarishini bildiradi.

2-ta'rifga ko'ra $a = b$ bo'lganida

$$\int_a^a f(x)dx = 0 \qquad (2.7)$$

bo'ladi. Bu tenglik bir xil chegaralari integrallashda aniq integralning nolga teng bo'lishini bildiradi.

(2.4) integral yig'indi berilgan funksiyaning argumenti qanday harf bilan belgilanishiga bog'liq bo'lmagani sababli, uning limiti va shuningdek aniq integral integrallash o'zgaruvchisining belgilanishiga bog'liq bo'lmaydi:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(z)dz.$$

Misol

$\int_0^1 x^2\,dx$ integralni uning ta'rifidan foydalanib hisoblaymiz. $[0;1]$ kesmada $y = x^2$ funksiya uzluksiz. $[0;1]$ kesmani $0 = x_0 < x_1 <\ldots< x_{i-1} < x_i <\ldots< x_{n-1} < x_n = 1$ nuqtalar bilan uzunliklari $\Delta x_i = \frac{1}{n}(i = \overline{1,n})$ bo'lgan $n$ta teng bo'lakka bo'lamiz. Bunda $d = \frac{1}{n}$, ξ_i nuqta sifatida qismiy kesmalarning oxirlarini olamiz:

$$\xi_i = x_i = \frac{i}{n}.$$

Tegishli integral yig'indini tuzamiz:

$$w_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \sum_{i=1}^{n} \frac{i^2}{n^2}\cdot\frac{1}{n} = \frac{1}{n^3}(1^2 + 22+\ldots+n^2) =$$

$$= \frac{n(n+1)(2n+1)}{6n^3} = \frac{(n+1)(2n+1)}{6n^2}.$$

Bundan

$$\lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2} = \frac{1}{3}.$$

Demak, ta'rifga ko'ra

$$\int_0^1 x^2\,dx = \lim_{\lambda\to 0\,(n\to\infty)} w_n = \frac{1}{3}.$$

Endi ξ_i nuqta sifatida qismiy kesmalarning boshlarini olamiz:

$$\xi_i = x_{i-1} = \frac{i-1}{n}.$$

Bundan

$$w_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \sum_{i=1}^{n} \frac{(i-1)^2}{n^2}\cdot\frac{1}{n} = \frac{(n-1)n(2n-1)}{6n^3}$$
$$= \frac{(n-1)(2n-1)}{6n^2}$$

yoki

$$\int_0^1 x^2\,dx = \lim_{\lambda\to 0(n\to\infty)} w_n = \lim_{n\to\infty}\frac{(n-1)(2n-1)}{6n^2} = \frac{1}{3}.$$

berilgan integralning qiymati $[0;1]$ kesmani bo'lish usuliga va bu kesmada ξ_i nuqtani tanlash usuliga bog'liq emas va $\int_0^1 x^2\,dx = \frac{1}{3}$.

Funksiya integrallanuvchi bo'ladigan shartlarni keltiramiz.

1-teorema (funksiya integrallanuvchi bo'lishining zaruriy sharti). Agar $y = f(x)$ funksiya $[a;b]$ kesmada integrallanuvchi bo'lsa, u holda u bu kesmada chegaralangan bo'ladi.

Isboti. $[a;b]$ kesmada integrallanuvchi $y = f(x)$ funksiya bu kesmada chegaralanmagan bo'lsin deb faraz qilamiz. U holda bu kesmaning istalgan $\{x_i\}$ bo'linishida $[x_{i-1};x_i]$ kesmalarning hech bo'lmaganida bittasida funksiya chegaralanmagan bo'ladi. Bu holda $\xi_i \in [x_{i-1};x_i]$ nuqtani turli usullar bilan tanlash orqali $f(\xi_i)\Delta x_i$ ko'paytmani yetarlicha katta qilish mumkin. Demak, integral faqat $\xi_i \in [x_{i-1};x_i]$ nuqtalarni tanlash hisobiga yig'indi yetartlicha katta bo'ladi va $d \to 0$da hech bir limitga intilmaydi. Shu sababli $y = f(x)$ funksiya $[a;b]$ kesmada integrallanuvchi bo'lmaydi. Olingan ziddiyatdan teoremaning isboti kelib chiqadi.

2-teorema. (funksiya integrallanuvchi bo'lishining yetarli sharti). $[a;b]$ kesmada uzlukziz bo'lgan funksiya bu kesmada integrallanuvchi bo'ladi.

Teoremani isbotsiz qabul qilamiz.

Funksiyaning uzluksizligi uning integrallanuvchi bo'lishini faqat yetarli sharti bo'lad. Boshqacha aytganda $[a;b]$ kesmada uzilishga ega bo'lgan, ammo bu kesmada integrallanuvchi funksiyalar mavjud bo'lishi ham mumkin.

Shuningdek, funksiya integrallanuvchi bo‘lishinngi kuchsizroq shartlarini ifodalovchi quyidagi teoremalar o‘rinli bo‘ladi.

3-teorema. $[a; b]$ kesmada uzlukziz bo‘lib, chekli sondagi birinchi tur uzilish nuqtalariga ega bo‘lgan funksiya bu kesmada integrallanuvchi bo‘ladi.

4-teorema. $[a; b]$ kesmada monoton funksiya bu kesmada integrallanuvchi bo‘ladi.

2.3.Aniq integralning geometrik va mexanik ma’nolari

Egri chiziqli trapetsiyaning yuzasi masalasiga qaytamiz. (2.2) tenglikning o‘ng tomoni integral yig‘indidan iborat. U holda (2.5) formuladan aniq integralning geometrik ma’nosi kelib chiqadi: agar $f(x)$ funksiya $[a; b]$ kesmada integrallanuvchi va manfiy bo‘lmasa, u holda $[a; b]$ kesmada $f(x)$ funksiyadan olingan aniq integral $y = f(x) \geq 0, y = 0, x = a,\quad x = b\ a < b$ chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzasiga teng.

Misol. $\int_{-3}^{3} \sqrt{9 - x^2}\, dx$ integralni uning geometrik ma’nosiga tayanib hisoblaymiz.

Bunda x ning -3 dan 3 gacha o‘zgarishida tenglamasi $y = \sqrt{9 - x^2}$ bo‘lgan chiziq $x^2 + y^2 = 9$aylananing yuqori bo‘lagidan iborat bo‘ladi. Shu sababli $x = -3, x = 3, y = 0, y = \sqrt{9 - x^2}$ chiziqlar bilan chegaralangan egri chiziqli trapetsiya $x^2 + y^2 = 9$ doiraning yuqori qismidan tashkil topadi. Uning yuzi $S = \frac{9\pi}{2}$ ga teng.

Demak,

$$\int_{-3}^{3} \sqrt{9 - x^2}\, dx = \frac{9\pi}{2}.$$

Endi bosib o‘tilgan yo‘l masalasiga o‘tamiz. (2.3) tenglikning o‘ng tomoni integral yig‘indidan iborat bo‘lgani uchun (2.5) formuladan ushbu xulosaga kelamiz: agar $v(t)$ funksiya $[a; b]$, $a < b$ kesmada integrallanuvchi va manfiy bo‘lmasa, u holda $v(t)$ tezlikdan $[a; b]$ vaqt oralig‘ida olingan aniq integral material nuqtaning $t = a$ dan $t = b$ gacha vaqt oralig‘ida bosib o‘tgan yo‘liga teng.

Bu jumla aniq integralning mexanik ma’nosini anglatadi.

2.4. Aniq integralning xossalari

1^o. Agar integral ostidagi funksiya birga teng bo'lsa, u holda

$$\int_a^b dx = b - a$$

bo'ladi.

Isboti. Aniq integralning ta'rifiga ko'ra

$$\int_a^b dx = \lim_{d\to 0}\sum_{i=1}^{n} 1\cdot \Delta x_i = \lim_{d\to 0}\sum_{i=1}^{n}(x_i - x_{i-1}) = \lim_{d\to 0}(b-a) = b - a.$$

2^o. Ozgarmas ko'paytuvchini aniq integral belgisidan tashqariga chiqarish mumkin, ya'ni

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx,\ k = const.$$

Isboti. $\int_a^b kf(x)dx = \lim_{d\to 0}\sum_{i=1}^{n} kf(\xi_i)\Delta x_i = k\lim_{d\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x = k\int_a^b f(x)dx.$

3^o. Chekli sondagi funktsiyalar algebraik yig'indisining aniq integrali qo'shiluvchilar aniq integrallarining algebraik yig'indisiga teng, ya'ni

$$\int_a^b (f(x) \pm \phi(x))dx = \int_a^b f(x)dx \pm \int_a^b \phi(x)dx.$$

Isboti. $\int_a^b (f(x) \pm \phi(x))dx = \lim_{\lambda\to 0}\sum_{i=1}^{n}(f(\xi_i) \pm \phi(\xi_i))\Delta x_i =$

$= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i \pm \lim_{\lambda\to 0}\sum_{i=1}^{n}\phi(\xi_i)\Delta x_i = \int_a^b f(x)dx \pm \int_a^b \phi(x)dx.$

4^o. Agar $[a; b]$ kesma bir necha qismga bo'lingan bo'lsa, u holda $[a; b]$ kesma bo'yicha olingan aniq integral har bir qism bo'yicha olingan aniq integrallar yig'indisiga teng bo'ladi. Masalan,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,\ c \in [a; b].$$

Isboti. $a < c < b$ bo'lsin deylik. Integral yig'indi $[a; b]$ kesmani bo'lish usuliga bog'liq emas. Shu sababli $c$ni $[a; b]$ kesmani bo'lish nuqtasi qilib olamiz. Masalan, agar $c = x$ deb olsak, u holda w_n ni ikki yig'indiga ajratish mumkin:

$$w_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \sum_{i=1} f(\xi_i)\Delta x_i + \sum^{n} f(\xi_i)\Delta x_i.$$

Bunda $d \to 0$ da limitga o'tamiz:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

a, b, c nuqtalarning boshqacha joylashishida xossa shu kabi isbotladi. Masalan, $a < b < c$ bo'lsa, $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$ bo'ladi.

Bundan

$$\int_a^b f(x)dx = \int_a^c f(x)dx - \int_b^c f(x)dx$$

yoki integrallash chegaralarining almashtirilishi xossaga ko'ra

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

5^o. Agar $[a; b]$ kesmada funksiya o'z ishorasini o'zgartirmasa, u holda funksiya aniq integralining ishorasi funksiya ishorasi bilan bir xil bo'ladi, ya'ni:

$[a; b]$ da $f(x) \geq 0$ bo'lganda $\int_a^b f(x)dx \geq 0$;

$[a; b]$ da $f(x) \leq 0$bo'lganda $\int_a^b f(x)dx \leq 0$.

Isboti. $f(x) \geq 0$ funksiya uchun integral yig'indi $w_n \geq 0$ bo'ladi, chunki $\Delta x_i > 0$. Bundan $\int_a^b f(x)dx \geq 0$. Shu kabi $\Delta x_i > 0$, $f(x) \leq 0$ ekanidan $w_n \leq 0$ va $\int_a^b f(x)dx \leq 0$ kelib chiqadi.

6^o. Agar $[a; b]$ kesmada $f(x) \geq \phi(x)$ bo'lsa, u holda

$$\int_a^b f(x)dx \geq \int_a^b \phi(x)dx$$

bo'ladi.

Isboti. $f(x) \geq \phi(x)$ dan $f(x) - \phi(x) \geq 0$bo'ladi. U holda 5^oxossaga ko'ra

$\int_a^b (f(x) - \phi(x))dx \geq 0$ yoki 3^o xossaga ko'ra $\int_a^b f(x)dx - \int_a^b \phi(x)dx \geq 0$.

Bundan

$$\int_a^b f(x)dx \geq \int_a^b \phi(x)dx.$$

7^o. Agar m va M sonlar $f(x)$ funksiyaning $[a; b]$ kesmadagi eng kichik va eng katta qiymatlarii bo'lsa, u holda

$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a)$$

bo'ladi.

Isboti. Shartga kora $m \leq f(x) \leq M$. U holda 6^oxossaga ko'ra

$$\int_a^b m dx \leq \int_a^b f(x)dx \leq \int_a^b M dx.$$

Bunda

$\int_a^b mdx = m\int_a^b dx = m(b-a)$, $\int_a^b Mdx = M\int_a^b dx = M(b-a)$.

U holda

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Bu xossa aniq integralni baholash haqidagi teorema deb yuritiladi.

8^o. Agar $f(x)$ funktsiya $[a;b]$ kesmada uzluksiz bo'lsa, u holda shunday $c \in [a;b]$ nuqta topiladiki,

$$\int_a^b f(x)dx = f(c)(b-a) \qquad (2.8)$$

bo'ladi.

Isboti. 7^o xossaga ko'ra

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Bundan

$$m \le \frac{\int_a^b f(x)dx}{b-a} \le M.$$

$\frac{\int_a^b f(x)dx}{b-a} = \mu$ $(m \le \mu \le M)$ deymiz. U holda Bolsano-Koshining ikkinchi teoremasiga ko'ra shunday $c \in [a;b]$ nuqta topiladiki, $f(c) = \mu$ bo'ladi.

Shu sababli

$$\frac{\int_a^b f(x)dx}{b-a} = f(c) \text{ yoki } \int_a^b f(x)dx = f(c)(b-a).$$

(2.8) formulaga o'rta qiymat formulasi, $f(c)$ ga $f(x)$ funktsiyaning $[a;b]$ kesmadagi o'rtacha qiymati deyiladi.

Bu xossa o'rta qiymat haqidagi teorema deb ataladi.

O'rta qiymat haqidagi teorema quyidagi geometrik talqinga ega: agar $f(x) > 0$ bo'lsa, u holda $\int_a^b f(x)dx$ integral qiymati balandligi $f(c)$ ga va asosi $(b-a)$ ga teng bo'lgan to'g'ri to'rtburchakning yuzasiga teng bo'ladi.

Aniq integralning xossalaridan quyidagi natijalar kelub chiqadi.

1-natija. $[a;b]$ kesmada aniqlangan $f(x)$ funktsiya uchun

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$$

bo'ladi.

2-natija. Agar $[a;b]$ kesmada $|f(x)| \le k$ bo'lsa, u holda

$$\left|\int_a^b f(x)dx\right| \le k(b-a), \quad (k = const.)$$

bo'ladi.

Misollar1. $\int_0^1 x^2 \cos^2 x\, dx$ va $\int_0^1 x \sin^2 x\, dx$ integrallarni taqqoslaymiz.

$x \in [0;1]$ nuqtalarda $x^2 \cos^2 x \le x \sin^2 x$ tengsizlik bajariladi.

U holda aniq integralning 8^o xossasiga ko'ra

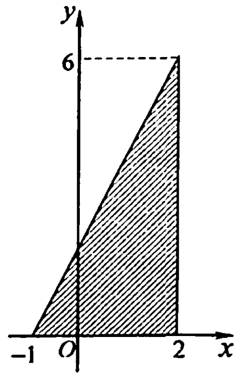
$$\int_0^1 x^2 \cos^2 x dx \le \int_0^1 x \sin^2 x\, dx$$

bo'ladi.

2. $\int_0^{\frac{\pi}{2}} \frac{dx}{4+3\sin^2 x}$ integralni baholaymiz. $0 \le \sin^2 x \le 1$ ekanidan $\frac{1}{7} \le \frac{1}{4+3\sin^2 x} \le \frac{1}{4}$ bo'ladi. U holda aniq integralni baholash haqidagi teoremaga ko'ra $\frac{1}{7}\left(\frac{\pi}{2}-0\right) \le \int_0^{\frac{\pi}{2}} \frac{dx}{4+3\sin^2 x} \le \frac{\pi}{4}\left(\frac{\pi}{2}-0\right)$

yoki

$$\frac{\pi}{14} \le \int_0^{\frac{\pi}{2}} \frac{dx}{4+3\sin^2 x} \le \frac{\pi}{8}.$$



3-shakl

3. $y = 2x + 2$ funksiyaning $[-1;2]$ kesmadagi o'rtacha qiymatini topamiz. Bunda o'rta qiymat haqidagi teoremadan foydalanamiz:

$$f_{o'rt} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Aniq integralning geometrik ma'nosiga ko'ra $\int_{-1}^{2}(2x+2)dx$ integralning qiymati uchburchakning yuzasiga teng, ya'ni

$$S = \frac{1}{2}\cdot(2+1)\cdot 6 = 9.$$

Bundan

$$f_{o,rt} = \frac{1}{2-(-1)}\cdot 9 = 3.$$

4. Uzunligi 12 *sm.* bo'lgan AB kesmada C nuqta olingan. Tomonlari AC va CB kesmalardan iborat to'g'ri to'rtburchakning o'rtacha yuzasini topamiz.

Bunda A nuqtani hisob (koordinata) boshi deb olamiz. U holda $AC = x$ va $CB = 12 - x$ bo'ladi. Bu kesmalarni tomonlar qilib yasalgan to'g'ri to'rtburchakning yuzasi $s = x(12-x)$ bo'ladi. Bu yusaning o'rta qiymatini aniq integralning 8^o – xossasi bilan topamiz:

$$S_{o'rt} = \frac{\int_0^{12} x(12-x)dx}{12-0} = \frac{1}{12}\left(\int_0^{12} 12x dx - \int_0^{12} x^2 dx\right) =$$

$$=\frac{1}{12}\left(6x^2-\frac{x^3}{3}\right)\Big|_0^{12}=\frac{1}{12}(864-576)=24\,sm^2.$$

Amaliy mashg'ulotda yechiladigan misollar.

Berilgan aniq integrallarni hisoblang.

1. $\int_1^4 \sqrt{x}dx$ $(Javob.\frac{14}{3})$
2. $\int_1^2 (2x^2+\frac{2}{x^4})dx$ $(Javob.\frac{21}{4})$
3. $\int_1^{e^3} \frac{1}{x\sqrt{1+\ln x}}dx$ $(Javob.2)$
4. $\int_0^1 \frac{1}{x^2+4x+5}dx$ $(Javob.arctg\frac{1}{7})$
5. $\int_0^4 \frac{1}{1+\sqrt{2x+1}}dx$ $(Javob.2-\ln 2)$
6. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x-\cos^3 x}\,dx$ $(Javob.\frac{4}{3})$
7. $\int_0^2 \sqrt{4-x^2}dx$ $(Javob.\pi)$
8. $\int_1^3 \frac{1}{x\sqrt{x^2+5x+1}}dx$ $(Javob.\ln\frac{7+2\sqrt{7}}{9})$
9. $\int_0^5 \frac{1}{2x+\sqrt{3x+1}}dx$ $(Javob.\frac{1}{5}\ln 1\,12)$

Mustaqil yechish uchun misollar.

Berilgan aniq integrallarni hisoblang.

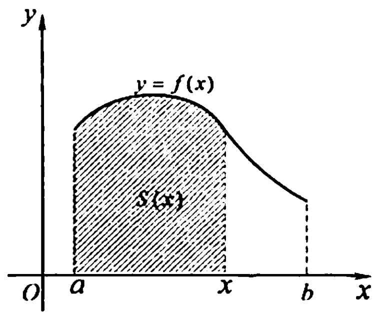
1. $\int_1^4 (2x+\frac{3}{\sqrt{x}})dx$ $(Javob.21)$
2. $\int_4^9 \frac{y-1}{\sqrt{y}+1}dy$ $(Javob.\frac{23}{3})$
3. $\int_0^9 \frac{\sqrt{x}}{\sqrt{x}+1}dx$ $(Javob.3+4\ln 2)$

2.5. Yuqori chegarasi o'zgaruvchi aniq integral

$y = f(x)$ funksiya $[a;b]$ kesmada aniqlangan va uzluksiz bo'lsin. U holda u ixtiyoriy $[a;x]$ ($a \leq x \leq b$) kesmada integrallanuvchi bo'ladi, ya'ni istalgan $x \in [a;b]$ uchun

$$F(x) = \int_a^x f(t)dt \qquad (2.9)$$

integral mabjud bo'ladi.



4-shakl

Agar ixtiyoriy $t \in [a;b]$ da $f(t) > 0$ bo'lsa, u holda $F(x) = \int_a^x f(t)dt$ integral asosi $[a;x]$ kesmadan iborat bo'lgan egri chiziqli trapetsiyaning o'zgaruvci yuzasi $S(x)$ ni ifodalaydi (1-shakl).

$[a;b]$ kesmada (2.9) tenglik bilan aniqlanuvchi $F(x)$ funksiya yuqori chegarasi o'zgaruvchi aniq integral deyiladi.

$F(x)$ funksiya $[a;b]$ kesmada uzluksiz va differensiallanuvchi bo'ladi. Shunindek, bunda $F(x)$ funksiya uchun quyidagi teorema o'rinli bo'ladi.

1-teorema $[a;b]$ kesmada uzluksiz $f(t)$ funksiyaning yuqori chegarasi o'zgaruvchi integrali $F(x)$ dan yuqori chegara bo'yicha olingan hosila mavjud va u integral ostidagi funksiyaning yuqori chegaradagi qiymatiga teng bo'ladi, ya'ni

$$F'(x) = \left(\int_a^x f(t)dt\right)'_x = f(x), \ x \in [a;b]. \qquad (2.10)$$

Isboti. $x \in [a;b]$ va $x + \Delta x \in [a;b]$ bo'lsin. U holda aniq integralning 4^o xossasini qo'llab, topamiz:

$$F(x + \Delta x) = \int_a^{x+\Delta x} f(t)dt = \int_a^x f(t)dt + \int_x^{x+\Delta x} f(t)dt.$$

Bundan (2.9) tenglik va o'rta qiymat haqidagi teoremaga ko'ra

$$\Delta F = F(x + \Delta x) - F(x) = \int_x^{x+\Delta x} f(t)dt = f(c)\Delta x, \text{ bu yerda } c \in [x, x + \Delta x].$$

$F(x)$ funksiyaning hosilasini aniqlaymiz:

$$F'(x) = \lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c)\Delta x}{\Delta x} = \lim_{\Delta x \to 0} f(c).$$

$\Delta x \to 0$da $x + \Delta x \to x$ va $c \to x$, chunki $c \in [x, x + \Delta x]$.

U holda $f(x)$ funksiyaning uzluksizligidan

$$F'(x) = \lim_{\Delta x \to 0} f(c) = f(x)$$

bo'ladi.

1-teorema matematik analizning asosiy teoremalaridan biri hisoblanadi. Bu teorema differensial bilan aniq integral tushunchalari orasidagi munosabatni ochib beradi.

Bu teoremadan $[a; b]$ kesmada uzluksiz har qanday $f(x)$ funksiya shu kesmada boshlang'ich funksiyaga ega bo'ladi va uning boshlang'ich funksiyalaridan biri yuqori chegarasi o'zgaruvchi $F(x)$ integral bo'ladi degan xulosa kelib chiqadi.

$f(x)$ funksiyaning boshqa bir boshlang'ich funksiyasi $F(x)$ funksiyadan C o'zgarmas songa farq qilgani uchun aniqmas va aniq integrallar orasidagi ushbu bog'lanish kelib chiqadi:

$$\int f(x)dx = \int_a^x f(t)dt + C, x \in [a; b].$$

2.6. Nyuton-Leybnis formulasi

Aniq integralni integral yig'indining limiti sifatida hisoblash hatto oddiy funksiyalar uchun ham ancha qiyinchiliklar tug'diradi. Shu sababli aniq integralni hisoblashning (2.10) formulaga asoslangan, amaliy jihatdan qulay bo'lgan hamda keng qo'llaniladigan usuli bilan tanishamiz.

2-teorema (integral hisobning asosiy teoremasi). Agar $F(x)$ funksiya $[a; b]$ kesmada uzluksiz bo'lgan $f(x)$ funksiyaning boshlang'ich funksiyasi bo'lsa, u holda $[a; b]$ kesmada $f(x)$ funksiyadan olingan aniq integral $F(x)$ funksiyaning integrallash oralig'idagi orttirmasiga teng bo'ladi, ya'ni

$$\int_a^b f(x)dx = F(b) - F(a). \qquad (2.11)$$

Isboti. $F(x)$ funksiya $f(x)$ funksiyaning boshlang'ich funksiyalaridan biri bo'lsin. U holda 1-teoremaga asosan $\int_a^x f(t)dt$ funksiya ham $f(x)$ funksiyaning $[a; b]$ kesmadagi boshlang'ich funksiyasi bo'ladi. Boshlang'ich funksiyalar C o'zgarmas songa farq qilganidan

$$\int_a^x f(t)dt = F(x) + C.$$

Bu tenglikka $x = a$ni qo'yamiz va bir xil chegarali aniq integralning xossasini qo'llaymiz:

$$\int_a^a f(t)dt = F(a) + C = 0.$$

Bundan $C = -F(a)$. U holda istalgan $x \in [a; b]$ uchun

$$\int_a^x f(t)dt = F(x) - F(a)$$

bo'ladi.

Oxirgi tenglikda $x = b$ deymiz va t o'zgaruvchini x bilan almashtiramiz. Natijada (2.11) formula kelib chiqadi.

(2.11) formulaga Nyuton-Leybnis formulasi deyiladi.

$F(b) - F(a)$ ayirmani shartli ravishda $F(x)|_a^b$ deb yozish kelishilgan.

Bu kelishuv natijasida Nuyton-Leybnis formulasi

$$\int_a^b f(x)dx = F(x)|_a^b \qquad (2.12)$$

ko'inishda ifodalanadi.

Misollar

1. $\int_0^3 \frac{dx}{\sqrt{1+x^2}} = ln\left|x + \sqrt{1 + x^2}\right|\Big|_0^3 = ln\left|3 + \sqrt{10}\right| - ln\,1 = ln\left|3 + \sqrt{10}\right|.$

2. $\int_1^4 \frac{dx}{x^2 - 2x + 10} = \int_1^4 \frac{dx}{(x-1)^2 + 3^2} = \frac{1}{3} arctg\,\frac{x-1}{3}\Big|_1^4 = \frac{1}{3}(arctg1 - arctg0) = \frac{\pi}{12}.$

Nyuton-Leybnis formulasidan uning qo'llanish shartlarini hisobga olmagan holda formal foydalanish xato hatijaga olib kelishi mumkin.

Masalan, $\frac{1}{4+x^2}$ funksiya uchun boshlang'ich funksiya sifatida $\frac{1}{2}arctg\,\frac{x}{2}$ni yoki $\frac{1}{2}arcctg\,\frac{2}{x}$ ni olish mumkin. Avval $F(x) = \frac{1}{2}arctg\,\frac{x}{2}$ deb olamiz:

$$\int_{-2}^2 \frac{dx}{4+x^2} = \frac{1}{2}arctg\,\frac{x}{2}\Big|_{-2}^2 = \frac{1}{2}(arctg1 - arctg(-1))$$
$$= \frac{1}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \frac{\pi}{4}.$$

Bunda Nyuton-Leybnis formulasi to'g'ri qo'llanildi, chunki $F(x) = arctgx$ funksiya $[-2; 2]$ kesmada uzluksiz va $F'(x) = f(x)$ tenglik butun kesmada bajariladi.

Endi $F(x) = \frac{1}{2}arcctg\,\frac{2}{x}$ deb olamiz:

$$\int_{-2}^2 \frac{dx}{4+x^2} = \frac{1}{2}arcctg\,\frac{2}{x}\Big|_{-2}^2 = \frac{1}{2}(arcctg1 - arcctg(-1)) = \frac{1}{2}\left(\frac{\pi}{4} - \frac{3\pi}{4}\right)$$
$$= -\frac{\pi}{4}.$$

Bunda Nyuton-Leybnis formulasi noto'g'ri (formal) qo'llanildi, chunki $x = 0$ da $arcctg\,\frac{1}{x}$ funksiya uzilishga ega va u $[-2; 2]$ kesmada boshlang'ich funksiya bo'la olmaydi. Natijada xatolik $\left(-\frac{\pi}{4} \neq \frac{\pi}{4}\right)$ kelib chiqdi.

Demak, Nyuton-Leybnis formulasini qo'llashda $F(x)$ boshlang'ich funksiya berilgan kesmada uzluksiz deb faraz qilinadi (ayrim shartlarda

Nyuton-Leybnis formulasi uzilishga ega bo'lgan funksiyalar uchun ham o'rinli bo'lishi mumkin).

2.7.Aniq integralda o'zgaruvchini almashtirish

1-teorema. $y = f(x)$ funksiya $[\boldsymbol{a}; \boldsymbol{b}]$ kesmada uzluksiz bo'lsin.

Agar: 1) $x = \phi(t)$ funksiya $[\alpha; \beta]$ kesmada differensiallanuvchi va $\phi'(t)$ funksiya $[\alpha; \beta]$ kesmada uzluksiz; 2) $x = \phi(t)$ funksiyaning qiymatlar sohasi $[\boldsymbol{a}; \boldsymbol{b}]$ kesmadan iborat; 3) $\phi(\alpha) = a$ va $\phi(\beta) = b$ bo'lsa, u holda

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(t))\phi'(t)dt \qquad (2.13)$$

bo'ladi.

Isboti. Nyuton-Leybnis formulasiga ko'ra

$$\int_a^b f(x)dx = F(b) - F(a),$$

bu yerda $F(x)$ funksiya $f(x)$ funksiyaning $[a; b]$ kesmadagi boshlang'ich funksiyalaridan biri. $\Phi(t) = F(\phi(t))$ murakkab funksiyani qaraymiz.

Murakkab funksiyani differensiallash qoidasiga asosan

$\Phi'(t) = F'(\phi(t))\phi'(t) = f(\phi(t))\phi'(t)$. $\Phi'(t) = F'(\phi(t))\phi'(t) = f(\phi(t))\phi'(t)$.

Demak, $\Phi(t)$ funksiya $[\alpha; \beta]$ kesmada $f(\phi(t))\phi'(t)$ uzluksiz funksiya uchun boshlang'ich funksiya bo'ladi. Nyuton-Leybnis formulasi bilan topamiz:

$$\int_\alpha^\beta f(\phi(t))\phi'(t)dt = \Phi(\beta) - \Phi(\alpha) = F(\phi(\beta)) - F(\phi(\alpha)) = F(b) - F(a) = \int_a^b f(x)dx.$$

(2.13) formula aniq integralda o'zgaruvchini almashtirish formulasi deb yuritiladi. Aniq integralni hisoblashning bu usulida aniq integralda o'rniga qo'yish usuli deyiladi.

Izoh. Aniq integralni (2.13) formula bilan hisoblashda yangi o'zgaruvchidan eski o'zgaruvchiga qaytish shart emas, chunki integrallash chegarasi o'rniga qo'yishga mos tarzda o'zgaradi.

Misollar

1. $\int_0^{\sqrt{3}} \sqrt{4 - x^2}\, dx$integralni hisoblaymiz. Bunda $x = 2\sin t$,

$0 \le t \le \frac{\pi}{3}$ belgilash kiritamiz. Bu o'zgaruvchini almashtirish 3-teoremaning barcha shartlarini qanoatlantiradi: birinchidan $f(x) = \sqrt{4 - x^2}$ funksiya $[0; \sqrt{3}]$ kesmada uzluksiz, ikkinchidan $x = 2\sin t$

funksiya $\left[0;\frac{\pi}{3}\right]$ kesmada differensiallanuvchi va $x' = 2\cos t$ bu kesmada uzluksiz, uchinchidan t o'zgaruvchi 0 dan $\frac{\pi}{3}$ gacha o'zgarganda $x = 2\sin t$ funksiya 0 dan $\sqrt{3}$gacha o'sadi va bunda $\phi(0) = 0$ va $\phi\left(\frac{\pi}{3}\right) = \sqrt{3}$. Bunda $dx = 2\cos t\, dt$.

(6) formuladan topamiz:

$$\int_0^{\sqrt{3}} \sqrt{4 - x^2}\, dx = 4\int_0^{\frac{\pi}{3}} \cos^2 t\, dt =$$

$$= 2\int_0^{\frac{\pi}{3}} (1 + \cos 2t) dt = 2\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^{\frac{\pi}{3}} = \frac{2\pi}{3} + \frac{\sqrt{3}}{2}.$$

2. $\int_0^1 x\sqrt{1 + x^2}\, dx$ integralni hisoblaymiz. Bunda $t = \sqrt{1 + x^2}$ o'rniga qo'yish bajaramiz. U holda

$$x = \sqrt{t^2 - 1}, dx = \frac{tdt}{\sqrt{t^2-1}}, x = 0 \text{ da } t = 1, x = 1\text{da } t = \sqrt{2}.$$

$[1;\sqrt{2}]$ kesmada $\sqrt{t^2 - 1}$ funksiya monoton o'sadi, demak o'rniga qo'yich to'g'ri bajarilgan. Bundan

$$\int_0^1 x\sqrt{1 + x^2}\, dx = \int_1^{\sqrt{2}} \sqrt{t^2 - 1} \cdot t \cdot \frac{tdt}{\sqrt{t^2 - 1}} = \int_1^{\sqrt{2}} t^2\, dt = \frac{t^3}{3}\Big|_1^{\sqrt{2}}$$

$$= \frac{2\sqrt{2} - 1}{3}.$$

Izoh. (2.13) formulani qo'llashda teoremada sanab o'tilgan shartlarning bajarilishini tekshirish lozim. Agar bu shartlar buzilsa keltirilgan formula bo'yicha o'zgaruvchini almashtirish xato natijaga olib kelishi mumkin.

2.8.Aniq integralni bo'laklab integrallash

1-teorema. Agar u va v funksiyalar u' va v' hosilalari bilan $[a, b]$ kesmada uzluksiz bo'lsa, u holda

$$\int_a^b udv = uv|_a^b - \int_a^b vdu; \qquad (2.14)$$

bo'ladi.

(2.14) formula aniq integralni bo'laklab integrallash formulasi deb ataladi.

Misol

1. $\int_0^1 \arcsin x\, dx = \left| \begin{matrix} \arcsin x = u, dv = dx \\ du = \frac{dx}{\sqrt{1-x^2}}, v = x \end{matrix} \right| = x \arcsin x|_0^1 -$

$- \int_0^1 \frac{xdx}{\sqrt{1-x^2}} = \frac{\pi}{2} + \frac{1}{2}\int_0^1 \frac{d(1-x^2)}{\sqrt{1-x^2}} = \frac{\pi}{2} + \sqrt{1-x^2}\Big|_0^1 = \frac{\pi}{2} + 1$

2.9. Aniq integralning geometrik masalalarga tatbiqlari

1. Aniq integralning qo'llanish sxemalari

x o'garuvchi aniqlangan $[a; b]$ kesmaga bog'liq biror geometrik yoki fizik A kattalikning qiymatini (tekis shakl yuzasini, jism hajmini, kuchning bajargan ishini va hokazo) hisoblash talab qilingan bo'lsin. Bunda A kattalik additiv deb faraz qilinadi, ya'ni $[a; b]$ kesmaning $c \in [a; b]$ nuqta bilan $[a; c]$ va $[c; b]$ qismlarga bo'linishida A kattalikning $[a; b]$ kesmaga mos qiymati uning $[a; c]$ va $[c; b]$ kesmalarga mos qiymatllarining yig'indisiga teng qilib olinadi.

A kattalikning qiymatini hisoblash ma'lum tartibda (sxema asosida) bajariladi. Bunda ikki sxemadan biriga amal qilish mumkin: I sxema (integral yig'indilar usuli) va IIsxema (differensial usuli).

I sxema aniq integralning ta'rifiga asoslanadi. Bunda:

1^o. $[a; b]$ kesma $a = x_0 < x_1 <\ldots< x_{i-1} < x_i <\ldots< x_{n-1} < x_n = b$ nuqtalar bilan n ta kichik kesmalarga bo'linadi. Bunda A kattalik mos n ta $\Delta A_i, (i = \overline{1, n})$ «elementar qo'shiluvchilar» ga bo'linadi: $A = \Delta A_1 + \Delta A_2 + \ldots + \Delta A_n$;

2^o. Har bir «elementar qo'shiluvchi» masalaning shartidan aniqlanuvchi funksiyaning mos kesma istalgan nuqtasida hisoblangan qiymati bilan kesmaning uzunligi ko'paytmasi ko'rinishiga keltiriladi: $\Delta A_i \approx f(\xi_i)\Delta x_i$;

ΔA_i ning taqribiy qiymatini topishda ayrim soddalashtirishlar qilish mumkin: kichik bo'lakda yoyni uning chekkalarini tortib turuvchi vatar bilan almashtirish mumkin; kichik bo'lakda o'zgaruvchi tezlikni o'zgarmas deyish mumkin va hokazo.

Bunda A kattalikning taqribiy qiymati integral yig'indidan iborat bo'ladi:

$$A \approx \sum_{i=1}^{n} f(\xi_i)\Delta x_i.$$

3^o. A kattalikning haqiqiy qiymati bu integral yig'indining $n \to \infty$ dagi (bunda $\lambda = \max_{1\le i\le n} \Delta x_i$) limitiga teng bo'ladi:

$$A = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)dx.$$

II sxema *I* sxemaning o‘zgargan ko‘rinishi hisoblanadi va «differensial usul» yoki «yuqori tartibli cheksiz kichiklarni tashlab yuborish usuli» deb ataladi. Bunda:

1°. $[a;b]$ kesmada ixtiyoriy x qiymatni tanlaymiz va o‘zgaruvchi $[a;x]$ kesmani qaraymiz. Bu kesmada A kattalik x ning funksiyasi bo‘ladi: $A = A(x)$, ya’ni A kattalikning bo‘lagi $A(x)$ noma’lum funksiya bo‘ladi, bu yerda $x \in [a;b]$ - A kattalikning parametrlaridan biri;

2°. x ning kichik $\Delta x = dx$ kattalikka o‘zgarishida ΔA orttirmaning bosh qismini topamiz: $dA = f(x)dx$, bu yerda $f(x)$ - x o‘zgaruvchining masala shartidan kelib chiquvchi funksiyasi (bunda mumkin bo‘lgan soddalashtirishlar qilinishi mumkin);

3°. $\Delta A \approx dA$ deb, dA ni a dan b gacha integrallash orqali A ning izlanayotgan qiymati topiladi:

$$A = \int_a^b f(x)dx.$$

2. Tekis shakl yuzasini hisoblash

Yuzani dekart koordinatalarida hisoblash

Aniq integralning geometrik ma’nosiga asosan abssissalar o‘qidan yuqorida yotgan, ya’ni yuqoridan $y = f(x)$ ($f(x) \geq 0$) funksiya grafigi bilan, quyidan Ox o‘q bilan, yon tomonlaridan $x = a$ va $x = b$ to‘g‘ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzasi

$$S = \int_a^b f(x)dx \qquad (2.15)$$

integtegralga teng bo‘ladi.

Shu kabi, abssissalar o‘qidan pastda yotgan, ya’ni quyidan $y = f(x)$ ($f(x) \leq 0$) funksiya grafigi bilan, yuqoridan Ox o‘q bilan, yon tomonlaridan $x = a$ va $x = b$ to‘g‘ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzasi

$$S = -\int_a^b f(x)dx \qquad (2.16)$$

integtegralga teng bo‘ladi.

(2.15) va (2.16) formulalarni bitta formula bilan umumlashtirish mumkin:

$$S = \left|\int_a^b f(x)dx\right|. \qquad (2.17)$$

Misol. $y = x^2$, $y = 0$ va $x = 1$ chiziqlar bilan chegaralangan tekis shakl yuzasini (2.15) formula bilan topamiz:

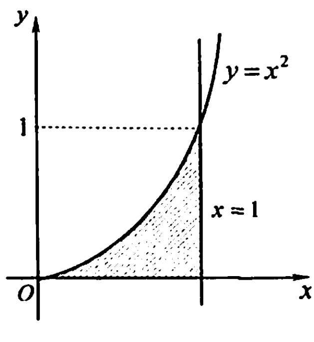
$$S = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$$

Yuzani hisoblashga oid murakkabroq masalalar yuzaning additivlik xossasiga asoslangan holda yechiladi. Bunda tekis shakl kesishmaydigan qismlarga ajratiladi va aniq integralning 4ʼxossasiga ko'ra tekis shaklning yuzasi qismlar yuzalarining yig'indisiga teng bo'ladi.

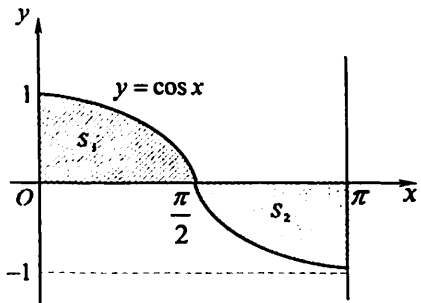
Misol. $y = cos\, x$, $y = 0$, $x = 0$ va $x = \pi$ chiziqlar bilan chegaralangan tekis shakl yuzasini hisoblaymiz. Bunda berilgan tekis shaklni yuzalari S_1 va S_2 bo'lgan kesishmaydigan qismlarga ajratamiz. U holda yuzaning additivlik xossasiga asosan berilgan tekis shaklning yuzasi qismlar yuzalarining yig'indisiga teng bo'ladi. Demak,

$$S = S_1 + S_2 = \int_0^{\frac{\pi}{2}} cos\, x\, dx - \int_{\frac{\pi}{2}}^{\pi} cos\, x\, dx = sinx|_0^{\frac{\pi}{2}} - sin\, x|_{\frac{\pi}{2}}^{\pi}$$

$$= 1 - 0 - (0 - 1) = 2$$



5- shakl.



6-shakl.

$[a;b]$ kesmada ikkita $y_1 = f_1(x)$ va $y_2 = f_2(x)$ uzliksiz funksiyalar berilgan va $x \in [a;b]$ da $f_2(x) \geq f_1(x)$ bo'lsin. Bu funksiyalarning grafiklari va $x = a$, $x = b$ to'g'ri chiziqlar bilan chegaralangan tekis shaklning yuzasini topamiz.

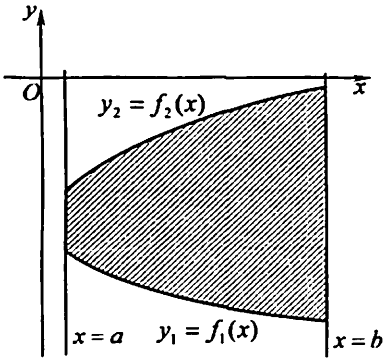
Har ikkala funksiya musbat bo'lganda bu tekis shaklning yuzasi yuqoridan $y_2 = f_2(x)$ va $y_1 = f_1(x)$ funksiyalar garfiklari bilan, quyidan Ox o'q bilan, yon tomonlardan $x = a$ va $x = b$ to'g'ri chiziqlar bilan chegaralangan egri chiziqli

trapetsiyalar yuzalarining ayirmasiga teng bo'ladi:

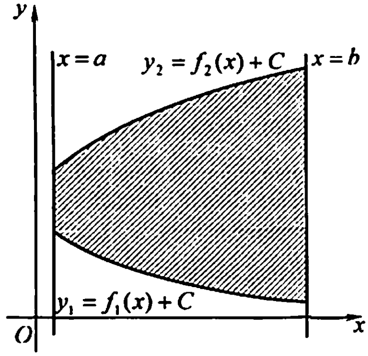
$$S = \int_a^b f_2(x)dx - \int_a^b f_1(x)dx = \int_a^b (f_2(x) - f_1(x))dx. \quad (2.18)$$

(2.18) formula $[a;b]$ kesmada uzluksiz va musbat bo'lmagan $y_2 = f_2(x)$ va $y_1 = f_1(x)$ funksiyalar uchun ham o'rinli bo'ladi.

Haqiqatan ham, agar $y_2 = f_2(x)$ va $y_1 = f_1(x)$ funksiyalar $[a;b]$ kesmada



7-shakl



8-shakl

manfiy qiymatlar qabul qilsa (bunda $y_2 \geq y_1$), har bir funksiyaga bir xil o'zgarmas $y = C$ qiymatlar qo'shish orqali $y_1 = f_1(x) + C$ va $y_2 = f_2(x) + C$ funksiyalar grafiglarini Ox o'qidan yuqorida joylashtirish mumkin (5-shakl).

tekis shakl tekis shaklni parallel ko'chirish orqali hosil qilindi. Shu sababli yuzaning ko'chishga nisbatan invariantlik xossasiga ko'ra bu tekis shakllar teng yuzalarga ega bo'ladi.

yuza uchun (2.18) formula o'rinli, ya'ni

$$S = \int_a^b (f_2(x) + C)dx - \int_a^b (f_1(x) + C)dx = \int_a^b ((f_2(x) + C) - (f_1(x) + C))dx.$$

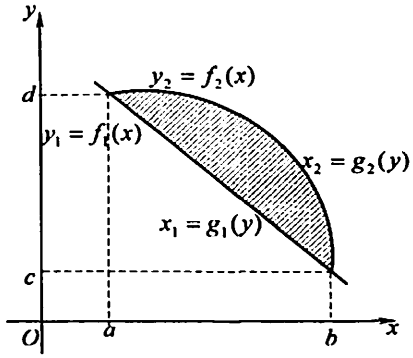
Bundan

$$S = \int_a^b (f_2(x) - f_1(x))dx.$$

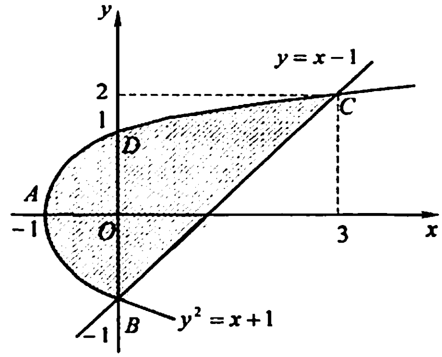
Demak, (2.17) formula 6-shakldagi tekis shakl uchun ham o'rinli bo'ladi.

Ayrim hollarda yuzani hisoblashga oid masalalar yuzaning ko'chishga nisbatan invariantlik xossasidan foydalangan holda soddalashtiriladi. Bunda tekis shakl yuzasi (2.18) formulada x va y o'zgaruvchilar (Ox va Oy o'qlar) ning o'rnini almashtirish yo'li bilan hisoblanadi, ya'ni

$$S = \int_a^b (f_2(x) - f_1(x))dx = \int_c^d (g_2(y) - g_1(y))dy. \qquad (2.19)$$



9-shakl



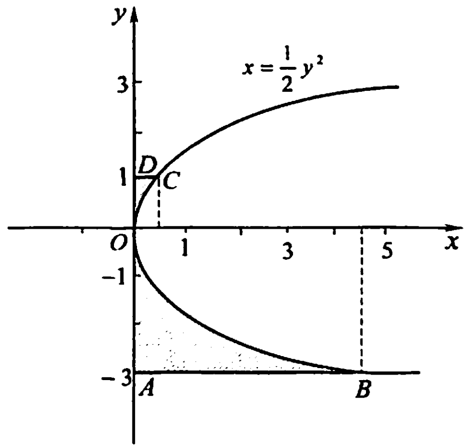
10-shakl.

Misollar

1. $y^2 = x + 1$ va $y = x - 1$ chiziqlar bilan chegaralangan tekis shaklning yuzasini hisoblaymiz.

Tekis shakl umumiy $B(0;-1)$ va $C(3;2)$ nuqtalarga ega bo‘lgan parabola va to‘g‘ri chiziq bilan chegaralangan. Tekis shaklni uchta qismga, ya’ni yuzalari S_1 ga teng bo‘lgan AOD va AOB parabolik sektorlarga va yuzasi S_2 ga teng bo‘lgan BCD parabolik uchburchakka ajratamiz.

Bunda (2.15) va (2.18) formulalarni qo‘llab, topamiz:



11-shakl

$$S = 2S_1 + S_2 = 2\int_{-1}^{0} \sqrt{x+1}dx + \int_{0}^{3} (\sqrt{x+1} - (x-1))dx =$$

$$= \frac{4}{3}\sqrt{(x+1)^3}\Big|_{-1}^{0} + \left(\frac{2}{3}\sqrt{(x+1)^3} - \frac{x^2}{2} + x\right)\Big|_{0}^{3} = \frac{9}{2}.$$

Bu yuza y o'zgaruvchi bo'yicha hisoblanganda tekis shaklni qismlarga ajratiish shart bo'lmaydi:

2. $x = \frac{1}{2}y^2$, $y = -3$, $y = 1$ chiziqlar va ordinatalar o'qi bilan chegaralangan tekis shakl yuzasini hisoblaymiz:

$$S = \int_{-3}^{1} \frac{1}{2} y^2 dy = \frac{1}{2} \cdot \frac{1}{3} y^3 \Big|_{-3}^{1} = \frac{1}{6}(1 + 27) = \frac{14}{3}.$$

Agar egri chiziqli trapetsiya yuqoridan $x = \varphi(t)$, $y = \psi(t)$, $\alpha \le t \le \beta$ parametrik tenglamalar bilan berilgan funksiya grafigi bilan chegaralangan bo'lsa (2.15) formulada $x = \varphi(t)$, $dx = \varphi'(t)dt$ o'rniga qo'yish orqali o'zgaruvchi almashtiriladi.

U holda

$$S = \int_{\alpha}^{\beta} \psi(t)\varphi'(t)dt \qquad (2.20)$$

bo'ladi, bu yerda, $a = \varphi(\alpha)$ va $b = \varphi(\beta)$.

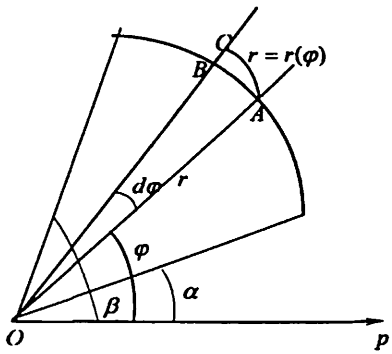
Misol. Radiusi R ga teng doira yuzasini hisoblaymiz. Buning uchun koordinatalar boshini doiraning markaziga joylashtiramiz. Bu doiraning aylanasi $x = R\cos t$, $y = R\sin t$ parametrik tenglamalar bilan aniqlanadi va doira koordinata o'qlariga nisbatan simmetrik bo'ladi. Shu sababli uning birinchi chorakdagi yuzasini hisoblaymiz (bunda x o'zgaruvchi 0 dan R gacha o'zgarganda t parametr $\frac{\pi}{2}$ dan 0 gacha o'zgaradi) va natijani to'rtga ko'paytiramiz:

$$S = 4S_1 = 4\int_{\frac{\pi}{2}}^{0} R\sin t(-R\sin t)dt = 4R^2 \int_{0}^{\frac{\pi}{2}} \sin^2 t dt =$$

$$= 2R^2 \int_{0}^{\frac{\pi}{2}} (1 - \cos 2t)dt = 2R^2 \left(t - \frac{\sin 2t}{2} \right) \Bigg|_{0}^{\frac{\pi}{2}} = \pi R^2.$$

3. Yuzani qutb koordinatalarida hisoblash

Qutb koordinatalar (r - qutb radiusi, φ - qutb burchagi) sistemasida berilgan $r = r(\varphi)$ funksiya $\varphi \in [\alpha;\beta]$ kesmada uzluksiz bo'lsin.



12-shakl

$r = r(\varphi)$ funksiya grafigi hamda qutbdan chiqqan $\varphi = \alpha$ va $\varphi = \beta$ nurlar bilan chegaralangan tekis shaklga egri chiziqli sektor deyiladi.

AOB egri chiziqli sektor yuzasini hisoblaymiz. Bunda II sxemadan foydalanamiz.

1°. Izlanayotgan yuza φ burchakning funksiyasi bo'lsin deymiz: $S = S(\varphi)$, bu yerda $\alpha \le \varphi \le \beta$ ($\varphi = \alpha$ bo'lganda $S(\alpha) = 0$, $\varphi = \beta$ bo'lganda $S(\beta) = S$).

2°. Joriy φ qutb burchagi $\Delta\varphi = d\varphi$ orttirma olganida ΔS yuza OAB «elementar egri chuiiqli sektor» yuzasiga teng orttirma oladi.

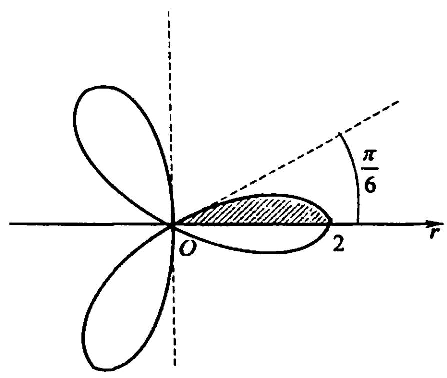
Bunda dS differensial ΔS orttirmaning $d\varphi \to 0$ dagi orttirmasining bosh qismini ifodalaydi va radiusi r ga, markaziy burchagi $d\varphi$ ga teng bo'lgan OAC doiraviy sektor yuzaasiga teng bo'ladi.

Shu sababli

$$dS = \frac{1}{2} r^2 d\varphi.$$

3°. Oxirgi ifodani $\varphi = \alpha$ dan $\varphi = \beta$ gacha integrallab izlanayotgan yuzani topamiz:

$$S = \frac{1}{2}\int_{\alpha}^{\beta} r^2(\varphi)d\varphi. \quad (2.21)$$



13-shakl

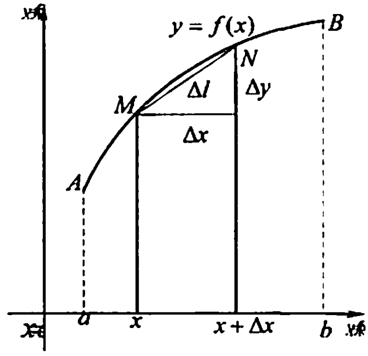
Misol

$r = 2\cos 3\varphi$ egri chiziq bilan chegaralangan figuraning yuzasini hisoblaymiz. Bu figura uch yaproqli gul deyiladi. (2.21) formula bilan topamiz:

$$\frac{1}{6}S = \frac{1}{2}\int_0^{\frac{\pi}{6}} 4\cos^2 3\varphi d\varphi = \int_0^{\frac{\pi}{6}} (1 + \cos 6\varphi)d\varphi = \left(\varphi + \frac{\sin 6\varphi}{6}\right)\Bigg|_0^{\frac{\pi}{6}} = \frac{\pi}{6}.$$

Bundan $S=\pi$.

4. Tekis egri chiziq yoyi uzunligini hisoblash



14-shakl

Tekislikda AB egri chiziq $[a;b]$ kesmada uzluksiz $y=f(x)$ funksiya grafigi bilan berilgan bo‘lsin. AB egri chiziq uzunligini ll sxemadan foydalangan holda topamiz.

1°. $[a;b]$ kesmada ixtiyoriy x qiymatni tanlaymiz va o‘zgaruvchi $[a;x]$ kesmani qaraymiz. Bu kesmada l kattalik x ning funksiyasi bo‘ladi: $l=l(x)$ $(l(a)=0$ va $l(b)=l)$.

2°. x ning kichik $\Delta x=dx$ kattalikka o‘zgarishida dl differensialni topamiz: $dl=l'(x)dx$. $\breve{MN}$ yoyni uni tortib turuvchi vatar bilan almashtiramiz va $l'(x)$ ni topamiz:

$$l'(x)=\lim_{\Delta x\to 0}\frac{\Delta l}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta x}==\lim_{\Delta x\to 0}\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}=\sqrt{1+(y'_x)^2}.$$

Demak, $dl=\sqrt{1+(y'_x)^2}dx$ yoki $y'_x=\dfrac{dy}{dx}$ ekanidan $dl=\sqrt{(dx)^2+(dy)^2}$.

3°. dl ni a dan b gacha integrallab, topamiz:

$$l=\int_a^b\sqrt{1+(y'_x)^2}dx \qquad (2.22)$$

(2.22) tenglikka yoy differensialining to‘g‘ri burchakli koordinatalardagi formulasi deyiladi.

Agar egri chiziq $x=g(y)$, $y\in[c;d]$ tenglama bilan berilgan bo‘lsa, yuqorida keltirilganlarni takrorlab, AB yoy uzunligini hisoblashning quyidagi formulasini hosil qilamiz:

$$l=\int_c^d\sqrt{1+(x'_y)^2}dy\,. \qquad (2.23)$$

Agar egri chiziq $x=\varphi(t)$, $y=\psi(t)$, $\alpha\le t\le\beta$ parametrik tenglamalar bilan berilgan bo‘lsa, (2.22) formulada $x=\varphi(t)$, $y=\psi(x)$, $dx=\varphi'(t)dt$ o‘rniga qo‘yish orqali o‘zgaruvchi almashtiriladi.

Bunda

$$l=\int_{\alpha}^{\beta}\sqrt{\varphi'^{2}(t)+\psi'^{2}(t)}dt \qquad (2.24)$$

kelib chiqadi, bu yerda $a=\varphi(\alpha)$ va $b=\varphi(\beta)$.

Egri chiziq qutb koordinatalar sistemasida $r=r(\varphi)$, $\alpha\le\varphi\le\beta$ tenglama bilan berilgan bo'lsin, bunda $r(\varphi)$, $r'(\varphi)$ funksiyalar $[\alpha;\beta]$ kesmada uzluksiz va A,B nuqtalarga qutb koordinatalarida α,β burchaklar mos keladi.

$x=r\cos\varphi$, $y=r\sin\varphi$ ekaninidan

$x'(\varphi)=r'(\varphi)\cos\varphi-r(\varphi)\sin\varphi$, $y'(\varphi)=r'(\varphi)\sin\varphi-r(\varphi)\cos\varphi$.

(2.24) formulaga $x'(\varphi)$ va $y'(x)$ hosilalarni qo'yamiz va almashtirishlar bajarib, topamiz:

$$l=\int_{\alpha}^{\beta}\sqrt{r^{2}(\varphi)+r'^{2}(\varphi)}d\varphi. \qquad (2.25)$$

Misollar

1. $y=x^{\frac{3}{2}}$ yarim kubik parabolaning $x=0$ dan $x=5$ gacha yoyi uzunligini topamiz. Bunda $y=x^{\frac{3}{2}}$ dan $y'=\frac{3}{2}x^{\frac{1}{2}}$ kelib chiqadi.

U holda (2.22) formula bilan topamiz:

$$l=\int_{0}^{5}\sqrt{1+\frac{9}{4}x}dx=\frac{8}{27}\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}\Bigg|_{0}^{5}=\frac{335}{27}.$$

2. $y=\frac{3}{8}x\sqrt[3]{x}-\frac{3}{4}\sqrt[3]{x^{2}}$ egri chiziq yoyining Ox o'q bilan kesishish nuqtalari orasidagi uzunligini hisoblaymiz. Buning uchun avval $y=0$ deb egri chiziqning Ox oq bilan kesishish nuqtalarini aniqlaymiz: $x_1=0$, $x_2=2\sqrt{2}$.

Hosilani topamiz:

$$y'=\frac{3}{8}\cdot\frac{4}{3}x^{\frac{1}{3}}-\frac{3}{4}\cdot\frac{2}{3}x^{-\frac{1}{3}}=\frac{1}{2}\left(x^{\frac{1}{3}}-x^{-\frac{1}{3}}\right).$$

Yoy uzunligini hisoblaymiz:

$$l=\int_{0}^{2\sqrt{2}}\sqrt{1+\frac{1}{4}\left(x^{\frac{1}{3}}-x^{-\frac{1}{3}}\right)^{2}}dx=\frac{1}{2}\int_{0}^{2\sqrt{2}}\sqrt{\left(x^{\frac{1}{3}}+x^{-\frac{1}{3}}\right)^{2}}dx=$$

$$=\frac{1}{2}\int_{0}^{2\sqrt{2}}\left(x^{\frac{1}{3}}+x^{-\frac{1}{3}}\right)dx=\frac{1}{2}\left(\frac{3}{4}x^{\frac{4}{3}}+\frac{3}{2}x^{\frac{2}{3}}\right)\Bigg|_{0}^{2\sqrt{2}}=3.$$

3. $\begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t \end{cases}$ tenglama bilan berigan egri chiziq uzunligini topamiz.
Berilgan tenglama astroidani ifodalaydi.

Astroidaning uzunligini (2.23) formula bilan topamiz:

$$l = 4\int_0^{\frac{\pi}{2}} \sqrt{(-3a\cos^2 t\sin t)^2 + (3a\sin^2 t\cos t)^2}\,dt =$$

$$= 4\int_0^{\frac{\pi}{2}} 3a\sqrt{\cos^2 t\sin^2 t\cdot(\cos^2 t + \sin^2 t)}\,dt =$$

$$= 12a\int_0^{\frac{\pi}{2}} \cos t\sin t\,dt = 6a\sin^2 t\Big|_0^{\frac{\pi}{2}} = 6a.$$

4. $r = a(1+\cos\varphi)$, $a > 0$ kardioida uzunligini topamiz. Bunda egri chiziqning simmetrikligini hisobga olamiz. U holda

$$l = 2l = 2\int_0^{\pi} \sqrt{a^2(1+\cos\varphi)^2 + a^2(-\sin\varphi)^2}\,d\varphi = 4a\int_0^{\pi}\sqrt{\frac{1+\cos\varphi}{2}}\,d\varphi =$$

$$= 4a\int_0^{\pi}\cos\frac{\varphi}{2}\,d\varphi = 8a\sin\frac{\varphi}{2}\Big|_0^{\pi} = 8a.$$

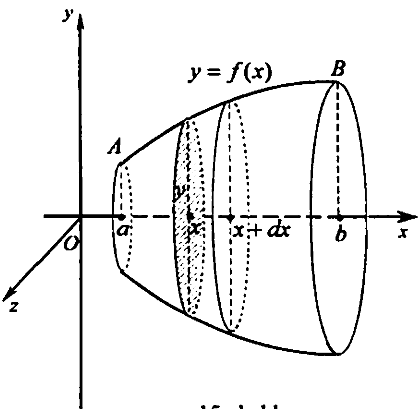
5. Aylanish sirti yuzasini hosoblash

AB egri chiziq $y = f(x) \geq 0$ funksiyaning grafigi bo'lsin. Bunda $x \in [a;b]$, $y = f(x)$ funksiya va uning $y' = f'(x)$ hosilasi bu kesmada uluksiz bo'lsin.

AB egri chiziqning Ox o'q atrofida aylanishidan hosil bo'lgan jism sirti yuzasini hisoblaymiz. Buning uchun II sxemani qo'llaymiz.

1°. Istalgan $x \in [a;b]$ nuqta orqali Ox o'qqa perpendikular tekislik o'tkazamiz. Bu tekislik aylanish sirtini radiusi $y = f(x)$ bo'lgan aylana bo'ylab kesadi. Bunda aylanish sirtidan iborat S kattalik x ning funksiyasi bo'ladi: $S = S(x)$ $(S(a) = 0$ va $S(b) = S)$.

2°. x argumentga $\Delta x = dx$ orttirma beramiz va $x + \Delta x \in [a;b]$ nuqta orqali Ox o'qqa perpendikular tekislik o'tkazamiz. Bunda $S = S(x)$ funksiya «belbog'» ko'rinishida ΔS orttirma oladi.



15-shakl

Kesimlar orasidagi jismni yasovchisi dl bo'lgan va asoslarining radiuslari y va $y + dy$ bo'lgan kesik konus bilan almashtiramiz. Bu kesik konusning yon sirti $dS = \pi(y + y + dy)dl = 2\pi y dl + \pi dy dl$ ga teng. $dydl$ ko'paytmani dS ga nisbatan yuqori tartibli cheksiz kichik sifatida tashlab yuboramiz: $dS = 2\pi y dl$. Bunda $dl = \sqrt{1 + (y'_x)^2}dx$ ekanini hisobga olamiz: $dS = 2\pi y\sqrt{1 + (y'_x)^2}dx$.

3°. dS ni a dan b gacha integrallab, topamiz:

$$S = 2\pi\int_a^b y\sqrt{1 + (y'_x)^2}dx \qquad (2.26)$$

Shu kabi $x = g(y)$, $y \in [c;d]$ funksiya grafigining Oy o'q atrofida aylantirshdan hosil bo'lgan jism sirtining yuzasi ushbu

$$S = 2\pi\int_c^d x\sqrt{1 + (x'_y)^2}dy \qquad (2.27)$$

formula bilan hisoblanadi.

Agar sirt $x = \varphi(t)$, $y = \psi(t)$, $\alpha \le t \le \beta$ parametrik tenglamalar bilan berilgan bo'lsa, u holda AB egri chiziqning $Ox(Oy)$ o'q atrofida aylanishidan hosil bo'lgan jism sirti yuzasi quyidagicha hisoblanadi:

$$S = 2\pi\int_\alpha^\beta \psi(t)\sqrt{\varphi'^2(t) + \psi'^2(t)}dt \left(S = 2\pi\int_{\alpha_1}^{\beta_1} \varphi(t)\sqrt{\psi'^2(t) + \varphi'^2(t)}dt \right). \quad (2.28)$$

bu yerda $a = \varphi(\alpha)$ va $b = \varphi(\beta)$ $(c = \psi(\alpha_1)$ va $d = \psi(\beta_1))$.

AB egri chiziq qutb koordinatalar sistemasida $r = r(\varphi)$, $\alpha \le \varphi \le \beta$ tenglama bilan berilgan bo'lganida quyidagi formulalar o'rinli bo'ladi:

$$S = 2\pi\int_\alpha^\beta r\sin\varphi\sqrt{r^2 + r'^2}d\varphi \ (Ox),\ S = 2\pi\int_\alpha^\beta r\cos\varphi)\sqrt{r^2 + r'^2}d\varphi \ (Oy). \ (2.29)$$

Misollar

1. Radiusi R ga teng bo'lgan shar sirti yuzaini hisoblaymiz..Shar parametrik tenglamasi $x = R\cos t$, $y = R\sin t$ bo'lgan yarim aylananing Ox

o‘q atrofida aylanishidan hosil bo‘ladi. Sharning koordinata o‘qlariga simmetrik bo‘lishini inobatga olib, hisoblaymiz:

$$S = 2 \cdot 2\pi \int_0^{\frac{\pi}{2}} R \sin t \sqrt{(-R \sin t)^2 + (R \cos t)^2}\, dt =$$

$$= 4\pi R^2 \int_0^{\frac{\pi}{2}} \sin t dt = -4\pi R^2 \cos t \Big|_0^{\frac{\pi}{2}} = 4\pi R^2 .$$

2. zanjir chizig‘i $x = 0$ dan $x = a$ gacha bo‘lagining Ox o‘qi atrofida aylanishidan hosil bo‘lgan sirt yuzasini hisoblaymiz.

Buning uchun avval $y' = sh\frac{x}{a}$ hosilani va $\sqrt{1 + (y'_x)^2} = ch\frac{x}{a}$ ifodani topamiz.

U holda (2.26) formulaga ko‘ra

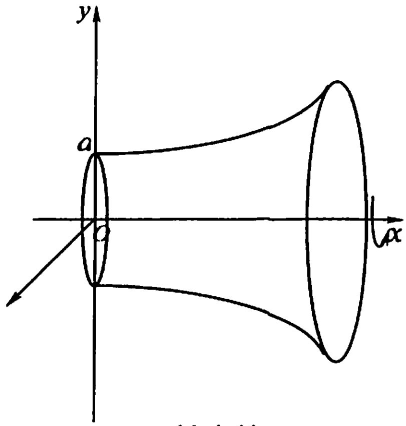
$$S = 2\pi \int_0^a a ch^2 \frac{x}{a} dx = \pi a \int_0^a \left(1 + ch\frac{2x}{a} \right) dx =$$

$$= \pi a \left(\frac{a}{2} sh\frac{2x}{a} + x \right) \Big|_0^a = \pi a^2 \left(\frac{1}{2} sh 2 + 1 \right).$$

3. $y = \frac{x^2}{2}, x > 0$ parabola bo‘lagining $y = \frac{3}{2}$ to‘g‘ri chiziq bilan kesilgan qismining Oy o‘qi atrofida aylanishidan hosil bo‘lgan sirt yuzasini hisoblaymiz. Misol shartidan topamiz: $x = \sqrt{2y},\ x' = \frac{1}{\sqrt{2y}}$.

(2 .27) formula bilan topamiz:

$$\sigma = 2\pi \int_0^{\frac{3}{2}} \sqrt{2y}\sqrt{1 + \frac{1}{2y}} dy = 2\pi \int_0^{\frac{3}{2}} \sqrt{2y + 1} dy = 2\pi \frac{1}{3}(2y + 1)^{\frac{3}{2}} \Big|_0^{\frac{3}{2}} = \frac{14\pi}{3}.$$



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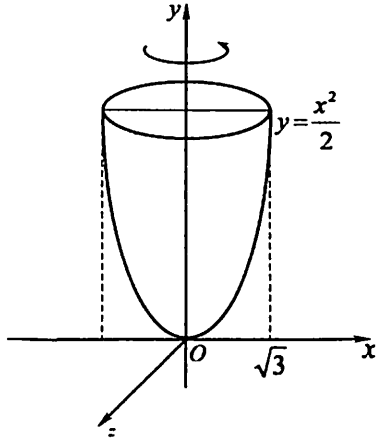


17-shakl

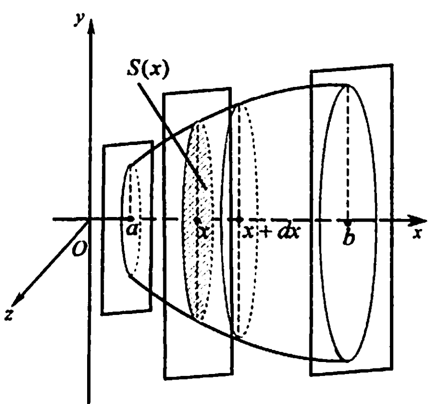
6. Hajmlarni hisoblash

Hajmni ko‘ndalang kesim yuzasi bo‘yicha hisoblash

Hajmi hisoblanishi lozim bo‘lgan qandaydir jism uchun uning istalgan ko‘ndalang kesim yuzasi S ma'lum bo‘lsin. Bu yuza ko‘ndalang kesim joylashishiga bog‘liq bo‘ladi: $S = S(x)$, $x \in [a;b]$, bu yerda $S(x)$ - $[a;b]$ kesmada uzluksiz funksiya.

Izlanayotgan hajmni II sxema asosida topamiz.

1°. Istalgan $x \in [a;b]$ nuqta orqali Ox o‘qqa perpendikular tekislik o‘tkazamiz. Jismning bu tekislik bilan kesimi yuzasini $S(x)$ bilan va jismning bu tekislikdan chapda yotgan bo‘lagining hajmini $V(x)$ bilan belgilaymiz. Bunda V kattalik x ning funksiyasi bo‘ladi: $V = V(x)$ $(V(a) = 0$ va $V(b) = V)$.



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2°. $V(x)$ funksiyaning dV differensialini topamiz. Bu differensial Ox o‘q bilan x va $x + \Delta x$ nuqtalarda kesishuvchi parallel tekisliklar orasidagi «elementar qatlam» dan iborat bo‘ladi. Bu differensialni asosi

$S(x)$ ga va balandligi dx ga teng silindr bilan taqriban almashtirish mumkin. Demak, $dV = S(x)dx$.

3°. dV ni a dan b gacha integrallab, izlanayotgan hajmni topamiz:

$$V = \int_a^b S(x)dx. \qquad (2.30)$$

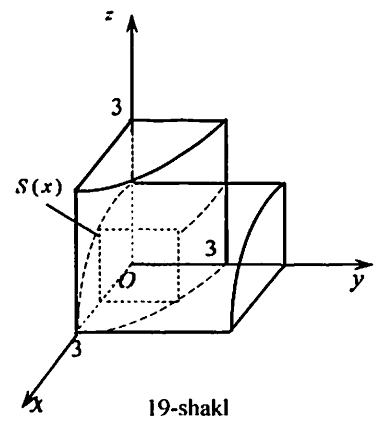
Misollar

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ellipsoidning hajmini hisoblaymiz.

Ellipsoidning koordinatalar boshidan x $(-a \le x \le a)$ masofada o‘tuvchi Ox o‘qqa perpendikulyar tekislik bilan kesamiz. Kesimda yarim o‘qlari $b(x) = b\sqrt{1 - \frac{x^2}{a^2}}$ va $c(x) = c\sqrt{1 - \frac{x^2}{a^2}}$ bo‘lgan ellips hosil bo‘ladi. Uning yuzasi $S(x) = \pi b(x)c(x) = \pi bc\left(1 - \frac{x^2}{a^2}\right)$. U holda

$$V = \int_{-a}^{a} \pi bc\left(1 - \frac{x^2}{a^2}\right)dx = \pi bc\left(x - \frac{x^3}{3a^2}\right)\Bigg|_{-a}^{a} = \frac{4}{3}\pi abc.$$

2. $x^2 + y^2 = 9$ va $x^2 + z^2 = 9$ silindrlar bilan chegaralangan jism hajmini hisoblaymiz. Berilgan jismning I oktantda $(x \ge 0, y \ge 0, z \ge 0)$ joylashgan sakkizdan bir bo‘lagini qaraymiz. Uning Ox o‘qqa perpendikular tekislik bilan kesimi kvadratdan iborat. Kesim abssissasi $(x;0;0)$ nuqtadan o‘tganda kvadratning tomonlari $a = y = z = \sqrt{9 - x^2}$ ga va yuzasi $S(x) = 9 - x^2$ ga teng bo‘ladi, bu yerda $0 \le x \le 9$.



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Jismning hajmni (2.30) formula bilan hisoblaymiz:

$$V = 8\int_0^3 (9 - x^2)dx = 8\left(9x - \frac{x^3}{3}\right)\Bigg|_0^3 = 144.$$

Aylanish jismlarining hajmini hisoblash

Yuqoridan $y = f(x)$ uzluksiz funksiya grafigi bilan, quyidan Ox o‘q bilan, yon tomonlaridan $x = a$ va $x = b$ to‘g‘ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning Ox o‘q atrofida aylantirishdan hosil bo‘lgan jism hajmini hisoblaymiz. Bu jismning ixtiyoriy ko‘ndalang kesimi doiradan iborat. Shu sababli jismning $X = x$ tekislik bilan kesimining yuzasi $S(x) = \pi y^2$ bo'ladi.

U holda (2.30) formulaga ko'ra

$$V = \pi \int_a^b y^2 dx. \qquad (2.31)$$

Shu kabi yuqoridan $y = f(x)$ uzluksiz funksiya grafigi bilan, quyidan Ox o'q bilan, yon tomonlaridan $x = a$ va $x = b$ to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyani Oy o'qi atrofida aylantirishdan hosil bo'lgan jismning hajmi quyidagi formula bilan hisoblanadi:

$$V = 2\pi \int_a^b yx dx. \qquad (2.32)$$

Agar egri chiziqli trapetsiya $x = \varphi(y)$ uzluksiz funksiya grafigi, Oy o'qi, $y = c$ va $y = d$ to'g'ri chiziqlar bilan chegaralangan bo'lsa, u holda

$$V = \pi \int_c^d x^2 dy \ (Oy) \left(V = 2\pi \int_c^d xy dy \ (Ox) \right) \qquad (2.33)$$

bo'ladi.

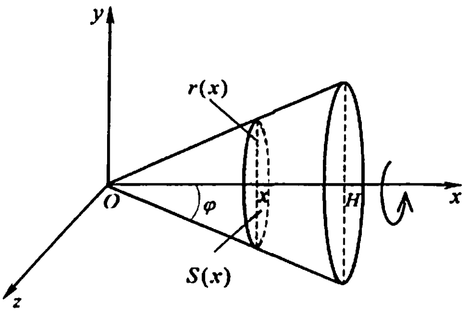
$r = r(\varphi)$ egri chiziq va $\varphi = \alpha$, $\varphi = \beta$ nurlar bilan chegaralangan egri chiziqli sektorning qutb o'qi atrofida aylanishidan hosil bo'lgan jismning hajmi

$$V = \frac{2\pi}{3} \int_\alpha^\beta r^3 \sin \varphi d\varphi \qquad (2.34)$$

formula bilan topiladi.

Misollar

1. $x = y^2$, $y = 0$ va $x = a$ $(a > 0)$ chiziqlar bilan chegaralangan tekis shaklning Ox o'q aylanishidan hosil bo'lgan jismning hajmini (2.31) formula bilan hisoblaymiz:



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$$V = \pi \int_0^a (\sqrt{x})^2 dx = \pi \int_0^a x dx = \pi \frac{x^2}{2}\Big|_0^a = \frac{\pi a^2}{2}.$$

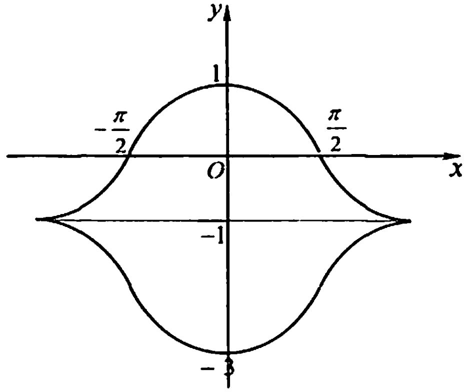
2. Radiusi R ga va balandligi H ga teng bo'lgan konusning hajmini hisoblaymiz. Bunda konusnni katetlari R va H bo'lgan to'g'ri burchakli

uchburchakning balandlik bo'ylab yo'nalgan Ox o'q atrofida aylanishidan hosil bo'lgan jism deyish mumkin. Gipotenuza tenglamasi $y = kx$ bo'lsin deymiz.

U holda

$$y = kx\ , \ k = tg\,\varphi = \frac{R}{H},\ \ y = \frac{R}{H}x\ .$$

Bundan



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$$V = \pi\int_0^H y^2dx = \pi\int_0^H \frac{R^2}{H^2}x^2dx = \frac{\pi R^2}{H^2}\cdot\frac{x^3}{3}\Bigg|_0^H = \frac{1}{3}\pi R^2 H\ .$$

3. $y = \cos x$ va $y = -1$ chiziqlar bilan chegaralangan tekis shaklning $-\pi \le x \le \pi$ da $y = -1$ to'g'ri chiziq atrofida aylanishidan hosil bo'lgan jismning hajmini hisoblaymiz. Egri chiziq $y = -1$ to'g'ri chiziq atrofida aylangani uchun yangi koordinatalar sistemasiga o'tish maqsadga muvofiq bo'ladi: $x' = x,\ y' = y + 1$.

U holda aylanish jismining hajmi

$$V' = \pi\int_{-\pi}^{\pi}(y')^2dx' = \pi\int_{-\pi}^{\pi}(y+1)^2dx =$$

$$= \pi\int_{-\pi}^{\pi}(\cos x + 1)^2dx = = \pi\int_{-\pi}^{\pi}(1 + 2\cos\varphi + \cos^2\varphi)d\varphi =$$

$$= \pi(\varphi + 2\sin\varphi)\Big|_{-\pi}^{\pi} + \frac{1}{2}\pi\int_0^{\pi}(1 + \cos 2\varphi)d\varphi = 2\pi^2 + \frac{1}{2}\pi\left(\varphi + \frac{1}{2}\sin 2\varphi\right)\Bigg|_{-\pi}^{\pi} = 3\pi^2.$$

2.10. Aniq integralning mexanika masalalariga tatbiqlari

1. Momentlar va og'irlik markazini hisoblash

Tekis shakl va aylanish sirti yuzalarini, tekis egri chiziq yoyi uzunligini, hajmlarini hisoblashga oid yuqorida keltirilgan masalalarni I sxema asosida yechishda aynan bir xil usul qo'llaniladi. Bunda izlanilayotgan Q kattalikni hisoblash integral yig'indi limitini topishga keltiriladi. Barcha masalalarda Q kattalik biror $[a;b]$ kesma va bu kesmadagi uzluksiz funksiaylarga bog'liq holda o'rganiladi.

Shu kabi Q kattalik quyidagi xossalarga ega deb faraz qilamiz:

1°. Additivlik xossasi. $[a;b]$ kesma istalgancha bo'laklarga bo'linganida ham

$Q=\sum_{i=1}^{n} Q_i$ bo'ladi, bu yerda Q_i - Q ning i - bo'lakdagi qiymati;

2°. Kichiklikdagi chiziqlilik xossasi. $[a;b]$ kesmadagi istalgan kichik $[x_{i-1};x_i]$ kesmada $Q_i \approx k\Delta x_i$ bo'ladi, bu yerda $\Delta x_i = x_i - x_{i-1}$.

Quyida keltiriladigan momentlarni, og'irlik markazi va kuchning bajargan ishini hisoblash formulalari ham yuqoridagi singari hosil qilinadi. Shu sababli bu formulalarni keltirib chiqarmaymiz va ulardan masalalarni yechishda foydalanamiz.

Oxy tekislikda massalari mos ravishda $m_1, m_2, \ldots, m_n$ bo'lgan $A_1(x_1;y_1), A_2(x_2;y_2), \ldots, A_n(x_n;y_n)$ nuqtalar sistemasi berilgan bo'lsin.

Sistemaning Ox (Oy) o'qqa nisbatan statik momenti M_x (M_y) deb nuqtalar massalarini ularning ordinatalariga (absissalariga) ko'paytmalari yig'indisiga aytiladi, ya'ni

$$M_x=\sum_{i=1}^{n} m_i y_i \quad \left(M_y=\sum_{i=1}^{n} m_i x_i\right).$$

Sistemaning Ox (Oy) oqqa nisbatan inersiya momenti J_x (J_y) deb nuqtalar massalarini ularning ordinatalari (absissalari) kvadratiga ko'paytmalari yig'indisiga aytiladi, ya'ni

$$J_x=\sum_{i=1}^{n} m_i y_i^2 \quad \left(J_y=\sum_{i=1}^{n} m_i x_i^2\right).$$

Sistemaning og'irlik markazi deb koordinatalari $\left(\frac{M_y}{m};\frac{M_x}{m}\right)$ bo'lgan nuqtalarga aytiladi, bu yerda $m=\sum_{i=1}^{n} m_i$.

2. Tekis egri chiziqning momentlari va og‘irlik markazi

Oxy tekislikda AB egri chiziq $y = f(x)$ $(a \le x \le b)$ tenglama bilan berilgan va egri chiziqning har bir nuqtasida $\gamma = \gamma(x)$ zichlik va $f(x)$ funksiya $f'(x)$ hosilasi bilan birga uzluksiz bo‘lsin.

U holda AB egri chiziqning momentlari va og‘irlik markazining koordinatalari quyidagi formulalar bilan aniqlanadi:

$$M_x = \int_a^b \gamma y dl, \quad M_y = \int_a^b \gamma x dl; (1)$$

$$J_x = \int_a^b \gamma y^2 dl, \quad J_y = \int_a^b \gamma x^2 dl; (2)$$

$$x_c = \frac{\int_a^b \gamma x dl}{\int_a^b \gamma dl}, \quad y_c = \frac{\int_a^b \gamma y dl}{\int_a^b \gamma dl}, (3)$$

bu yerda $y = f(x)$, $\gamma = \gamma(x)$, $dl = \sqrt{1 + y'^2}dx$, $a \le x \le b$.

Misol

Zichligi $\gamma = 1$ ga teng bo‘lgan $y = \sqrt{R^2 - x^2}$, $|x| \le R$ yarim aylananing momentlari va og‘irlik markazini topamiz. Bunda $y' = -\frac{x}{\sqrt{R^2 - x^2}}$ bo‘lgani uchun $dl = \frac{Rdx}{\sqrt{R^2 - x^2}}$.

U holda (1) - (3) formulalardan topamiz:

$$M_x = \int_{-R}^{R} \sqrt{R^2 - x^2} \frac{Rdx}{\sqrt{R^2 - x^2}} = R\int_{-R}^{R} dx = Rx\Big|_{-R}^{R} = 2R^2,$$

$$M_y = \int_{-R}^{R} \frac{xRdx}{\sqrt{R^2 - x^2}} = -R\sqrt{R^2 - x^2}\Big|_{-R}^{R} = 0.$$

$$J_x = \int_{-R}^{R} (R^2 - x^2)\frac{Rdx}{\sqrt{R^2 - x^2}} = R\int_{-R}^{R} \sqrt{R^2 - x^2}dx = R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R^2 \cos^2 t dt =$$

$$= R^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos 2t}{2} dt = \frac{R^3}{2}\left(t + \frac{\sin 2t}{2}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi R^3}{2},$$

$$J_y = \int_{-R}^{R} \frac{x^2 Rdx}{\sqrt{R^2 - x^2}} = R\left(-x\sqrt{R^2 - x^2}\Big|_{-R}^{R} + \int_{-R}^{R} \sqrt{R^2 - x^2}dx\right) = R\left(0 + \frac{\pi R^2}{2}\right) = \frac{\pi R^3}{2}.$$

$$\int_{-R}^{R} dl = \int_{-R}^{R} \frac{Rdx}{\sqrt{R^2 - x^2}} = R \arcsin \frac{x}{R}\Big|_{-R}^{R} = \pi R,$$

$$x_c = 0,\ y_c = \frac{2R^2}{\pi R} = \frac{2R}{\pi}.$$

3. Tekis shaklning momentlari va og'irlik markazi

Oxy tekislikda $[a;b]$ kesmada uzluksiz bo'lgan $y = f(x)$ funksiya grafigi, Ox o'q, $x = a$ va $x = b$ to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiya (tekis shakl) berilgan va tekis shaklning har bir nuqtasida $\gamma = \gamma(x)$ zichlik uzluksiz bo'lsin. U holda tekis shaklning momentlari va og'irlik markazining koordinatalari quyidagi formulalar orqali topiladi:

$$M_x = \frac{1}{2}\int_a^b \gamma y^2 dx,\quad M_y = \int_a^b \gamma xy dx\ ;(4)$$

$$J_x = \frac{1}{2}\int_a^b \gamma y^3 dx,\quad J_y = \int_a^b \gamma x^2 y dx\ ;(5)$$

$$x_c = \frac{\int_a^b \gamma xy dx}{\int_a^b \gamma y dx},\quad y_c = \frac{\frac{1}{2}\int_a^b \gamma y^2 dx}{\int_a^b \gamma y dx},(6)$$

bu yerda $\gamma = \gamma(x)$, $y = y(x)$, , $a \le x \le b$.

Misollar

1. $y = \sin x$ sinusoida yoyi va Ox o'qining $0 \le x \le \pi$ bo'lagi bilan chegaralangan, zichligi $\gamma = 1$ ga teng figuraning og'irlik markazini topamiz. Bunda. sinusoidaning simmetrikligidan $x_c = \frac{\pi}{2}$ bo'ladi.

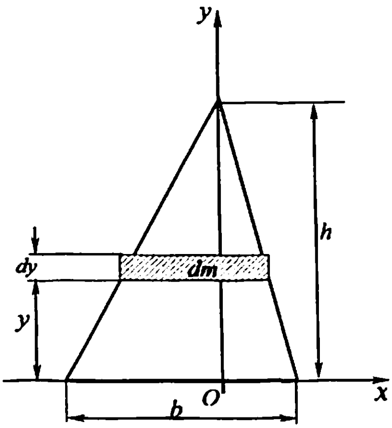
U holda

$$M_x = \frac{1}{2}\int_0^\pi y^2 dx = \frac{1}{2}\int_0^\pi \sin^2 x dx = \frac{1}{2}\int_0^\pi \frac{1 - \cos 2x}{2} dx = \frac{1}{4}\left(x - \frac{\sin 2x}{2}\right)\Bigg|_0^\pi = \frac{\pi}{4}.$$

$$\int_0^\pi y dx = \int_0^\pi \sin x dx = -\cos x\Big|_0^\pi = 2,\ y_c = \frac{\frac{\pi}{4}}{2} = \frac{\pi}{8}.$$

Demak,

$$x_c = \frac{\pi}{2},\ y_c = \frac{\pi}{8}.$$



22-shakl

2. Asosi b ga va balandligi h ga teng uchburchakning asosiga nisbatan momentini toping.

Ox o‘qni uchburchakning asosi bo‘ylab va Oy o‘qni balandlik bo‘ylab yo‘naltiramiz. Uchburchakni qalinligi dy ga teng cheksiz yupqa gorizontal tasmalarga bo‘lamiz. Bu tasmalar elementar massalar rolini o‘ynaydi.

Uchburchaklarning o‘xshashlik alomatiga ko‘ra:

$$dm = b\frac{h-y}{h}dy,$$

$$dJ_x = y^2 dm = \frac{b}{h}y^2(h-y)dy.$$

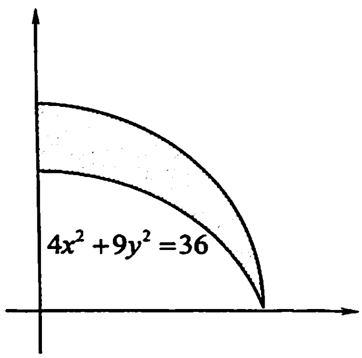
Bundan

$$J_x = \frac{b}{h}\int_0^h y^2(h-y)dy = \frac{b}{3}y^3\Big|_0^h - \frac{b}{4h}y^4\Big|_0^h = \frac{1}{12}bh^2.$$

3. $4x^2+9y^2=36$ ellips va $x^2+y^2=9$ aylanalar bilan chegaralangan tekis shaklning birinchi charokdagi bo‘lagi og‘irlik markazini topamiz:

$$M_y = \int_0^3 x(y_2-y_1)dx = \int_0^3 x\left(\sqrt{9-x^2} - \frac{2}{3}\sqrt{9-x^2}\right)dx = \frac{1}{3}\int_0^3 x\sqrt{9-x^2}dx =$$

$$= -\frac{1}{6}\int_0^3\sqrt{9-x^2}\,d(9-x^2) = -\frac{1}{9}(9-x^2)^{\frac{3}{2}}\Big|_0^3 = 3.$$



23-shakl

$$M_x = \frac{1}{2}\int_0^3 (y_2^2 - y_1^2)dx = \frac{1}{2}\int_0^3 x\left((9-x^2) - \frac{4}{9}(9-x^2)\right)dx =$$

$$= \frac{5}{18}\int_0^3 (9-x^2)dx = \frac{5}{18}\left(9x - \frac{x^3}{3}\right)\Big|_0^3 = 5.$$

Radiusi 3 ga teng doiraning yuzasi $\frac{9\pi}{4}$ ga, yarim o'qlari $a=3,\ b=2$ bo'lgan ellips chorak qismining yuzasi $\frac{3\pi}{2}$ ga teng bo'lgani uchun qaralayotgan tekis shakl yuzasi $S=\frac{9\pi}{4}-\frac{3\pi}{2}=\frac{3\pi}{4}$ ga teng bo'ladi. Sunday qilib,

$$x_c=\frac{M_y}{S}=\frac{4}{\pi},\quad y_c=\frac{M_x}{S}=\frac{20}{3\pi}.$$

4. Kuchning bajargan ishini hisoblash

Material nuqta o'zgaruvchan $\vec{F}$ kuch tasirida Ox o'qi bo'ylab harakatlanayotgan bo'lsin va bunda kuchning yo'nalishi harakat yo'nalishi bilan bir xil bo'lsin. U holda $\vec{F}$ kuchning material nuqtani Ox o'qi bo'ylab $x=a$ nuqtadan $x=b\ (a<b)$ nuqtaga ko'chirishda bajargan ishi quyidagi formula bilan hisoblanadi:

$$A=\int_a^b F(x)dx, \qquad (7)$$

bu yerda $F(x)$ funksiya $[a;b]$ kesmada uzluksiz.

Misollar:

1. Agar prujina $1H$ kuch ostida $1\ sm$ ga cho'zilsa, uni $6\ sm$ cho'zish uchun qancha ish bajarish kerak bo'lishini topamiz. Guk qonuniga muvofiq F kuch va x cho'zilish o'zaro $F=kx$ bog'lanishga ega. Proporsionallik koeffitsiyentini masalaning shartidan topamiz:

$x=1\ sm=0{,}01\ m$ da $F=1H$, ya'ni $1=k\cdot 0{,}01$.

Bundan, $k=100$ va $F=100x$.

U holda

$$A=\int_0^{0,06}100\,xdx=50x^2\Big|_0^{0,06}=0{,}18\ (J).$$

2. m massali kosmik kemani erdan h masofaga uchurish uchun qancha ish bajarish kerak bo'lishini topamiz. Butun olam tortishish qonuniga ko'ra yerning jismni tortish kuchi $F=k\frac{mM}{x^2}$ ga teng, bu yerda M -yerning massasi, x - yer markazidan kosmik kemagacha bo'lgan masofa, k - gravtasiya doimiyligi. Yer sirtida, ya'ni $x=R$ da $F=mg$ ga teng, bu yerda g - erkin tushish tezlanishi.

U holda

$$mg = k\frac{mM}{R^2}.$$

Bundan $kM = gR^2$ va $F = mg\frac{R^2}{x^2}$.

Izlanayotgan ishni (7) formula bilan topamiz:

$$A = \int_R^{R+h} mg\frac{R^2}{x^2}dx = -mgR^2\frac{1}{x}\Big|_R^{R+h} = -mgR^2\left(\frac{1}{R+h} - \frac{1}{R}\right) = mgR\frac{h}{R+h}.$$

Agar kosmik kema cheksizlikka ketsa, ya'ni $h \to \infty$ da $A = mgR$ bo'ladi.

3. Ikkita e_0 va e elektr zaryadi mos ravishda Ox o'qining $x_0 = 0$ va $x_1 = a$ nuqtalrida joylangan. Ikkinchi zaryadni $x_2 = b$ $(b > a)$ masofaga ko'chirish uchun kerak bo'ladigan ishni topamiz. Kulon qonuniga ko'ra e_0 zaryad e zaryadni

$F = \frac{e_0 e}{x^2}$ kuch bilan itaradi, bu yerda x -zaryadlar orasidagi masofa.

Izlanayotgan ishni (18.4) formula bilan topamiz:

$$A = \int_a^b e_0 e\frac{dx}{x^2} = -e_0 e\frac{1}{x}\Big|_a^b = -e_0 e\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{e_0 e(b-a)}{ab}.$$

5. Jismning bosib o'tgan yo'li

Material nuqta (jism) to'g'ri chiziq bo'ylab o'zgaruvchan $v = v(t)$ tezlik bilan harakatlanayotgan bo'lsin. Bu nuqtaning t_1 dan t_2 gacha vaqt oralig'ida bosib o'tgan yo'lini topamiz.

Hosilaning fizik ma'nosiga ko'ra nuqtaning bir tomonga harakatida «to'g'ri chiziqli harakat tezligi yo'ldan vaqt bo'yicha olingan hosilaga teng», ya'ni $v(t) = \frac{dS}{dt}$. Bundan $dS = v(t)dt$. Bu tenglikni t_1 dan t_2 gacha integrallaymiz:

$$S = \int_{t_1}^{t_2} v(t)dt. \qquad (8)$$

Izoh. Bu formulani aniq integralni qo'llash sxemalari bilan topish mumkin.

Misol

Material nuqtaning tezligi $v = 2(6 - t)$ m/s qonun bilan o'zgaradi. Nuqtaning harahat boshidan eng katta uzoqlashishini topami:

$$S = \int_0^t 2(6 - t)dt = 12t - t^2.$$

Nuqtaning eng katta uzoqlashishini yo'lni vaqtning funksiyasi sifatida qarab, topamiz: $S' = 12 - 2t$. $t = 6$ da $S' = 0$ bo'ladi.

Bundan

$$S_{max} = 12 \cdot 6 - 6^2 = 36 \ m.$$

6. Suyuqlikning vertikal plastinkaga bosimi

Paskal qonuniga ko'ra suyuqlikning gorizontal plastinkaga bosimi

$$P = g \cdot \gamma \cdot S \cdot h$$

formula bilan topiladi, bu yerda g - erkin tushish tezlanishi, γ - suyuqlik zichligi, S - plastinkaning yuzasi, h - plastinkaning botish chuqurligi.

Plastinkaning vertikal botishida suyuqlikning plastinkaga bosimini bu formula bilan topib bo'lmaydi, chunki plastinkaning har xil nuqtalari turli chuqurlikda yotadi.

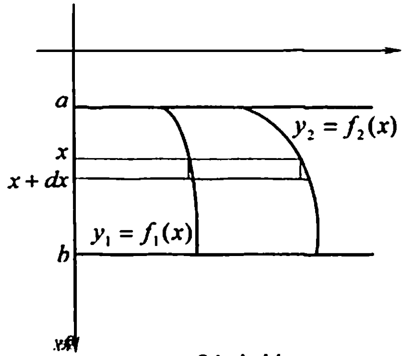
Suyuqlikka $x = a$, $x = b$, $y_1 = f_1(x)$, $y_2 = f_2(x)$ chiziqlar bilan chegaralangan plastinka vertikal botirilayotgan bo'lsin. Bunda koordinatalar sistemasi 24-shaklda ko'rsatilganidek tanlangan bo'lsin. Shuyuqlikning plastinkaga P bosimini topish uchun II sxemadan foydalanamiz.

Bunda: 1°. Izlanayotgan P kattalikning bir qismi x ning funksiyasi bo'lsin:

$p = p(x)$, ya'ni $p = p(x)$ - plastinkaning x o'zgaruvchi $[a;x]$ kesmasiga mos qismiga suyuqlik bosimi, bu yerda $x \in [a;b]$ $(p(a) = 0$ va $p(b) = P)$.

2°. x argumentga $\Delta x = dx$ orttirma beramiz . Bunda $p(x)$ funksiya Δp orttirma oladi (24-shakldа - dx qalinlikdagi tasma). Funksiyaning dp differensialni topamiz. dx kichik ekanidan tasmani barcha nuqtalari bitta chuqurlikda yotuvchi to‘g‘ri to‘rtburchak deb hisoblaymiz, ya’ni plastinka gorisontal bo‘lsin deymiz. U holda Paskal qonuniga ko‘ra

$$dp = g \cdot \gamma \cdot \underbrace{(y_2 - y_1) \cdot dx}_{s} \cdot \underbrace{x}_{h}.$$



24-shakl

3°. dp ni $x = a$ dan $x = b$ gacha integrallaymiz:

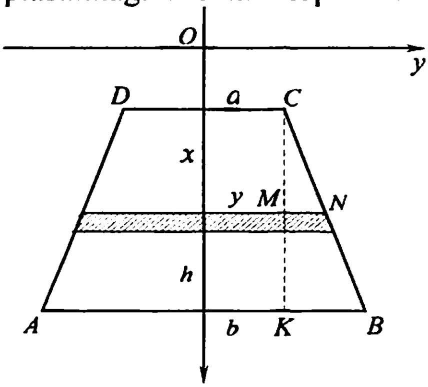
$$P = g \cdot \gamma \cdot \int_a^b (y_2 - y_1) x dx$$

yoki

$$P = g \cdot \gamma \cdot \int_a^b (f_2(x) - f_1(x)) x dx . \qquad (9)$$

Misol

Asoslari a va b ga, balandligi h ga teng bo‘lgan teng yonli trapetsiya shaklidagi plastinka suyuqlikka c chuqurlikda botirilgan. Suyuqlikning plastinkaga bosimini topamiz.



25-shakl

Izlanayotgan bosim (9) formulaga ko‘ra

$$P = g\gamma \int_c^{c+h} yxdx$$

bo‘ladi.

y o‘zgaruvchini x o‘zgaruvchi orqali ifodalash uchun CMN va CKB uchburchaklarning o‘xshashligidan foydalanamiz:

$$\frac{y - a}{b - a} = \frac{x - c}{h}.$$

Bundan $y = a + \dfrac{b - a}{h}(x - c)$.

U holda

$$P = g\gamma \int_c^{c+h} \left(a + \frac{b - a}{h}(x - c) \right) xdx = g\gamma \left(\frac{ax^2}{2} + \frac{b - a}{h} \left(\frac{x^3}{3} - \frac{cx^2}{2} \right) \right) \Bigg|_c^{c+h} =$$

$$= g\gamma\left(\frac{a+b}{2}ch + \frac{h^2}{6}(a+2b)\right).$$

Amaliy mashg'ulotda yechiladigan misollar.

1. $r = 2R\sin\varphi$ bir jinsli aylananing og'irlik markazini toping.

2. $x = a\cos^3 t,\ y = a\sin^3 t$ bir jinsli astroidaning Ox o'qdan yuqorida yotgan yoyining og'irlik markazini toping.

3. $4x + 3y - 12 = 0$ bir jinsli to'g'ri chiziqning koordinata o'qlari orasida joylashgan kesmasining koordinata o'qlariga nisbatan statik momentlarini toping.

4. $x = 0,\ y = 0,\ x + y = 2$ ciziqlar bilan chegaralangan bir jinsli tekis shaklning koordinata o'qlariga nisbatan statik va inersiya momentlarini, og'irlik markazini toping.

5. $y = 4 - x^2$ va $y = 0$ bir jinsli chiziqlar bilan chegaralangan figuraning og'irlik markazini toping.

6. Yarim o'qlari $a = 5$ va $b = 4$ bo'lgan bir jinsli ellipsning koordinata o'qlariga nisbatan inersiya momentini toping.

7. $x^2 + y^2 = R^2$ aylananing birinchi chorakda joylashgan bo'lagining o'girlik markazini toping. Bunda aylananing har bir nuqtasidagi chiziqli zichligi shu nuqta koordinatalarining ko'paytmasiga proporsional.

8. $x = 8\cos^3 t,\ 8 = 4\sin^3 t$ astroida birinchi chorakda yotgan yoyining koordinata o'qlariga nisbatan statik momentlarini va massasini toping. Bunda astroidaning har bir nuqtasidagi chiziqli zichligi x ga teng.

9. Prujinani 4 *sm*. ga cho'zish uchun 24 *J* ish bajariladi. 150 *J* ish bajarilsa, prujinana qanday uzunlikka cho'ziladi?

10. Agar prujinani 1 *sm*. ga siqish uchun 1 *kG* kuch sarf qilinsa, prujinaning 8 *sm*. ga siqishda sarf bo'ladigan F kuch bajargan ishni toping.

11. Uzunligi 0,5 *m*. va radiusi 4 *mm*. bo'lgan mis simni 2 *mm*. cho'zish uchun qancha ish bajarish kerak? Bunda $F = E\frac{Sx}{l}$, $E = 12\cdot 10^4\ N/mm^2$.

12. Og'irligi $P = 1{,}5\ T$ bo'lgan kosmik kemani yer sirtidan $h = 2000$ *km*. masofaga uchirish uchun bajarilishi kerak bo'ladigan ishni toping.

Mustaqil yechish uchun misollar.

1. Jismning to'g'ri chiziqli harakat tezligi $v = 2t + 3t^2\ (m/s)$ formula bilan ifodalanadi. Jismning harakat boshlanishidan 5 *s*. davomida bosib o'tgan yo'lini toping.

2. Nuqtaning harakat tezligi $v = 0{,}1t^3$ (m/s) ga teng. Nuqtaning 10 $s.$ davomidagi o'rtacha tezligini toping.

3. Sportchining parashutdan tushish tezligi $v = \frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right)$ formula bilan ifodalanadi, bu yerda g -erkin tushish tezlanishi, m - sportchining massasi, k - proporsionallik koeffitsiyenti. Agar parashutdan tushish 3 min . davom etgan bo'lsa, sportchi qanday balandlikdan sakragan?

4. Nuqtaning harakat tezligi $v = 0.1e^{-0{,}01t}$ (m/s) ga teng. Nuqtaning harakat boshlanishidan harakat to'xtaguncha bosib o'tgan yo'lini toping.

5. Suyuqlikka vertikal botirilgan asoslari a va b $(b > a)$ ga, balandligi h ga teng bo'lgan teng yonli trapetsiya shaklidagi plastinkaga suyuqlikning bosimini toping.

6. Asosi 18 m. Va balandligi 6 $m.$ bo'lgan to'rt burchakli shluzga suv bosimini toping.

7. Diametri 6 $m.$ bo'lgan va suv sathida joylashgan yarim doira shaklidagi vertikal devorga suv bosimini toping. Suv zichligi $\gamma = 1000$ kg/m^3.

2.11. Xosmas integrallar

$\int_a^b f(x)dx$ integral mavjud bo'lishi uchun ikkita shartning bajarilishi talab qilingan edi: 1) integrallash chegarasi chekli $[a;b]$ kesmadan iborat bo'lishi; 2) integral ostidagi funksiya $[a;b]$ kesmada aniqlangan va chegaralangan bo'lishi.

Agar aniq integral uchun keltirilgan shartlardan biri bajarilmasa, u holda integralga xosmas integral deyiladi. Agar faqat birinchi shart o'rinli bo'lmasa, integralga cheksiz chegarali xosmas integral (yoki I tur xosmas integral) deyiladi. Agar faqat ikkinchi shart bajarilmasa, ya'ni integral ostidagi funksiya $[a;b]$ kesmada uzilishga ega bo'lsa, integralga chegaralanmagan funksiyaning xosmas integrali (yoki II tur xosmas integral) deyiladi.

1. Cheksiz chegarali xosmas integrallar (I tur xosmas integrallar)

1-ta'rif. $f(x)$ funksiya $[a;+\infty)$ oraliqda uzluksiz bo'lsin. Agar $\lim_{b\to+\infty}\int_a^b f(x)dx$ chekli limit mavjud bo'lsa, bu limitga yuqori chegarasi cheksiz xosmas integral deyiladi va $\int_a^{+\infty} f(x)dx$ kabi belgilanadi, ya'ni

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx. \quad (1)$$

Bu holda $\int_a^{+\infty} f(x)dx$ integralga yaqinlashuvchi integral deyiladi.

Agar $\lim_{b\to+\infty}\int_a^b f(x)dx$ limit mavjud bo'lmasa yoki cheksiz bo'lsa $\int_a^{+\infty} f(x)dx$ integralga uzoqlashuvchi integral deyiladi.

Quyi chegarasi cheksiz va har ikkala chegarasi cheksiz xosmas integrallar shu kabi aniqlanadi:

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx, \quad (2)$$

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{a\to-\infty}\int_a^c f(x)dx + \lim_{b\to+\infty}\int_c^b f(x)dx, \quad (3)$$

bu yerda c - o'qning istalgan fiksirlangan nuqtasi.

Bunda (3) tenglikning chap tomonidagi xosmas integral, o'ng tomondagi har ikkala xosmas integral yaqinlashganda yaqinlashadi.

Misollar

1. $\int_1^{+\infty}\frac{dx}{x^\alpha} (\alpha > 0)$ integralni yaqinlashishga tekshiramiz. $\alpha \neq 1$ bo'lsin. U holda

$$\int_1^{+\infty}\frac{dx}{x^\alpha} = \lim_{b\to+\infty}\int_1^b\frac{dx}{x^\alpha} = \lim_{b\to+\infty}\left.\frac{x^{1-\alpha}}{1-\alpha}\right|_1^b = \frac{1}{1-\alpha}(\lim_{b\to+\infty} b^{1-\alpha} - 1).$$

Bunda: 1) $\alpha < 1$ bo'lganda $\int_1^{+\infty}\frac{dx}{x^\alpha} = \frac{1}{1-\alpha}(\lim_{b\to+\infty} b^{1-\alpha} - 1) = +\infty$,

2) $\alpha > 1$ bo'lganda,

3) $\alpha = 1$ bo'lganda $\int_1^{+\infty}\frac{dx}{x^\alpha} = \lim_{b\to+\infty}\int_1^b\frac{dx}{x} = \lim_{b\to+\infty}\ln x\Big|_1^b = \lim_{b\to+\infty}\ln b = +\infty$.

Demak, $\int_1^{+\infty}\frac{dx}{x^\alpha}$ xosmas integral $\alpha > 1$ da yaqinlashadi va $0 < \alpha \le 1$ da uzoqlashadi.

2. $\int_{-\infty}^{0} x\cos x dx$ integralni yaqinlashishga tekshiramiz.

$$\int_{-\infty}^{0} x\cos xdx = \lim_{a\to-\infty}\int_a^0 x\cos xdx = \lim_{a\to-\infty}\left(x\sin x\Big|_a^0 - \int_a^0 \sin xdx\right) =$$

$$= \lim_{a\to-\infty}(-a\sin a + \cos x\Big|_a^0 = \lim_{a\to-\infty}(-a\sin a - \cos a + 1).$$

Bu limit mavjud emas. Shu sababli $\int_{-\infty}^{0} x\cos xdx$ integral uzoqlashadi.

3. $\int\limits_{-\infty}^{+\infty}\frac{dx}{x^2+6x+10}$ integralni yaqinlashishga tekshiramiz. Oraliq nuqtani $c=0$ deymiz. U holda(3) tenglikga ko‘ra

$$\int\limits_{-\infty}^{+\infty}\frac{dx}{x^2+6x+10}=\int\limits_{-\infty}^{0}\frac{dx}{x^2+6x+10}+\int\limits_{0}^{+\infty}\frac{dx}{x^2+6x+10}.$$

Bundan

$$\int\limits_{-\infty}^{0}\frac{dx}{x^2+6x+10}=\lim_{a\to-\infty}\int\limits_{a}^{0}\frac{dx}{(x+3)^2+1}=\lim_{a\to-\infty}arctg\,(x+3)\Big|_a^0=$$

$$=arctg\,3-\lim_{a\to-\infty}arctg\,(a+3)=arctg\,3+\frac{\pi}{2},$$

$$\int\limits_{0}^{+\infty}\frac{dx}{x^2+6x+10}=\lim_{b\to+\infty}\int\limits_{0}^{b}\frac{dx}{(x+3)^2+1}=\lim_{b\to+\infty}arctg\,(x+3)\Big|_0^b=$$

$$=\lim_{b\to+\infty}arctg\,(b+3)-arctg\,3=\frac{\pi}{2}-arctg\,3.$$

U holda

$$\int\limits_{-\infty}^{+\infty}\frac{dx}{x^2+6x+10}=\int\limits_{-\infty}^{0}\frac{dx}{x^2+6x+10}+\int\limits_{0}^{+\infty}\frac{dx}{x^2+6x+10}==arctg\,3+\frac{\pi}{2}+\frac{\pi}{2}-arctg\,3=\pi.$$

Demak, xosmas integral yaqinlashadi.

2. Chegaralanmagan funksiyalarning xosmas integrallari (II tur xosmas integrallar)

2-ta’rif. $f(x)$ funksiya $[a;b)$ oraliqda aniqlangan va uzluksiz bo‘lib, $\lim\limits_{\varepsilon\to0}\int\limits_a^{b-\varepsilon}f(x)dx$ chekli limit mavjud bo‘lsa, bu limitga f(x) funksiyadan olingan xosmas integral deyiladi va $\int\limits_a^b f(x)dx$ kabi belgilanadi.

Shunday qilib,

$$\int\limits_a^b f(x)dx=\lim_{\varepsilon\to0}\int\limits_a^{b-\varepsilon}f(x)dx.\quad(4)$$

Shunga o‘xshash: 1) agar $f(x)$ funksiya x ning a ga o‘ngdan yaqinlashishda uzilishga ega bo‘lsa

$$\int\limits_a^b f(x)dx=\lim_{\varepsilon\to0}\int\limits_{a+\varepsilon}^{b}f(x)dx;\quad(5)$$

bo‘ladi;

2) agar $f(x)$ funksiya $c\in[a;b]$ da uzilishga ega bo‘lsa, u holda

$$\int\limits_a^b f(x)dx=\lim_{\varepsilon\to0}\int\limits_a^{c-\varepsilon}f(x)dx+\lim_{\varepsilon\to0}\int\limits_{c+\varepsilon}^{b}f(x)dx\quad(6)$$

bo‘ladi.

Bu xosmas integrallar uchun yaqinlashish (uzoqlashish) tushunchalari cheksiz chegarali integrallardagi kabi kiritiladi.

Misollar

1. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ integralni yaqinlashishga tekshiramiz. $x=1$ da integral ostidagi funksiya ikkinchi tur uzilishga ega.

U holda (4) tenglikka ko‘ra

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\frac{dx}{\sqrt{1-x^2}} = \lim_{\varepsilon\to 0}\arcsin x\Big|_0^{1-\varepsilon} =$$

$$= \lim_{\varepsilon\to 0}(\arcsin(1-\varepsilon)-0) = \arcsin 1 = \frac{\pi}{2}.$$

Demak, xosmas integral yaqinlashadi

2. $\int_{-1}^{1}\frac{dx}{x\sqrt[3]{x}}$ integralni yaqinlashishga tekshiramiz. $x=0$ da integral ostidagi funksiya uzilishga ega.

U holda (6) tenglikka ko‘ra

$$\int_{-1}^{1}\frac{dx}{x\sqrt[3]{x}} = \lim_{\varepsilon\to 0}\int_{-1}^{-\varepsilon}\frac{dx}{x\sqrt[3]{x}} + \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1}\frac{dx}{x\sqrt[3]{x}} = -3\lim_{\varepsilon\to 0} x^{-\frac{1}{3}}\Big|_{-1}^{-\varepsilon} - 3\lim_{\varepsilon\to 0} x^{-\frac{1}{3}}\Big|_{\varepsilon}^{1} =$$

Demak, xosmas integral uzoqlashadi. Bunga Nyuton-Leybnits formulasi formal tarzda qo‘llanilsa, u holda xato natija kelib chiqadi:

$$\int_{-1}^{1}\frac{dx}{x\sqrt[3]{x}} = -\frac{3}{\sqrt[3]{x}}\Big|_{-1}^{1} = -6.$$

3. Xosmas integrallarning yaqinlashish alomatlari

Ko‘pincha xosmas integralni (1) - (6) formulalar orqali hisoblash shart bo‘lmasdan faqat uning yaqinlashuvchi yoyi uzoqlashuvchi bo‘lishini bilish yetarli bo‘ladi. Bunday hollarda berilgan integralning yaqinlashuvchi yoki uzoqlashuvchi bo‘lishi yaqinlashuvchi yoki uzoqlashuvchiligi oldindan ma’lum bo‘lgan boshqa xosmas integral bilan taqqoslash orqali aniqlanadi. Xosmas integrallarning taqqoslash alomatlarini ifodalovchi teoremalarni isbotsiz keltiramiz.

1-teorema (I tur xosmas integralning yaqinlashish alomati). $[a;+\infty)$ oraliqda $f(x)$ va $\varphi(x)$ funksiyalar uzluksiz bo‘lsin va $0\le f(x)\le\varphi(x)$ tengsizlikni qanoatlantirsin. U holda:

1) agar $\int_a^{+\infty}\varphi(x)dx$ integral yaqinlashsa $\int_a^{+\infty}f(x)dx$ integral yaqinlashadi;

2) agar $\int_a^{+\infty}f(x)dx$ integral uzoqlashsa $\int_a^{+\infty}\varphi(x)dx$ integral uzoqlashadi.

Misollar

1. $\int_0^{+\infty} e^{-x^2}dx$ integralni yaqinlashishga tekshirami. Puasson integrali deb ataluvchi bu integral boshlang'ich funksiya ega emas.

Bunda

$$\int_0^{+\infty} e^{-x^2}dx = \int_0^{1} e^{-x^2}dx + \int_1^{+\infty} e^{-x^2}dx.$$

$\int_0^{1} e^{-x^2}dx$ integral xosmas integral emas va u chekli son qiymatiga ega.

$\int_1^{+\infty} e^{-x^2}dx$ integralni qaraymiz. $[1;+\infty)$ oraliqda $0 < e^{-x^2} \le e^{-x}$ bo'ladi, e^{-x^2} va e^{-x} funksiyalar uzluksiz. U holda

$$\int_1^{+\infty} e^{-x}dx = \lim_{b\to+\infty}\int_1^{b} e^{-x}dx = \lim_{b\to+\infty}(-e^{-x})\Big|_1^b = \frac{1}{e} - \lim_{b\to+\infty}\frac{1}{e^b} = \frac{1}{e}.$$

Demak, bu integral yaqinlashadi va 1-teoremaning birinchi bandiga asosan

Puasson integrali ham yaqinlashadi.

2-teorema (II tur xosmas integralning yaqinlashish alomati). $[a;b)$ oraliqda $f(x)$ va $\varphi(x)$ funksiyalar uzluksiz bo'lsin va $0 \le f(x) \le \varphi(x)$ tengsizlikni qanoatlantirsin, $x = b$ da $f(x)$ va $\varphi(x)$ funksiyalar aniqlanmagan yoki uzilishga ega bo'lsin. U holda:

1) agar $\int_a^b \varphi(x)dx$ integral yaqinlashsa $\int_a^b f(x)dx$ integral yaqinlashadi;

2. $\int_0^1 \frac{dx}{e^x - \cos x}$ integralni yaqinlashishga tekshiramiz. Integral ostidagi funksiya $x = 0$ da uzilishga ega.

$$x \in (0;1] \text{ da } \frac{1}{e^x - \cos x} \ge \frac{1}{xe}.$$

Bundan

$$\int_0^1 \frac{dx}{xe} = \frac{1}{e}\lim_{\varepsilon\to 0}\int_{+\varepsilon}^1 \frac{dx}{x} = \frac{1}{e}\lim_{\varepsilon\to 0}\ln x\Big|_{+\varepsilon}^1 = \frac{1}{e}(0 - \lim_{\varepsilon\to 0}\ln|\varepsilon|) = +\infty.$$

Demak, $\int_0^1 \frac{dx}{xe}$ integral uzoqlashadii va 2-teoremaning ikkinchi bandiga asosan berilgan integral ham uzoqlashadi.

2) agar $\int_a^b f(x)dx$ integral uzoqlashsa $\int_a^b \varphi(x)dx$ integral uzoqlashadi.

Taqqoslash teoremasi faqat nomanfiy funksiyalarga tegishli. Ishorasi almashadigan funksiyalarning xosmas integrallari uchun quyidagi alomat o'rinli bo'ladi.

3-teorema. Agar $\int_a^{+\infty}|f(x)|dx$ $\left(\int_a^b|f(x)|dx\right)$ integral yaqinlashuvchi bo'lsa, u holda $\int_a^{+\infty}f(x)dx$ $\left(\int_a^b f(x)dx\right)$ integral yaqinlashuvchi bo'ladi.

Agar $\int_a^{+\infty}|f(x)|dx$ $\left(\int_a^b|f(x)|dx\right)$ integral yaqinlashuvchi bo'lsa $\int_a^{+\infty}f(x)dx$ $\left(\int_a^b f(x)dx\right)$ xosmas integral absolyut yaqinlashuvchi xosmas integral deyiladi.

Agar $\int_a^{+\infty}f(x)dx$ $\left(\int_a^b f(x)dx\right)$ integral yaqinlashuvchi bo'lib, $\int_a^{+\infty}|f(x)|dx$ $\left(\int_a^b|f(x)|dx\right)$ integral uzoqlashuvchi bo'lsa $\int_a^{+\infty}f(x)dx$ $\left(\int_a^b f(x)dx\right)$ integral shartli yaqinlashuvchi xosmas integral deyiladi.

Misol

$\int_1^{+\infty}\frac{\cos x}{x^2}dx$ integralni yaqinlashishga tekshiramiz. Integral ostidagi funksiya $[1;+\infty)$ oraliqda ishorasini almashtiradi.

Ma'lumki, $\left|\frac{\cos x}{x^2}\right|\le\frac{1}{x^2}$. va 1-misolga ko'ra $\int_1^{+\infty}\frac{dx}{x^2}$ integral yaqinlashuvchi.

U holda, 1-teoremaga asosan $\int_1^{+\infty}\left|\frac{\cos x}{x^2}\right|dx$ integral yaqinlashuvchi va 3-teorema va 3-ta'rifga ko'ra $\int_1^{+\infty}\frac{\cos x}{x^2}dx$ integral absolyut yaqinlashuvchi bo'ladi.

(2), (3) ko'rinishdagi ((5),(6) ko'rinishdagi) xosmas integrallar uchun taqqoslash alomatlari hamda absolyut va shartli yaqinlashish tusunchalari yuqorida (1) ko'rinishdagi ((4) ko'rinishdagi) integrallar uchun keltirilgandagi kabi kiritiladi.

III BOB. IKKI KARRALI INTEGRALLAR

3.1. Ikki karrali integralga oid asosiy tushuncha va ta'riflar

Chegaralangan $z=f(x,y)$ funksiya $z=f(x,y)$ Oxy tekislikning qandaydir yopiq D sohasida aniqlangan bo'lsin. Agar

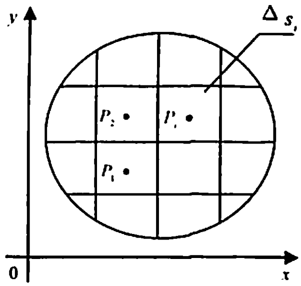
$$\sum_{k=1}^{n} f(x_k,y_k)\Delta s_k$$

integral yig'indining D sohaning D_k bo'laklarga bo'linishlari va M_k nuqtalarning tanlanish usuliga bog'liq bo'lmagan holda $d\to 0(n\to\infty)$ dagi limiti mavjud bo'lsa, bu limitga $f(x,y)$ funksiyaning D soha bo'yicha olingan ikki o'lchovli integrali deyiladi

$$\lim_{\substack{d\to 0\\(n\to\infty)}}\sum_{k=1}^{n} f(x_k,y_k)\Delta s_k=\iint_D f(x,y)dx\,dy,$$

$f(x,y)$ funksiya esa D sohada integrallanuvchi deyiladi(bu yerda Δs_k - D_k bo'lakning yuzi, $d=\max\limits_{1\le k\le n} d_k$ - bo'laklar diametrlarining maksimumi).

Agar $f(x,y)$ funksiya yopiq D sohada uzluksiz bo'lsa, u holda bu fuksiya D sohada integrallanuvchi bo'ladi. Ikki o'lchovli integrallar ham aniq integrallardagi kabi xossalarga ega. Ikki o'lchovli integralni hisoblash ikkita takroriy integralni hisoblashga keltiriladi. $z=f(x,y)=f(P)$ funksiya L chiziq bilan chegaralangan yopiq D sohada aniqlangan va uzliksiz bo'lib, $\Delta s_1,\Delta s_2,\Delta s_3,\dots,\Delta s_n$ - D sohani n ta elementar bo'laklarga bo'lish natijasida hosil bo'lgan yuzchalar bo'lsin.



26-chizma.

Har qaysi Δs_i elementar sohada ixtiyoriy $P_i(x_i, y_i)$ nuqtani tanlaymiz va funksiyaning P_i nuqtadagi qiymatini hisoblab, ushbu ko'paytmani tuzamiz:

$$f(P_i) \cdot \Delta s_i = f(x_i, y_i)\Delta s_i.$$

Shunday ko'paytmalarning barchasining

$$\sum_{n=1}^{\infty} f(P_i) \cdot \Delta s_i = \sum_{n=1}^{\infty} f(x_i, y_i)\Delta s_i$$

yig'indisi $z = f(x, y) = f(P)$ funksiya uchun D sohadagi *integral yig'indi* deyiladi.

s_i elementar yuzchalar soni cheksiz orttirilsa, u holda ular diametrlarining eng kattasi nolga intilgandagi integral yig'indining limiti $z = f(x, y)$ funksiyadan D soha bo'yicha olingan *ikki o'lchovli integral* deyiladi va bunday belgilanadi:

$$\iint_D f(P)ds \; yoki \; \iint_D f(x, y)ds$$

Shunday qilib,

$$\iint_D f(x, y)ds = \lim_{\max diam \Delta s_i \to 0} \sum_{n=1}^{\infty} f(x, y)\Delta s_i$$

Bunda D – integrallash sohasi, $z = f(x, y)$ integral ostidagi funksiya, ds – yuz elementi deyiladi.

Teorema(mavjudlik haqidagi). Agar chegaralangan yopiq **(*D*)** sohada $z = f(P) = f(x;y)$ funksiya uzluksiz bo'lsa, u holda bu sohani qismiy sohalarga bo'lishlar sonini ΔS_i yuzalar diametrlarining eng kattasi nol'ga intiladigan qilib kattalashtirilganda $(n \to \infty)$

$$\sum_{i=1}^{n} f(x_i;y_i)\cdot \Delta S_i = \sum_{i=1}^{n} f(P_i)\cdot \Delta S_i \qquad (1)$$

ko'rinishdagi integral yig'indining limiti mavjud bo'ladi.

Bu limit **(*D*)** sohani ΔS_i yuzalarga bo'lish usuliga ham va har bir qism ichida $P_i(x_i;y_i)$ nuqtani tanlash usuliga ham bog'liq bo'lmaydi. Bu limit qiymatga $z = f(P) = f(x;y)$ funksiyadan **(*D*)** soha bo'yicha olingan ***ikki karrali integral*** deyiladi va quyidagicha belgilanadi:

$$\lim_{\max diam \Delta S_i \to 0} \sum_{i=1}^{n} f(x_i;y_i)\cdot \Delta S_i =$$
$$= \lim_{\max diam \Delta S_i \to 0} \sum_{i=1}^{n} f(P_i)\cdot \Delta S_i = \iint_{(D)} f(P)\,dS = \iint_{(D)} f(x;y)\,dS \qquad (2)$$

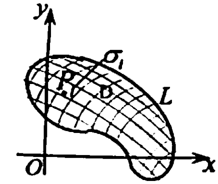
bu yerda **(*D*)** - integrallash sohasi, $f(x;y)$ - integral ostidagi funksiya, x,y - lar integrallash o'zgaruvchilari, dS yuz elementi deyiladi. Ikki karrali integral sohani qismlarga bo'lish usuliga bog'liq bo'lmaganligi uchun uni koordinatalar o'qlariga parallel to'g'ri chiziqlar bilan tomonlari Δx_i, Δy_i ga teng bo'lgan to'g'ri to'rtburchaklarga bo'lish mumkin (26-chizma.), bunda $\Delta S_i = \Delta x_i \cdot \Delta y_i$.

Ikki karrali integral ta'rifiga ko'ra:

$$\lim_{\max diam \Delta S_i \to 0} \sum_{i=1}^{n} f(x_i;y_i)\,\Delta x_i \Delta y_i =$$
$$= \iint_{(D)} f(x;y)\,dS = \iint_{(D)} f(x;y)\,dxdy \qquad (3)$$

3.2. Ikki karrali integral va uning xossalari

Oxy tekislikning L silliq (yoki bo'lakli silliq) yopiq chiziq bilan chegaralangan D sohasida $z=f(x,y)$ yoki $z=f(P)$ funksiya berilgan bo'lsin.



27- chizma

Quyidagi ishlarni bajaramiz.

1. D sohani ixtiyoriy ravishda umumiy ichki nuqtalarga ega bo'lmagan va yuzalari $\Delta\sigma_1, \Delta\sigma_2, ..., \Delta\sigma_n$ bo'lgan n ta bo'lakka bo'lamiz.

$\sigma=\sum_{i=1}^{n}\Delta\sigma_i$, bunda σ - D sohaning yuzasi.

2. $\Delta\sigma_i$ yuzalarning har birida $P_i(x_i;y_i)$ nuqtani tanlab, bu nuqtada $z=f(x,y)$ funksiyaning qiymatini hisoblaymiz va uni $\Delta\sigma_i$ ga ko'paytiramiz:

$$f(x_i,y_i)\Delta\sigma_i.$$

3. Barcha shunday ko'paytmalarning yig'indisini tuzamiz:

$$I_n=\sum_{i=1}^{n}f(x_i,y_i)\Delta\sigma_i. \qquad (1)$$

Bu yig'indiga $f(x,y)$ *funksiya uchun* D *sohadagi integral yig'indi* deyiladi.

$\Delta\sigma_i$ yuza chegaraviy nuqtalari orasidagi masofalarning (vatarlarning) eng kattasiga shu *yuzaning diametri* deyiladi va d_i bilan belgilanadi, bunda $n\to\infty$ da $d_i\to 0$.

1-ta'rif. Agar (1) integral yig'indining $\max d_i\to 0$ dagi chekli limiti D sohani bo'laklarga bo'lish usuliga va bu bo'laklarda $P_i(x_i,y_i)$ nuqtani tanlash usuliga bog'liq bo'lmagan holda mavjud bo'lsa, u holda bu limitga $f(x,y)$ funksiyadan D soha bo'yicha olingan ***ikki karrali integral*** deyiladi va $\iint_D f(x,y)d\sigma$ bilan belgilanadi.

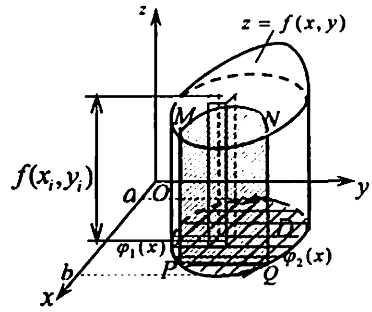
Demak,

$$\iint_D f(x,y)d\sigma=\lim_{\max d_i\to 0}\sum_{i=1}^{n}f(x_i,y_i)\Delta\sigma_i, \qquad (2)$$

bu yerda, D - *integrallash sohasi*, $f(x,y)$ - *integral ostidagi funksiya*, $f(x,y)d\sigma$ - *integral ostidagi ifoda*, x, y - *integrallash o'zgaruvchilari*, $d\sigma$ - *yuza elementi* deb ataladi.

Ikki karrali integralning mavjudlik teoremasi deb ataluvchi teoremani isbotsiz qabul qilamiz.

1-teorema. Chegaralangan yopiq sohada uzluksiz har qanday $z = f(x,y)$ funksiya uchun ikki karrali integral mavjud bo'ladi.



28-chizma

Mavjudlik teoremasidan D sohani istalgan ravishda bo'laklarga, masalan, koordinata o'qlariga parallel chiziqlar bilan tomonlari Δx_i va Δy_i ga teng bo'lgan to'g'ri to'rtburchaklarga bo'lish mumkinligi kelib chiqadi.

Bunday bo'lishda $\Delta\sigma_i = \Delta x_i \cdot \Delta y_i$ ekanidan

$$\iint_D f(x,y)d\sigma = \iint_D f(x,y)dxdy \ .$$

2-ta'rif. Quyidan D soha bilan, yuqoridan tenglamasi $z = f(x,y)$ bo'lgan sirt bo'lagi bilan, yon tomondan Oz o'qqa parallel yasovchilardan tashkil topgan silindrik sirt bilan chegaralangan jism *silindrik jism* deyiladi.

Agar D sohada $f(x,y) \geq 0$ bo'lsa, u holda (2.1) integral yig'indidagi har bir $f(x_i,y_i)\Delta\sigma_i$ qo'shiluvchi asosi $\Delta\sigma_i$ ga va balandligi $f(x_i,y_i)$ ga teng bo'lgan silindrik jism hajmiga teng bo'ladi, ya'ni $\Delta V_i = f(x_i,y_i)\Delta\sigma_i$. Bunda $I_n = \sum_{i=1}^{n} f(x_i,y_i)\Delta\sigma_i$ integral yig'indi ΔV_i hajmlar yig'indisini, boshqacha aytganda zinasimon silindrik jismlar hajmlari yig'indisini aniqlaydi. U holda $f(x,y)$ funksiyadan D soha bo'yicha olingan ikki karrali integral quyidan D soha bilan, yuqoridan tenglamasi $z = f(x,y)$ bo'lgan sirt bo'lagi bilan chegaralangan silindrik jismning V hajmiga teng bo'ladi, ya'ni

$$V = \iint_D f(x,y)dxdy \ . \quad (3)$$

Bu ifoda ikki karrali integralning *geometrik ma'nosi*ni bildiradi.

Agar D sohada $f(x,y) = 1$ bo'lsa, u holda ikki karrali integral D soha yuzasiga teng bo'ladi, ya'ni

$$\sigma = \iint_D dxdy \ . \qquad (4)$$

Agar integral ostidagi funksiya D yassi plastinkada massa taqsimotining zichligi $\gamma(x,y)$ bo'lsa, u holda ikki karrali integral D yassi plastinkaning massasiga teng bo'ladi, ya'ni

$$m = \iint_D \gamma(x,y)dxdy \ . \qquad (5)$$

Bu ifoda ikki karrali integralning *mexanik ma'nosi*ni anglatadi.

Ikki karrali integral aniq integralning hamma xossalariga ega bo'lib, u aniq integralning umumlashmasidir. Ikki karrali integral xossalarining isboti aniq integral xossalarining isboti kabi bajariladi. Shu sababli ikki karrali integralning quyidagi xossalarini isbotsiz keltiramiz.

1°. $\iint\limits_D kf(x,y)d\sigma = k\iint\limits_D f(x,y)d\sigma,\ k\in R.$

2°. $\iint\limits_D (f(x,y)\pm g(x,y))d\sigma = \iint\limits_D f(x,y)d\sigma \pm \iint\limits_D g(x,y)d\sigma.$

3°. Agar D soha umumiy ichki nuqtaga ega bo'lmagan chekli sondagi $D_1, D_2, ..., D_n$ sohalardan tashkil topgan bo'lsa, u holda

$$\iint\limits_D f(x,y)d\sigma = \iint\limits_{D_1} f(x,y)d\sigma + \iint\limits_{D_2} f(x,y)d\sigma + ... + \iint\limits_{D_n} f(x,y)d\sigma .$$

4°. Agar D sohada $f(x,y)\ge 0$ $(f(x,y)\le 0)$ bo'lsa, u holda

$$\iint\limits_D f(x,y)d\sigma \ge 0 \left(\iint\limits_D f(x,y)d\sigma \le 0\right).$$

5°. Agar D sohada $f(x,y)\ge g(x,y)$ $(f(x,y)\le g(x,y))$ bo'lsa, u holda

$$\iint\limits_D f(x,y)d\sigma \ge \iint\limits_D g(x,y)d\sigma \left(\iint\limits_D f(x,y)d\sigma \le \iint\limits_D g(x,y)d\sigma\right).$$

6°. Agar D sohada $f(x,y)$ funksiya uzluksiz bo'lsa, u holda shunday $P_0(x_0;y_0)\in D$ nuqta topiladiki

$$\iint\limits_D f(x,y)d\sigma = f(x_0,y_0)\sigma .$$

Bu xossa *o'rta qiymat haqidagi teorema* deb yuritiladi.

$f(x_0,y_0) = \dfrac{\iint\limits_D f(x,y)d\sigma}{\sigma}$ qiymatga $f(x,y)$ *funksiyaning* D *sohadagi o'rta qiymati* deyiladi.

7°. Agar D sohada $f(x,y)$ funksiya uzluksiz bo'lib, m va M funksiyaning shu sohadagi eng kichik va eng katta qiymatlari bo'lsa, u holda

$$m\sigma \le \iint\limits_D f(x,y)d\sigma \le M\sigma .$$

Bu xossa *integralning chegaralanganligi haqidagi teorema* deb yuritiladi

3.3. Ikki karrali integralning geometrik va mexanik ma'nosi

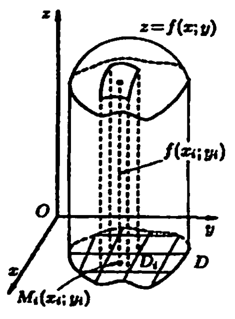
Bu yerda ikkita holni qarab chiqamiz.

3.3.1. Jism hajmini hisoblash

Agar D sohada $f(x,y) \geq 0$ bo'lsa, u holda ikki karrali integral son jihatidan asosi D bo'lgan va yon tarafidan yasovchilari Oz o'qiga parallel bo'lgan va yo'naltiruvchisi D soha chegarasi bo'lgan silindrik sirt, yuqoridan $z = f(x,y)$ sirt bilan chegeralangan *Q silindrik jismning hajmi*ga teng

$$V = \iint_D f(x,y)dxdy \qquad (1)$$

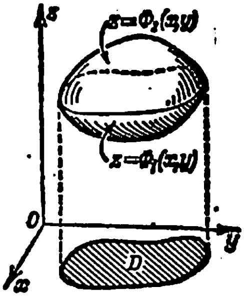
Yuqoridan $z = f(x,y)$ funksiya bilan quyidan $z = 0$ chegaralangan yasovchilari Oz o'qiga parallel bo'lgan silindrik sirt jismning *V hajmi quyidagi formula bilan aniqlanadi (*1.2-chizma*).*



29-chizma.

Xususan, $f(x,y) \equiv 1$, bo'lganda ikki karrali integral D *sohaning* $S(D)$ *yuzi*ga teng, ya'ni

$$S(D) = \iint_D dxdy. \qquad (2)$$



30-chizma.

Agar jismimiz yuqoridan $z = \Phi_2(x,y)$ pastdan $z = \Phi_1(x,y)$ sirtlar bilan chegaralangan bo'lib, bu sirtlarning XOY dagi proyeksiyasi D bo'lsa bu jismning hajmi (1.3-chizma) quyidagi formula bilan hisoblanadi.

$$V = \iint_D [\Phi_2(x,y) - \Phi_1(x,y)]dxdy =$$

$$= \iint_D \Phi_2(x,y)dxdy - \iint_D \Phi_1(x,y)dxdy$$

1-misol. $\boldsymbol{y^2 + z^2 = 4ax}$, $\boldsymbol{y^2 = ax}$ va $x = 3a$ sirtlar bilan chegaralangan jismning hajmi topilsin (silindrdan tashqarida).

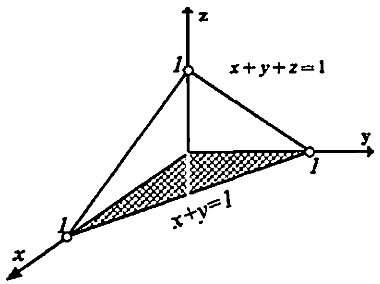
Yechish. 1) $y^2 + z^2 = 4ax$ simmetriya o'qi OX bo'lgan aylanma parabaloid; 2) $y^2 = ax$ yasovchisi XOY o'qiga parallel bo'lib, yo'naltiruvchisi OX tekisligida $y^2 = ax$ paraboladan iborat bo'lgan parabolik silindr; 3) $x = 3a$, OX -o'qidan 3a-ga teng kesma ajratib, $\boldsymbol{YOZ}$ tekisligiga parallel bo'lgan tekisliklar bilan chegaralangan;

Jism XOZ tekisligiga nisbatan simmetrik joylashganligi uchun uning 1-oktantadagi qismining hajmini hisoblab natijani 2 ga ko'paytiramiz. Jism XOY tekisligida $y^2 = 4ax$, $y^2 = ax$ parabolalar va $x = 3a$to'g'ri chiziq bilan chegaralangan OX o'qiga nisbatan simmetrik joylashgan (D) sohani ajratadi.Izlanayotgan hajm quyidagicha hisoblanadi:

$$V = \iint_D \sqrt{4ax - y^2}\, dxdy = 2\int_0^{3a} dx \int_{\sqrt{ax}}^{2\sqrt{ax}} \sqrt{4ax - y^2}\, dy = 3a^3(4\pi - 3\sqrt{3}).$$

2-misol. $x = 0, y = 0, z = 0, x + y + z = 1$ sirtlar bilan chegaralangan jismni hajmini toping.

Yechish. Dekart koordinatalar sistemasida $x = 0, y = 0, z = 0, x + y + z = 1$ sirtlarninng kesishishidan hosil qilingan yopiq sohani aniqlaymiz (31-chizma).



$$V = \int_0^1 dx \int_0^{1-x} (1 - x - y)dy =$$

$$= \int_0^1 \frac{(1-x)^2}{2} dx = \frac{1}{2}\frac{(1-x)^3}{3}\Big|_0^1$$

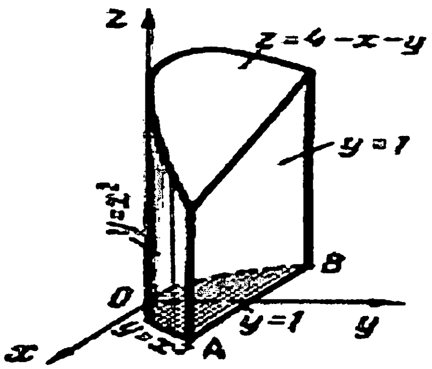
$$= \frac{1}{6}$$

kub birlik.

31-chizma.

3-misol. $y = x^2, y = 1, x + y + z = 4$ va $z = 0$ sirtlar bilan chegeralangan jism hajmini hisoblansin.

Yechilishi. Qaralayotgan jism vertikal silindr hisoblanib (32-chizma) yuqoridan $z = 4 - x - y$ tekislik bilan va quyidan esa XOY tekislikdagi $y = x^2$ parabola va $y = 1$ to‘g‘ri chiziqlar bilan chegaralangan yopiq sohadan iborat. U holda



32-chizma.

$$V = \iint_{OAB} zdxdy = \int_{-1}^{1} dx \int_{x^2}^{1} (4 - x - y)dy$$

$$= \int_{-1}^{1} dx \left(4y - xy - \frac{y^2}{2}\right)\Bigg|_{x^2}^{1} =$$

$$= \int_{-1}^{1} \left(\left(4 - x - \frac{1}{2}\right) - \left(4x^2 - xx^2 - \frac{(x^2)^2}{2}\right)\right) dx$$

$$= \int_{-1}^{1} \left(\frac{7}{2} - x - 4x^2 + x^3 + \frac{x^4}{2}\right) dx = \frac{7}{2}x - \frac{x^2}{2} - 4\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{10}\Big)\Bigg|_{-1}^{1}$$

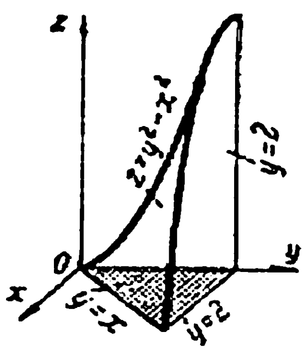
$$= \left(\frac{7}{2} - \frac{1}{2} - 4\frac{1}{3} + \frac{1}{4} + \frac{1}{10}\right) - \left(-\frac{7}{2} - \frac{1}{2} + 4\frac{1}{3} + \frac{1}{4} - \frac{1}{10}\right) = \frac{68}{15}$$

2 –usul tartibini o'zgartirib hajmini hisoblaganda

$V = \iint_{0AB} z dx dy = \int_0^1 dy \int_{-\sqrt{y}}^{\sqrt{y}} (4 - x - y) dx = \int_0^1 \Big[(4 - y)x -$
$\frac{x^2}{2}\Big]\Big|_{x=-\sqrt{y}}^{x=\sqrt{y}} dy == 2\int_0^1 (4 - y)\sqrt{y} dy = \frac{68}{15}.$

4-misol. $z = y^2 - x^2, y = \pm 2$ va $z = 0$ sirtlar bilan chegeralangan jism hajmini hisoblansin.

Yechilishi. Qaralayotgan jism Giperbolik paraboloid $z = y^2 - x^2$ $XOY(z = 0)$ tekislik, $y = \pm 2$ ikkita to'g'ri chiziqlar $XOZvaYOZ$ tekisliklarga simmetrik. Shuning uchun hajmning to'rtdan biri birinchi oktantada joylashgan (33-chizma)



33-chizma.

$$\frac{1}{4}V = \iint_{OAB} zdxdy = \int_0^2 dy \int_0^y (y^2 - x^2)dx = \int_0^2 dy\,(y^2x - \frac{x^3}{3})\Big/_{0}^{y}$$

$$= \int_0^2 (y^3 - \frac{y^3}{3})dy == \frac{2}{3}\int_0^2 y^3 dy = \frac{2}{3}\frac{y^4}{4}\Big|_0^2 = \frac{8}{3}.$$

U holda $V = \frac{32}{3}$.

Ushbu sirtlar bilan chegaralangan hajmlar topilsin.

1. $z = 9 - y^2$silindr va $3x + 4y = 12, \quad y \geq 0$ tekislik

2. $\frac{x^2}{a^2} + \frac{z^2}{a^2} = 1, \quad y = \frac{b}{a}x, \quad y = 0, \quad z = 0, \quad x > 0$

3. $z = 5xy, \quad y^2 = 2x, \quad y^2 = 3x, \quad x^2 = y, \quad x^2 = 2y, \quad z = 0$

4. $z = \frac{x^2+y^2}{4}, \quad x^2 + y^2 = 8y, \quad z = 0$

5. $z = x^2 + y^2 + 1, \quad z = 0, \quad x = 0, \quad y = 0, \quad x = 4, \quad y = 4$

3.3.2.Tekis plastinka massasini hisoblash

Agar D soha modda taqsimotining $\rho(x,y)$ sirt zichligiga ega, *xOy* tekislikda yotuvchi qalinligi bir birlik bo'lgan yassi jism(plastinka) bo'lsa, u holda *yassi jismninig massasi*ni quyidagi formula bilan hisoblanadi:

$$m = \iint_D \rho(x,y)dx\,dy. \tag{4}$$

Ikki karrali integralning **mexanik** ma'nosi shundan iborat.

3.4 Ikki karrali integralning xossalari

1^0. $\iint_{(D)} k \cdot f(x,y)dS = k \cdot \iint_{(D)} f(x;y)dS\,, k \in R,$

2^0.$\iint_{(D)}\big(f(x,y) \pm \varphi(x;y)\big)\,dS = \iint_{(D)} f(x;y)dS \pm \iint_{(D)} \varphi(x;y)dS$

3^0. Agar *D* soha umumiy ichki nuqtaga ega bo'lmagan chekli sondagi $D_1, D_2, \dots, D_n$ sohalardan tashkil topgan bo'lsa, u holda

$$\iint_D f(x,y)dS = \iint_D f(x,y)dS =$$
$$= \iint_{D_1} f(x,y)dS + \iint_{D_2} f(x,y)dS + \ldots + \iint_{D_n} f(x,y)dS$$

4^0. Agar D sohada $f(x,y) \geq 0 (f(x,y) \leq 0)$ bo'lsa, u holda

$$\iint_D f(x;y)dS \geq 0 \left(\iint_D f(x;y)dS \leq 0\right).$$

5^0. Agar D sohada $f(x,y) \geq g(x,y) (f(x,y) \leq g(x,y))$ bo'lsa, u holda

$$\iint_D f(x;y)dS \geq \iint_D g(x;y)dS$$
$$\left(\iint_D f(x;y)dS \leq \iint_D g(x;y)dS\right).$$

6^0. Agar D sohada $f(x,y)$ funksiya uzluksiz bo'lsa, u holda shunday $P_0(x_0;y_0;z_0) \in D$nuqta topiladiki

$$\iint_D f(x;y)dS \geq= f(x_0;y_0) \cdot S.$$

Bunda $f(x_0;y_0) = \frac{1}{S}\iint_D f(x;y)dS$ qiymatga $f(x,y)$ *funksiyaning* D *sohadagi o'rta qiymati* deyiladi.

7^0. Agar D sohada $f(x,y)$ funksiya uzluksiz bo'lsa, u holda

$$m \cdot V \leq \iint_D f(x;y)dS \leq M \cdot S$$

bo'ladi, bu yerda$m$ va M $f(x,y)$ funksiyaning D sohadagi eng kichik va eng katta qiymatlari.

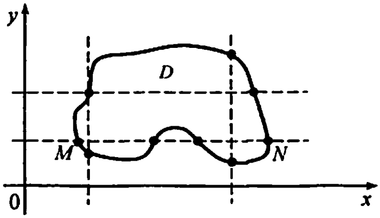
3.5. Ikki karrali integralni dekart koordinatalarida hisoblash

Dekart koordinatalarida $ds = dxdy$ bo'lganligi uchun ikki o'lchovli integral

$$\iint_D f(x,y)ds = \iint_D f(x,y)dxdy$$

Agar $f(x,y) \geq 0$ bo'lib, v – pastdan integrallash sohasi D bilan, yuqoridan D ga proeksiyalanuvchi $z = f(x,y)$ sirtning bo'lagi bilan, yon tomondan esa yasovchilari Oz o'qqa parallel va yo'naltiruvchisi D soha chegarasi L dan iborat silindrik sirt bilan chegaralangan jism hajmi bo'lsin. U holda

$$v = \iint_D f(x,y)dxdy\,.$$



34-chizma.

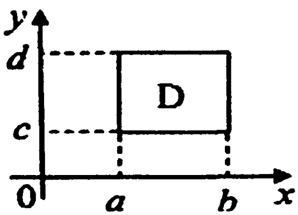
Agar $f(x,y) \equiv 1$ bo‘lsa, u holda ikki o‘lchovli integralning qiymati son jihatdan integrallash sohasi D ning S s yuziga teng bo‘ladi, ya’ni

$$\iint_D dxdy = \iint_D ds = S$$

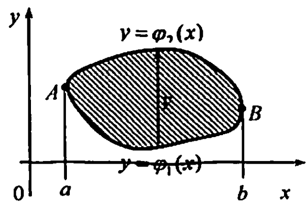
Agar $f(x,y)$ funksiya D sohaga joylashgan plastinka massasi taqsimlanishining zichligini ifodalasa, u holda ikki o‘lchovli integral shu plastinka moddasining massasi M ni beradi:

$$M = \iint_D f(x,y)dxdy = \iint_D f(x,y)ds.$$

Ikki o‘lchovli integralni hisoblash ikkita aniq integralni ketma-ket hisoblashga keltiradi.



35-chizma.



36- chizma.

Hisoblash usullari.

1)Agar D soha 35- chizma ko‘rsatilgandek $x = a, y = c, x = b, y = d$ chiziqlar bilan faqat bitta nuqtada kesishsa, u holda ikki o‘lchovli integralni hisoblashda yuqorida keltirilgan har ikkala formuladan foydalanish mumkin bo‘lib,

$$\iint_D f(x,y)dxdx = \int_a^b dx \int_c^d f(x,y)dy = \int_c^d dy \int_a^b f(x,y)dx$$

tenglik o'rinli bo'ladi.

2) Agar D integrallash sohasi Oy o'qiga nisbatan muntazam bo'lsa (36- chizma), ya'ni integrallash sohasi yon tarafdan $x = a$ va $x = b$ to'g'ri chiziqlar, quyidan va yuqoridan har biri vertikal chiziqlar bilan faqat bitta nuqtada kesishadigan $y = y_1(x)$ va $y = y_2(x)(y_1(x) \le y_2(x))$ uzluksiz chiziqlar bilan chegaralangan soha bo'lsa, ikki o'lchovli integral quyidagicha hisoblanadi:

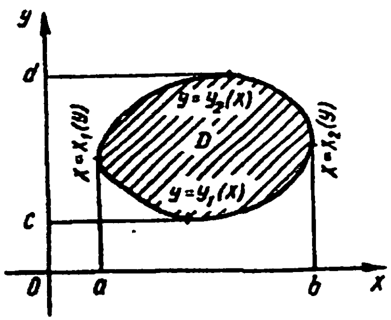
$$\iint_D f(x,y)\,dx\,dy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy. \qquad (1)$$

Avval ichki integralda x o'zgaruvchini o'zgarmas kattalik sifatida qabul qilib integral hisoblanadi. (1) integralning qiymati o'zgarmas son bo'ladi.

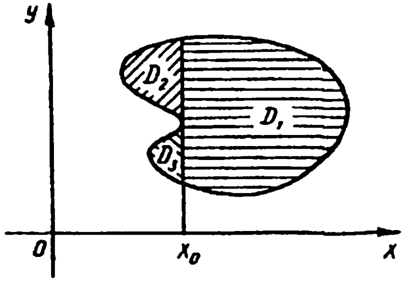
3) Agar D integrallash sohasi Ox o'qiga nisbatan muntazam bo'lsa(37-chizma), ya'ni integrallash sohasi quyidan va yuqoridan $y = c$ va $y = d$ to'g'ri chiziqlar, yon tarafdan har biri vertikal chiziqlar bilan faqat bitta nuqtada kesishadigan $x = x_1(y)$ va $x = x_2(y)(x_1(y) \le x_2(y))$ uzluksiz chiziqlar bilan chegaralangan soha bo'lsa, ikki o'lchovli integral quyidagicha hisoblanadi:

$$\iint_D f(x,y)\,dx\,dy = \int_c^d dy \int_{x_1(y)}^{x_2(y)} f(x,y)\,dx. \qquad (2)$$

Avval ichki integralda y o'zgaruvchini o'zgarmas kattalik sifatida qabul qilib integral hisobalanadi.



37-chizma.



38-chizma.

4) Agar integrallash sohasi 38- chizma ko'rsatilgandagidek konturga ega bo'lsa, u holda ikki o'lchovli integralni hisoblah uchun soha $x = x_0$ chiziq bilan bo'laklarga bo'linib, yuqoridagi formulalardan foydalaniladi.

D integrallash sohasi *Oy* o'qiga nisbatan ham, *Ox* o'qiga nisbatan ham muntazam bo'lmasa, u holda uni *Oy* (yoki *Ox*) o'qiga nisbatan muntazam bo'lgan chekli sondagi $D_1, D_2, ..., D_m$ sohalarga bo'linadi va ikki karrali integralni *D* soha bo'yicha hisoblashda *additivlik* xossasidan foydalaniladi.

Yuqoridagi formulalarda o'ng qismi ***ikki karrali integral*** (yoki ***takroriy integral***) deyiladi. Ikki o'lchovli integralni ***ikki karrali integral*** deb aytiladi.

Eslatma. Agar *D* ni koordinat o'qlariga parallel to'g'ri chiziqlar ikki nuqtadan ortiq nuqtada kessa u holda *D* ni *murakkab soha* deyiladi. Agar *D* murakkab soha bo'lsa uni bir necha to'g'ri sohalarga ajratib olinib, shu to'g'ri sohalar bo'yicha olingan integralarning yig'indisi shu *D* soha bo'yicha olingan integralni beradi.

1-misol. Ikki karrali integralni hisoblang:

$$\int_1^2 \int_0^1 \frac{1}{(x+2y)^3} dxdy.$$

Yechilishi. Integrallash chegaralari o'zgarmas bo'lganligi sababli ichki integralni istalgan o'zgaruvchi bo'yicha hisoblash mumkin. Integralni quyidagicha yozib olamiz:

$$\int_1^2 dx \int_0^1 \frac{1}{(x+2y)^3} dy.$$

x ni o'zgarmas deb, ichki integralni y bo'yicha hisoblaymiz:

$$\int_0^1 \frac{dy}{(x+2y)^3} = \frac{1}{2}\int_0^1 (x+2y)^{-3} d_y(x+2y) = \frac{1}{2}\cdot\frac{(x+2y)^{-2}}{-2}\Big|_{y=1}^{y=0} =$$

$$= -\frac{1}{4}((x+2)^{-2} - x^{-2}) = -\frac{1}{4}\left(\frac{1}{(x+2)^2} - \frac{1}{x^2}\right).$$

Endi tashqi integralni x bo'yicha hisoblaymiz:

$$-\frac{1}{4}.\int_1^2 \left(\frac{1}{(x+2)^2} - \frac{1}{x^2}\right) dx = -\frac{1}{4}.-\frac{1}{(x+2)} + \frac{1}{x}\Big|_1^2 =$$

$$= -\frac{1}{4}.\left(-\left(\frac{1}{4} - \frac{1}{3}\right) + \left(\frac{1}{2} - 1\right)\right) = \frac{5}{48}.$$

2-misol. Ikki karrali integralni hisoblang:

$z = x + 3y^2$ bunda $D: \{0 \le x \le 1, 0 \le y \le 1\}$.

Yechilishi.

$$\iint_D (x+3y^2)dxdy = \int_0^1 dy \int_0^1 (x+3y^2)dx = \int_0^1 \left(\frac{x^2}{2} + 3y^2x\right)\Big|_{x=1}^{x=0} dy$$

$$== \int_0^1 \left(\frac{1}{2} + 3y^2\right) dy = \left(\frac{1}{2}y + y^3\right)\Big|_1^0 = \frac{3}{2}.$$

3-misol. $\iint_D x \ln y \, dxdy$ integralni hisoblang, bu yerda D: $0 \le x \le 4, 1 \le y \le e$ to'g'ri to'rtburchak bo'lgan soha.

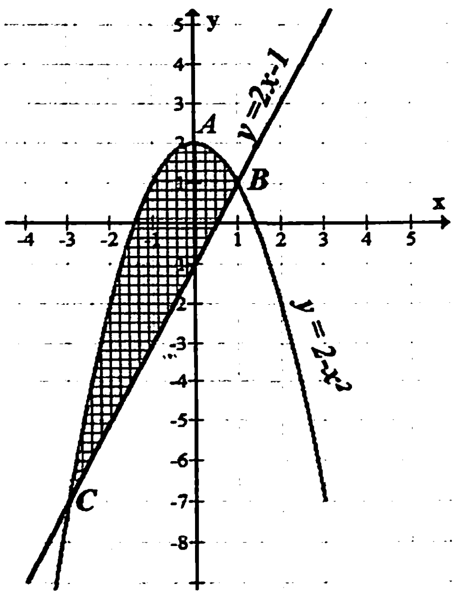
Yechish. (1) formulaga asosan,

$$\iint_D x\ln y\,dxdy = \int_0^4 xdx\int_1^e \ln y\,dy = \int_0^4 xdx[y\ln y - y]_1^e = 8.$$

4-misol. Ikki o‘lchovli integralni hisoblang:

$\iint_D (x-y)dxdy$, bu yerda D soha $y = 2 - x^2, y = 2x - 1$ chiziqlar bilan chegaralangan.

Yechish. D sohani hosil qilamiz. Uchi $A(0,2)$ nuqtada bo‘lgan



39-chizma.

$y = 2 - x^2$ parabola OY o‘qiga nisbatan simmetrik bo‘ib, $y = 2x - 1$ to‘g‘ri chiziq bilan ikkita $B(1,1)$ va $C(-3,-7)$ nuqtalarda kesishadi (1.12-chizma). Integrallash sohasi D ushbu tengsizliklar sistemasi bilan aniqlanadi:

$$\begin{cases} -3 \le x \le 1, \\ 2x - 1 \le y \le 2 - x^2. \end{cases}$$

Ikki o‘lchovli integralni hisoblaymiz:

$$\iint_D (x-y)dxdy = \int_{-3}^{1} dx\int_{2x-1}^{2-x^2}(x-y)dy = \int_{-3}^{1}\left[xy - \frac{y^2}{2}\right]_{2x-1}^{2-x^2} dx =$$

$$= \int_{-3}^{1}\left[\left(x\cdot(2-x^2) - \frac{(2-x^2)^2}{2}\right) - \left(x\cdot(2x-1) - \frac{(2x-1)^2}{2}\right)\right]dx =$$

$$\int_{-3}^{1}\left(2x - x^3 - \frac{4-4x^2+x^4}{2} - 2x^2 + x + \frac{4x^2-4x+1}{2}\right)dx =$$

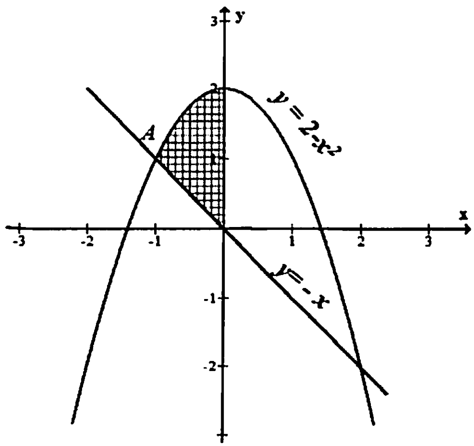
$$= \int_{-3}^{1}\left(-\frac{1}{2}x^4 - x^3 + 2x^2 + x - \frac{3}{2}\right)dx =$$

$$= \left[-\frac{1}{10}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 - \frac{3}{2}x\right]_{-3}^{1} = 4\frac{4}{15}.$$

5- misol. Ikki o'lchovli integralni hisoblang:

$\iint_D xdxdy$, bu yerda D soha $x = 0, y = -x, y = 2 - x^2$ chiziqlar bilan chegaralangan.

Yechilishi. ***D*** sohani quramiz. $\begin{cases} y = -x, \\ y = 2 - x^2, \end{cases}$ tenglamalar sistemasidan chiziqlarning kesishish nuqtalarini aniqlaymiz $-x = 2 - x^2, x^2 - x - 2 = 0$, $D = 1 + 8 = 9$, $x_{1,2} = \frac{1 \pm 3}{2}$ $\Rightarrow$ $x_1 = 2$, $x_2 = -1$. Xuddi shunday, $A(-1;1)$ nuqta qaralayotgan sohada chiziqlarning kesishish nuqtasi hisoblanadi.



40-chizma.

D soha chapdan $x = -1$, va o'ngdan $x = 0$ to'g'ri chiziqlar bilan, quyidan $y = -x$ to'g'ri chiziq bilan va yuqoridan $y = 2 - x^2$ parabola bilan chegaralangan (40-chizma). Shu sababli integralni quyidagicha hisoblaymiz:

$$\iint_D x\,dxdy = \int_{-1}^{0} dx \int_{-x}^{2-x^2} x\,dy = \int_{-1}^{0} xy\Big|_{-x}^{2-x^2} dx = \int_{-1}^{0} x\left(2 - x^2 - (-x)\right) dx =$$

$$= \int_{-1}^{0} x(2 - x^2 + x)dx = \int_{-1}^{0}(2x - x^3 + x^2)dx = \left(\frac{2x^2}{2} - \frac{x^4}{4} + \frac{x^3}{3}\right)\Big|_{-1}^{0} =$$

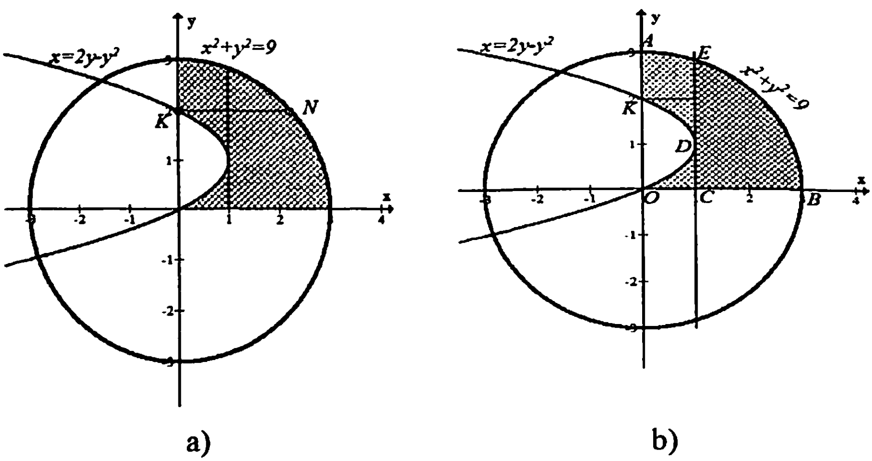
$$= 0 - \left((-1)^2 - \frac{(-1)^4}{4} + \frac{(-1)^3}{3}\right) = -\left(1 - \frac{1}{4} - \frac{1}{3}\right) = -\frac{5}{12}.$$

Agar tashqi integralni y o'zgaruvchi bo'yicha olib boradigan bo'lsak, u holda D soha $y = 1$ to'g'ri chiziq bilan ikkita to'g'ri muntazam sohalarga ajratish zarur va berilgan integral ikkita takroriy integrallar yig'indisi shaklida ifodalanadi.

6-misol. Ikki o'lchovli $\iint_D f(x,y)dxdy$ integralning integrallash sohasi $D: x = 2y - y^2, x^2 + y^2 = 9, y = 0 va x \geq 0$ chiziqlar bilan chegaralangan. Ichki va tashqi integrallarning integrallash chegaralarini aniqlang.

Yechilishi. Qaralayotgan soha hisoblash usullariga ko'ra nomuntazam. Shuning uchun sohani KN to'g'ri chiziq orqali ikki qismga ajratiladi (41-chizma).

$$\iint_D f(x,y)dxdy = \int_0^2 dy \int_{2y-y^2}^{\sqrt{9-y^2}} f(x,y)dx + \int_2^3 dy \int_0^{\sqrt{9-y^2}} f(x,y)dx$$



41-chizma.

Ikkinchi punkti bo'yicha qaralganda ham nomuntazam soha. *EC* to'g'ri chiziq orqali uchta sohaga ajratiladi: *ODC, AEDK* va *CBE*. U holda

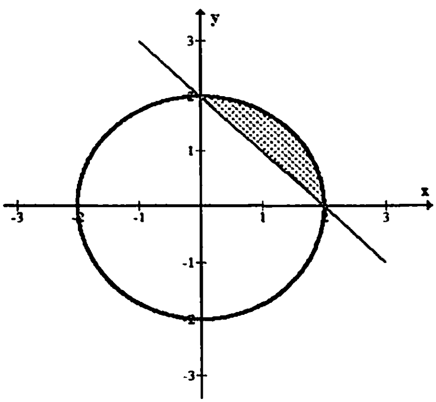
$$\iint_D f(x,y)dxdy = \int_0^1 dx \int_0^{1-\sqrt{1-x}} f(x,y)dy +$$

$$+\int_0^1 dx \int_{1-\sqrt{1-x}}^{\sqrt{9-y^2}} f(x,y)dx + \int_1^3 dx \int_0^{\sqrt{9-y^2}} f(x,y)dx$$

7-misol. Ikki o'lchovli integralni hisoblang:

$\iint_D xy^2dxdy$, bu yerda$D$ soha $x^2+y^2\le 4, x+y-2\ge 0$ chiziqlar bilan chegaralangan.

Yechilishi. *D* sohani chizamiz (42-chizma).



42-chizma.

$$\iint_D xy^2dxdy = \int_0^2 dy \int_{2-y}^{\sqrt{4-y^2}} xy^2dx = \int_0^2 y^2dy \cdot \left(\frac{x^2}{2}\right)\Bigg|_{2-y}^{\sqrt{4-y^2}} =$$

$$= \frac{1}{2}\int_0^2 y^2dy(4-y^2-(2-y)^2)$$

$$= \frac{1}{2}\int_0^2 (4y^2-y^4-4y^2+4y^3-y^4)dy =$$

$$= \frac{1}{2}\int_0^2 y^2 dy(4 - y^2 - (2-y)^2) =$$

$$= \frac{1}{2}\int_0^2 (4y^2 - y^4 - 4y^2 + 4y^3 - y^4)dy =$$

$$= \frac{1}{2}\int_0^2 (4y^3 - 2y^4)dy = \frac{1}{2}\left(y^4 - \frac{2y^5}{5}\right)\Big|_0^2 = \frac{1}{2}\cdot\frac{16}{5} = \frac{8}{5}.$$

8-misol. $I = \int_0^{1/2} dx \int_0^{\sqrt{1-4x^2}} (x+y)dy$ karrali integral hisoblansin.

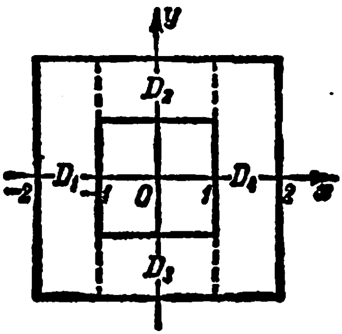
Yechish. Integrallash sohasi $\sigma: x = 0, x = 1/2, y = 0$ to‘g‘ri chiziqlar va $y = \sqrt{1-4x^2}$ egri chiziq (yarim ellips) bilan chegaralangan OY o‘qiga nisbatan simmetrik sohadir.

$$I = \int_0^{1/2} dx \int_0^{\sqrt{1-4x^2}} (x+y)dy = \int_0^{1/2} (xy + \frac{y^2}{2}) \Big|_0^{\sqrt{1-4x^2}} dx =$$

$$= \int_0^{1/2} (x\sqrt{1-4x^2} + \frac{1}{2} - 2x^2)dx = \frac{1}{4}.$$

9-Misol. Ikki karrali integralni hisoblang: $\iint_D e^{x+y} dS$ $D -$ soha markazi koordinatalar boshida, tomonlari koordinatalar o‘qiga parallel hamda tomonlar mos ravishda 2 va 4ga teng bo‘lgan kvadratlar orasida joylashgan soha.

Yechilishi.



43-chizma.

D –soha nomuntazam. $x = -1$ va $x = 1$ to‘g’ri chiziqlar to‘rtta muntazam D_1, D_2, D_3, D_4 sohalarga ajratadi (43-chizma). Shundan

$$\iint_D e^{x+y}dS = \iint_{D_1} e^{x+y}dS + \iint_{D_2} e^{x+y}dS +$$

$$+\iint_{D_3} e^{x+y}dS + \iint_{D_3} e^{x+y}dS$$

Hosil qilingan sohalarning chegaralarini qo‘yib ikki karrali integralni hisoblaymiz:

$$\iint_D e^{x+y}dS = \int_{-2}^{-1}\left(\int_{-2}^{2} e^{x+y}dy\right)dx + \int_{-1}^{-1}\left(\int_{1}^{2} e^{x+y}dy\right)dx +$$

$$+\int_{-1}^{-1}\left(\int_{-2}^{-1} e^{x+y}dy\right)dx + \int_{1}^{2}\left(\int_{-2}^{2} e^{x+y}dy\right)dx =$$

$$= (e^2 - e^{-2})(e^{-1} - e^{-2}) + (e^2 - e^1)(e^1 - e^{-1}) +$$

$$+(e^{-1} - e^{-2})(e^1 - e^{-1}) + (e^2 - e^{-2})(e^2 - e^1) =$$

$$= (e^3 - e^{-3})(e^1 - e^{-1}) = 4 \cdot sh3 \cdot sh1.$$

1. Ushbu ikki o‘lchovli integrallar hisoblansin.

1. $\int_0^1 dx \int_0^1 (x^2 + y^2)dy$

2. $\int_0^1 dx \int_x^1 (x + y)dy$

3. $\int_0^2 x^2 dx \int_0^{\frac{x}{2}} y dy$

4. $\int_0^a dx \int_0^{\sqrt{x}} dy$

5. $\int_2^4 dx \int_x^{2x} \frac{y}{x} dy$

6. $\iint_D x^3 y^2 dxdy$, D –aylana $x^2 + y^2 \le R^2$

7. $\iint_D xydxdy$, D –soha $0 \le x \le 1$, $0 \le y \le 2$

8. $\iint_D e^{x+y} dxdy$, D –soha $0 \le x \le 1$, $0 \le y \le 1$

3.6. Ikki karrali integralda o'zgaruvchilarni almashtirish

$\iint\limits_D f(x,y)\,dx\,dy$ ikki o'lchovli integralda x, y to'g'ri burchakli koordinatalar x, y bilan quyidagicha munosabatlar orqali bog'langan yangi u, v koordinatalarga o'tkaziladi

$$x = x(u,v), \quad y = y(u,v). \tag{1}$$

Agar D va D^* sohalar (44-chizma) o'rtasida (1) munosabatlar orqali o'zaro bir qiymatli akslantirish o'rnatilgan bo'lsa, shu bilan birga akslantirish yakobiani

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \ne 0$$

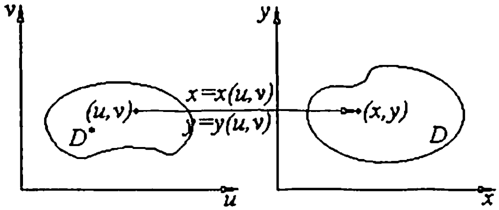
bo'lsa, quyidagi formula o'rinlidir:

$$\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_{D^*} f(x(u,v), y(u,v)) |J(u,v)|\,du\,dv. \tag{2}$$

(2) formulada $J(u,v)$ yakobianni hisoblashda

$$J(u,v)=\frac{1}{J(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial x}\end{vmatrix}^{-1} \qquad (3)$$

tengliklar bilan ifodalangan formulasidan foydalanish ham mumkin.



44-chizma.

1-misol. Ikki karrali integralni hisoblang: $\iint_D (x+y)^3\,(x-y)^2 dxdy$, bu yerda Dsoha $x+y=1, x-y=1, x+y=3, x-y=-1$ to‘g‘ri chiziqlar bilan chegaralangan kvadratdir.

Yechish.▶ $x+y=u, x-y=v$ almashtirishni bajaramiz, bundan $x=\frac{1}{2}(u+v), y=\frac{1}{2}(u-v)$. U holda almashtirishning Yakobiani

$$J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{-1}{2}\end{vmatrix}=-\frac{1}{2}.$$

Demak, $|J|=\frac{1}{2}$.

Bundan, $\iint_D (x+y)^3\,(x-y)^2 dxdy=\frac{1}{2}\iint_{D^*} u^3v^2 dudv.$

D^* soha $u=1; u=3, v=-1, v=1$ chiziqlar bilan chegaralangan kvadrat bo‘lgani uchun,

$$\iint_D (x+y)^3\,(x-y)^2 dxdy=\frac{1}{2}\int_1^3 u^3du\int_{-1}^1 v^2dv=$$

$$\frac{1}{2}\int_1^3 u^3\left[\frac{1}{3}v^3\right]_{-1}^1 du=\frac{1}{6}\int_1^3 u^3(1+1)du=\frac{1}{12}u^4\Big|_1^3=\frac{20}{3}.$$ ◀

3.7. Ikki karrali integralni qutb koordinatalarida hisoblash

Ma'lumki, to'g'ri burchakli x, y va qutb ρ, φ koordinatalar o'zaro

$$\begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi \end{cases}$$

munosabatlar bilan bog'langan. Bu yerda $r \geq 0, 0 \leq \varphi \leq 2\pi$.

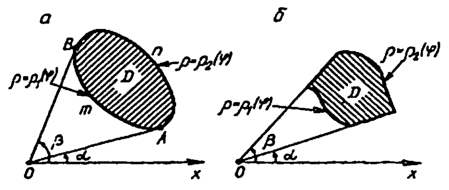
Ikki karrali integralda to'g'ri burchakli koordinatalardan qutb koordinatalarga o'tish quyidagi formula orqali amalga oshiriladi:

$$\iint_D f(x,y)dxdy = \iint_{\bar{D}} f(r\cos\varphi, r\sin\varphi)\rho d\rho d\varphi. \qquad (4)$$

Integrallash chegaralari O qutbning vaziyatiga bog'liq bo'ladi.

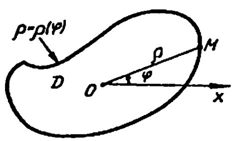
a) Agar O qutb $\varphi = \alpha$ va $\varphi = \beta (\alpha < \beta)$ nurlar, hamda $r = r_1(\varphi)$ va $r = r_2(\varphi)(r_1(\varphi) < r_2(\varphi))$ chiziqlar bilan chegaralangan D soha tashqarisida yotsa (45-chizma), shuningdek, $\varphi = \alpha$ va $\varphi = \beta (\alpha < \beta)$ nurlar soha chegarasini ikki nuqtada kesib o'tsa, ikki karrali integral quyidagi formula bilan hisoblanadi:

$$\iint_D f(x,y)dxdy = \int_\alpha^\beta d\varphi \int_{r_1(\varphi)}^{r(\varphi)} f(r\cos\varphi, r\sin\varphi)rdr. \qquad (5)$$



45-chizma.

b) Agar O qutb D soha ichida joylashgan bo'lsa va bu soha chegarasi qutb

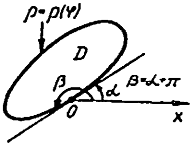


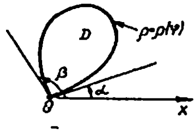
koordinatalar sistemasida $r = r(\varphi)$ ko'rinishiga ega bo'lsa (46-chizma), u holda

46-chizma. ikki karrali integral quyidagi formula bilan hisoblanadi

$$\iint_D f(x,y)dxdy = \int_0^{2\pi} d\varphi \int_0^{r(\varphi)} f(r\cos\varphi, r\sin\varphi)rdr. \qquad (6)$$

c) Agar O qutb $\varphi = \alpha$ va $\varphi = \beta(\alpha < \beta)$ nurlar bilan chegaralangan D soha chegarasida yotsa, shu bilan birga, chegaraning qutb koordinatalar sistemasida tenglamasi $r = r(\varphi)$ ko‘rinishiga ega bo‘lsa





47-chizma.

(47-chizma), u holda ikki karrali integral quydagi formula bilan hisoblanadi:

$$\iint_D f(x,y)dxdy = \int_\alpha^\beta d\varphi \int_0^{r(\varphi)} f(r\cos\varphi, r\sin\varphi)rdr. \qquad (7)$$

1-misol. $\iint_D f(x^2+y^2)\,dxdy$ integralda, qutb koordinatalariga o‘tib, integral chegarasini qo‘ying. Bu yerda

$$D: \{x^2+y^2 = x\sqrt{6}, (x^2+y^2)^2 = 9(x^2-y^2), y = 0(y \geq 0, x \leq \sqrt{6})\}$$

Yechish. ► $x^2+y^2 = x\sqrt{6} \Rightarrow r^2 = r\sqrt{6}\cos\varphi \Rightarrow r = \sqrt{6}\cos\varphi$

$$(x^2+y^2)^2 = 9(x^2-y^2) \Rightarrow r^4 = 9r^2\cos 2\varphi \Rightarrow$$

$$\Rightarrow r^2 = 9\cos 2\varphi \Rightarrow r = 3\sqrt{\cos 2\varphi}$$

Kesishish nuqtalarini topamiz:

$$\begin{cases} r = \sqrt{6}\cos\varphi, \\ r^2 = 9\cos 2\varphi \end{cases} \Rightarrow 9\cos 2\varphi = 6\cos^2\varphi,$$

$$9(2\cos^2\varphi - 1) = 6\cos^2\varphi,$$

$$12\cos^2\varphi = 9, \cos\varphi = \pm\frac{\sqrt{3}}{2}$$

Bundan, $y \geq 0$ bo'lgani uchun, quyidagiga ega bo'lamiz: $\varphi = \frac{\pi}{6}$.

Demak, (7) ga ko'ra,

$$\iint_D f(x^2 + y^2)\, dxdy$$

$$= \int_0^{\frac{\pi}{6}} d\varphi \int_0^{\sqrt{6}\cos\varphi} rf(r^2)dr + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} d\varphi \int_0^{3\sqrt{\cos 2\varphi}} rf(r^2)dr$$

2-misol. Berilgan $\int_0^a dy \int_{\sqrt{ay-y^2}}^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\, dx$ integralni qutb koordinatalar sistemasiga o'tib hisoblang.

Yechish. ► $x = r\cos\varphi, y = r\sin\varphi, J = r$ dan foydalanamiz.

$$\int_0^a dy \int_{\sqrt{ay-y^2}}^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\, dx =$$

$$=\cdot\left[\begin{array}{c} x = \sqrt{a^2 - y^2} \Rightarrow x^2 + y^2 = a^2 \Rightarrow r = a \\ x = \sqrt{ay - y^2} \Rightarrow x^2 + y^2 = ay \Rightarrow \\ r^2 = ar\sin\varphi \Rightarrow r = a\sin\varphi \end{array}\right] =$$

$$= \int_0^{\frac{\pi}{2}} d\varphi \int_{a\sin\varphi}^a r\sqrt{a^2 - r^2}\, dr = -\frac{1}{2}\cdot\frac{2}{3}\int_0^{\frac{\pi}{2}} \left(\sqrt{a^2 - r^2}\right)^3 \Big|_{a\sin\varphi}^{a} d\varphi = =$$

$$\frac{a^3}{3}\int_0^{\frac{\pi}{2}} \left(\sqrt{1 - \sin^2\varphi}\right)^3 d\varphi = \frac{a^3}{3}\int_0^{\frac{\pi}{2}} \cos^3\varphi\, d\varphi = \frac{a^3}{3}\int_0^{\frac{\pi}{2}}(1 -$$

$$\sin^2\varphi)\, d(\sin\varphi) == \frac{a^3}{3}\left(\sin\varphi - \frac{\sin^3\varphi}{3}\right)\Big|_0^{\frac{\pi}{2}} = \frac{2a^3}{9}. ◄$$

3-misol. $I = \iint_S \frac{dxdy}{\sqrt{x^2+y^2}}$ integral hisoblansin.

Ikki o'lchovli integral qutb koordinatalari φ va r larga o'tib hisoblansin, bu yerda S –radiusi $R = 1$ bo'lib, markazi $O(0,0)$ nuqtada bo'lgan aylananing birinchi choragi .

Yechish. $\sqrt{x^2+y^2}=r$ fomulani qo'llab $I=\iint_S \frac{rdrd\phi}{r}=\iint_S d\phi dr$ ni chiqaramiz. S –soha $0\le\varphi\le\frac{\pi}{2}$, $0\le r\le 1$ tengliklar bilan aniqlanadi.

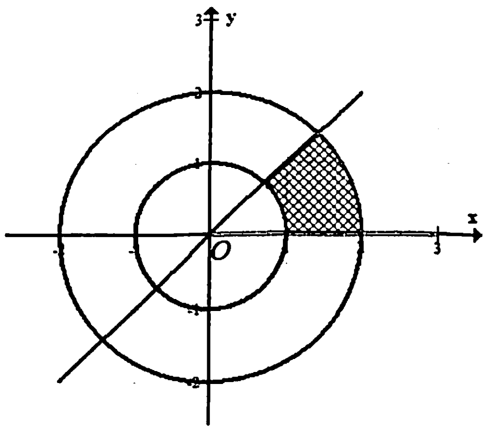
Shuning uchun formulaga asosan $I=\int_0^{\frac{\pi}{2}}d\varphi\int_0^1 dr=\frac{\pi}{2}\cdot 1=\frac{\pi}{2}$.

4-misol. $I=\iint_S(x^2+y^2+1)dxdy$ integral qutb kordinatalariga o'tib hisoblansin. integrallash sohasi S markazlari koordinatalar boshida, radiuslari esa $R=1$ va $R=2$ bo'lgan konsentrik aylanalar va $y=0, y=x$ to'g'ri chiziqlar bilan chegaralanib, birinchi chorakda yotgan qismi.

Yechish. $x^2+y^2=r^2, dxdy=rdrd\varphi$ formulaga asosan,

$I=\iint_S(r^2+1)rdrd\phi$, bu yerda S soha $0\le\varphi\le\frac{\pi}{4}, 1\le r\le 2$ tengsizliklar bilan aniqlanadi. (48-chizma).

Demak (7) formulaga asosan

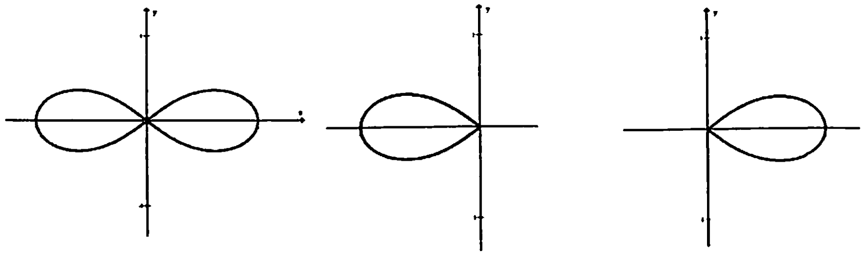


48-chizma

$$I=\int_0^{\frac{\pi}{4}}d\varphi\int_1^2(r^3+r)dr=\int_0^{\frac{\pi}{4}}\left(\frac{r^4}{4}+\frac{r^2}{2}\right)\Bigg|_1^2 d\varphi=\int_0^{\frac{\pi}{4}}\frac{21}{4}d\varphi=\frac{21\pi}{16}.$$

5-misol. $I=\iint_S f(x,y)dxdy$ integralda qutb koordinatalari r va φ larga o'tib, ularning chegaralari qo'yilsin, bu yerda S lemniskata $(x^2+y^2)^2=a^2(x^2-y^2)$

Yechish. $x = r\cos\varphi, y = \sin\varphi$ formuladan foydalansak bu holda lemniskata tenglamasi $r^2 = a^2\cos 2\varphi$ ko‘rinishga keladi. $dxdy = rdrd\varphi$



a) b)$\frac{3\pi}{4} \le \varphi \le \frac{5\pi}{4}$, c)$-\frac{\pi}{4} \le \varphi \le \frac{\pi}{4}$,

49-chizma.

ni hisobga olsak (49-chizma),

$$I = \iint_S f(r\cos\varphi, r\sin\varphi)rdrd\varphi$$

bu yerda $S = S_1 + S_2$ bo‘lib,

S_1 soha $\frac{3\pi}{4} \le \varphi \le \frac{5\pi}{4}$, $0 \le r \le a\sqrt{\cos 2\varphi}$,

S_2 soha $-\frac{\pi}{4} \le \varphi \le \frac{\pi}{4}$, $0 \le r \le a\sqrt{\cos 2\varphi}$

tengsizliklar bilan aniqlanadi.

Demak, $I_1 = \iint_{S_1} F(r,\varphi)rdrd\varphi$-$I_2 = \iint_{S_2} F(r,\varphi)rdrd\varphi =$

$$I_1 = \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} d\varphi \int_0^{a\sqrt{\cos 2\varphi}} F(r,\varphi)rdr + I_2 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int_0^{a\sqrt{\cos 2\varphi}} F(r,\varphi)rdr$$

bu yerda, $f(r\cos\varphi, r\sin\varphi) = F(r,\varphi)$ deb belgilanadi.

6-misol. $\iint_D \sqrt{4 - x^2 - y^2}dxdy$, $D: x^2 + y^2 = 4$ aylanadan iborat.

Yechish. Qutb koordinatalariga o'tadigan bo'lsak,

$$\begin{cases} x = \rho\cos\varphi \\ x = \rho\sin\varphi \end{cases} \Rightarrow \begin{matrix} \rho^2 = x^2 + y^2, \rho = 2, \\ 0 \le \rho \le 2 \quad , 0 \le \varphi \le 2\pi \end{matrix}$$

$$\int_0^2 \int_0^{2\pi} \sqrt{4 - \rho^2\cos^2\varphi - \rho^2\sin^2\varphi}\,\rho d\rho d\varphi = \int_0^2 \rho d\rho \int_0^{2\pi} \sqrt{4-\rho^2}\,d\varphi$$

$$= 2\pi\int_0^2 \sqrt{4-\rho^2}\,\rho d\rho = \left| \begin{matrix} 4-\rho^2 = t^2 \\ -2\rho d\rho = 2tdt \\ \rho = 0 da t = 2 \\ \rho = 2 da t = 0 \end{matrix} \right|$$

$$= -2\pi\int_2^0 t^2 dt = -2\pi\frac{t^3}{3}|_2^0 = \frac{16}{3}\pi.$$

Ushbu ikki o'lchovli integrallarda o'zgaruvchilarni almashtirib integrallang.

1. $\iint_{(P)} xydxdy,$ bunda $P -$ soha $x^2 = y,\ x^2 = 2y,\ \ y^2 = x,\ \ y^2 = 2x$ chiziqlar bilan chegaralangan.

2. $\iint_{(P)} dxdy,$ bunda $P -$ soha $xy = 1,\ xy = 4,\ \ x - 2y = 2,$ $x - 2y + 1 = 0,\ \ x > 0,\ \ y > 0.$

3. $\iint_{(P)}(x-2y)dxdy,$ bunda $P -$ soha $xy = 1,\ xy = 4,\ \ x - 2y = 2,\ \ \ x - 2y + 1 = 0,\ x > 0,\ y > 0.$

4. $\iint_{(P)} f(x,y)dxdy,$ bunda $P -$ soha $x^2 + y^2 \le R^2.$

5. $\iint_{(P)}(x^2+y^2)dxdy,$ bunda $P -$ soha $x^2 + y^2 = 2ax.$

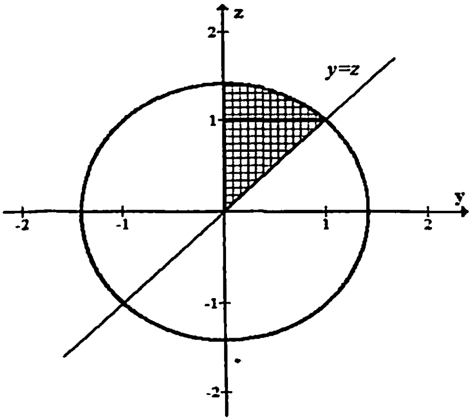
3.8. Integralda integrallash tartibini almashtirish

Integrallash chegaralarini tashqi va ichki integrallar uchun almashtirish integrallash *tartibini o'zgartirish* deyiladi. Ya'ni integralni (1) shakldan (2) ga o'tkazish va aksincha, (2) dan (1) ga almashtirish integrallash tartibini almashtirish deb ataladi.

1-misol. Integralda integrallash tartibini almashtiring:

$$\int_0^1 dy \int_y^{\sqrt{2-y^2}} f(y,z)dz.$$

Yechilishi. D soha yOz tekisligida $y = 0$ va $y = 1$ to'g'ri chiziqlar orasida joylashgan va uning quyi chegarasi $z = y$ yuqori chegarasi esa $z = \sqrt{2-y^2}$. D sohani Oz o'qiga proyeksiyalaymiz. Natijada $[0,\sqrt{2}]$ kesma hosil bo'ladi. D sohaning chap qismi esa $y = 0$ to'g'ri chiziq, o'ng qismida esa $[0,1]$ $y = z$ to'g'ri chiziq,



50-chizma.

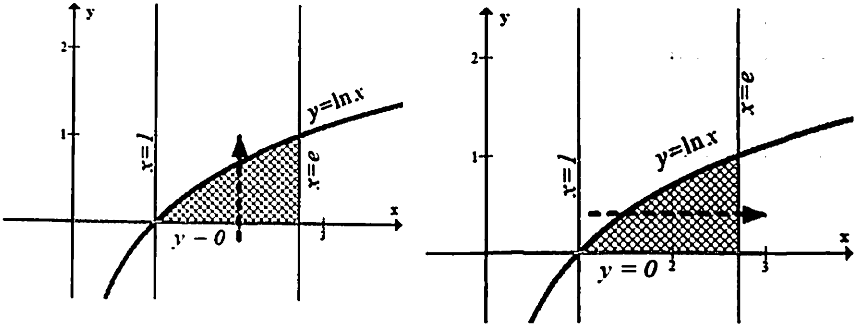
$[1,\sqrt{2}]$ qismida esa $y = \sqrt{2-z^2}$. aylananing yoyidan iborat bo'ladi (50-chizma). Shuning uchun Dsoha (D_1 va D_2) ikkita qismga ajratiladi. Integrali esa ikkita integraldan iborat bo'ladi:

$$\int_0^1 dy \int_y^{\sqrt{2-y^2}} f(y,z)dz = \int_0^1 dz \int_0^z f(y,z)dy +$$
$$\int_1^{\sqrt{2}} dz \int_0^{\sqrt{2-z^2}} f(y,z)dy.$$

2-Misol. $I = \int_1^e dx \int_0^{\ln x} f(x,y)dy$ karrali integrallarni integrallash tartibi o'zgartirilsin.

Yechish. Avval shaklida integrallash sohasi σ -ni tasvirlaymiz. Berilgan karrali integral chegaralaridan ko'rinib turibdiki, σ -soha $x =$

$1, x = e, y = 0, y = ln\,x$ chiziqlar bilan chegaralangan muntazam soha ekan.



a) b)

51-chizma.

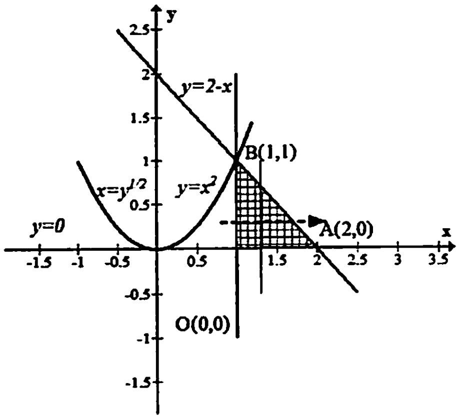
σ -sohaning eng pastki chet chizisi (OX) va $y = 0$ eng yuqori chet nuqtasi $y = 1$ OY -o'qining ixtiyoriy $\forall y \in]0,1[$ nuqtasiga o'tkazilgan gorizontalning «kirish» nuqta absissasi $x_1 = e^y$ (uni $y = ln\,x$ tenglamadan topdik) «chiqish» nuqta absissasi esa $x_2 = e$ (51-chizma). Demak

$$I = \int_0^1 dy \int_{e^y}^{e} f(x,y)dx,$$

ya'ni: $\int_1^e dx \int_0^{ln\,x} f(x,y)dy = \int_0^1 dy \int_{e^y}^{e} f(x,y)dx.$

10-misol. Ikki karrali $\iint_D f(x,y)dxdy$ integralda $y = x^2$, $x + y = 2$, $x \geq 1$ chiziqlar bilan chegaralangan bo'lsa, $\iint_D f(x,y)dxdy$ integrallash chegaralarini ikki usul bilan aniqlang.

Yechilishi. $y = x^2$, $x + y = 2$ tenglamalar sistemasini yechib, chiziqlarning kesishish nuqtalarini aniqlaymiz va D sohani quramiz (52-chizma),



52-chizma.

$\begin{cases} y = x^2, \\ x + y = 2, \end{cases}$ $x + x^2 = 2$, $x^2 + x - 2 = 0$, $D = 1 + 8 = 9,$

$x_{1,2} = \frac{-1 \pm 3}{2}$ $\Rightarrow$ $x_1 = 1$, $x_2 = -2.$

Masalan, birinchi tenglamadan $y_1 = (x_1)^2 = 1$, $y_2 = (x_2)^2 = 4$ topamiz.Xuddi shunday parabola to'g'ri chiziq bilan kesishishidan ikkita nuqta (1, 1) va (−2,4) larda kesishadi, ulardan biri $B(1,\ 1)$ nuqta D sohaga tegishli.

y o'zgaruvchi bo'yicha tashqi integral.

Dsoha bo'yicha integrallash chegarasi $y = 0$va $y = 1$ to'g'ri chiziqlar orasida joylashgan, x o'zgaruvchi bo'yicha esa y ning fiksirlangan qiymatida $x = \sqrt{y}$ parabola nuqtasidan $x = 2 - y$ to'g'ri chiziq nuqtasigacha o'zgaradi. U holda

$\iint_D f(x, y)dxdy = \int_0^1 dy \int_{\sqrt{y}}^{2-y} f(x, y)dx.$

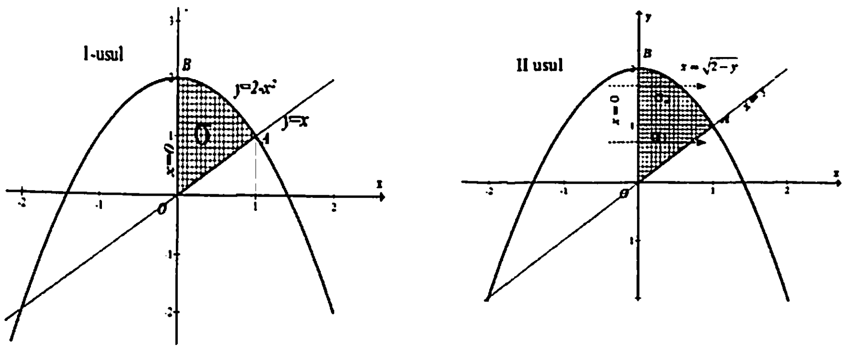
x o'zgaruvchi bo'yicha tashqi integral.

OBA sohaning oblasti *D* sohasi *OB* va *BA* chiziqlar bilan , $x = 1$ to'g'ri chiziq D sohani ikkita D_1: $0 \le x \le 1, 0 \le y \le x^2$ va D_2: $1 \le x \le 2,$ $0 \le y \le 2 - x$ sohalarga ajratadi. Natijada takroriy integrallar yig'indisini hosil qilamiz:

$\iint_D f(x, y)dxdy = \int_0^1 dx \int_0^{x^2} f(x, y)dy + \int_1^2 dx \int_0^{2-x} f(x, y)dy.$

6-misol. Ikki karrali integralni hisoblang $\int\int_\sigma xy^2 d\sigma$, agar integrallash sohasi σ quyidagi chiziqlar bilan chegaralangan $x = 0, y = x, y = 2 - x^2$ (53-chizma).

Yechilishi. Ikki karrali integralni hisoblash uchun (3) formuladan foydalanamiz, bunda



a) b)

53-chizma.

$$y = \phi_1(x) = x, y = \phi_2(x) = 2 - x^2$$

$$\iint_\sigma xy^2 d\sigma = \int_0^1 dx \int_x^{2-x^2} xy^2 dy.$$

$$\int_x^{2-x^2} xy^2 dy = \frac{xy^3}{3}\Big|_x^{2-x^2} = \frac{x(2-x^2)^3}{3} - \frac{x^4}{3}.$$

$$\iint_\sigma xy^2 d\sigma = \int_0^1 \left[\frac{x(2-x^2)^3}{3} - \frac{x^4}{3}\right] dx =$$

$$= -\frac{1}{6}\int_0^1 (2-x^2)^3 d(2-x^2) - \frac{1}{3}\int_0^1 x^4 dx = -\frac{1}{6}\frac{(2-x^2)^4}{4}\Big|_0^1 - \frac{x^5}{15}\Big|_0^1$$

$$= -\frac{1}{24} + \frac{2^4}{24} - \frac{1}{15} = \frac{67}{120}.$$

Ikkinchi usulda yechadigan bo‘lsak, $\iint_\sigma xy^2 d\sigma =$ ikki karrali integralni hisoblashda σ soha OAB ni ikkita sohaga σ_1 va σ_2 sohalarga ajratiladi. Additivlik xossasiga ko‘ra

$$\iint_\sigma xy^2 d\sigma = \iint_{\sigma_1} xy^2 d\sigma + \iint_{\sigma_2} xy^2 d\sigma.$$

$$\iint_{\sigma_1} xy^2 d\sigma = \int_0^1 dy \int_0^y xy^2 dx,$$

bu yerda $x = \varphi_1(y) = 0,\ x = \varphi_2(y) = y,\ c = 0,\ d = 1.$

$$\int_0^y xy^2 dx = \frac{x^2y^2}{2}\Big|_0^y = \frac{y^4}{2};$$

$$\iint_{\sigma_1} xy^2 d\sigma = \int_0^1 \frac{y^4}{2} dy = \frac{1}{10}.$$

$$\iint_{\sigma_2} xy^2 d\sigma = \int_1^2 dy \int_0^{\sqrt{2-y}} xy^2 dx,$$

bu yerda $x = 0,\ x = \sqrt{2-y},\ c = 1,\ d = 2;$

$$\int_0^{\sqrt{2-y}} xy^2 dx = \frac{x^2y^2}{2}\Big|_0^{\sqrt{2-y}} = \frac{(2-y)y^2}{2} = y^2 - \frac{y^3}{2}$$

$$\iint_{\sigma_2} xy^2 d\sigma = \int_1^2 \left(y^2 - \frac{y^3}{2}\right)dy = \frac{y^3}{3} - \frac{y^4}{8}\Big|_1^2 = \frac{11}{24};$$

Natijada $\iint_\sigma xy^2 d\sigma = \frac{1}{10} + \frac{11}{24} = \frac{67}{120}.$

Ushbu ikki o‘lchovli integrallarning integrallash tartibini o‘zgartiring.

1. $\int_0^3 dx \int_x^{3x} f(x,y)dy$
2. $\int_0^1 dy \int_y^{\sqrt{y}} f(x,y)dx$
3. $\int_{-1}^1 dx \int_{x^2}^{4-x^2} f(x,y)dy$
4. $\int_{-1}^1 dx \int_0^{\sqrt{1-x^2}} f(x,y)dy$
5. $\int_0^\pi dx \int_0^{\sin x} f(x,y)dy$

6. $\int_1^{10} dx \int_{-\lg x}^{\lg x} f(x,y)dy$

7. $\int_{-4}^{0} dy \int_{-2}^{y+2} f(x,y)dx + \int_0^4 dy \int_{-\sqrt{4-y}}^{\sqrt{4-y}} f(x,y)dx$

Ko`rsatilgan $D$ soha uchun $\iint_D f(x,y)dxdy$ integralda qutb koordinatalariga $(x = r\cos\phi, y = r\sin\phi)$ o`tib, integrallash chegaralari ikki xil tartibda qo`yilsin.

1. $D = \{(x,y): x^2 + y^2 \le 2y\}$.
2. $D = \{(x,y): (x^2 + y^2)^2 = a^2(x^2 - y^2), x \le 0\}$.
3. $D = \{(r,\phi): r \ge 2\cos\phi, r \le 4\cos\phi\}$.
4. $D = \{(x,y): x^2 + y^2 \ge 8, x^2 + y^2 \le 4x\}$.
5. $D = \{(x,y): x^2 + y^2 \ge 18, x^2 + y^2 \le 6y\}$.
6. $D = \{(x,y): x \ge y, x + y \le 6,\ y \ge 0\}$.

3.9. Ikki karrali integralning tatbiqlari

3.9.1. Jism hajmini hisoblash

Ikki o'lchovli integralning geometrik ma'nosi yuqoridan $z = f(x,y)$ sirt bilan yon tomonlaridan yasovchilari Oz o'qiga parallel bo'lgan silindrik sirt bilan, pastdan D soha bilan chegaralangan jismning hajmi quyidagi formula bilan hisoblanadi:

$$V = \iint_D f(x,y)dxdy.$$

1-misol. $y = 1 + x^2$, $z = 3x, y = 5, z = 0$ sirtlar bilan chegaralangan I oktantadagi jismning hajmini hisoblang.

Yechish. Hajmi hisoblanishi kerak bo'lgan jism yuqoridan $z = 3x$ tekislik, yondan $y = 1 + x^2$ parabologik silindr, tekislik bilan chegaralangan. Shunday qilib,

$$V=\iint_D 3xdxdy=3\int_0^2 xdx\int_{1+x^2}^5 dy=3\int_0^2 x[5-(1+x^2)]\,dx=$$

$$=3\int_0^2 (4x-x^3)dx=3\left(4\frac{x^2}{2}-\frac{x^4}{4}\right)_0^2=3\left(2\cdot 2^2-\frac{2^4}{4}\right)$$

$$=24-12=12$$

kub birlik.

2-misol. $z=0, z=x^2+y^2,\ y=x^2,\ y=1$ sirtlar bilan chegaralangan Q jismning hajmini hisoblang.

Yechish.▶ Berilgan jismni quyidagi ko‘rinishda tasvirlash kerak: $Q=\{(x,y,z):(x,y)\in D,\ \ 0\le z\le x^2+y^2\}$, bunda D — soha *Oxy* tekislikning $y=x^2$ va $y=1$ egri chiziqlari bilan chegaralangan, ya’ni. $D=\{(x,y):\ -1\le x\le 1,\ \ x^2\le y\le 1\}$. Ikki o‘lchovli integralning geometrik ma’nosiga ko‘ra, Q jismning hajmi quyidagicha topiladi:

$$V=\iint_D (x^2+y^2)dx\,dy=\int_{-1}^1 dx\int_{x^2}^1 (x^2+y^2)dy=\int_{-1}^1\left(x^2(1-x^2)+\frac{1}{3}(1-x^6)\right)dx=\frac{88}{105}.$$

3.9.2.Tekis figura yuzasini hisoblash

Agar $\sum_{i=1}^n f(x_i,y_i)\Delta s_i$ integral yig‘indida (D) sohada integral osti funksiya $f(x;y)\equiv 1$ va $\lambda\to 0$ da limitga o‘tadigan bo‘lsak, u holda ikki karrali integra lning qiymati son jihatdan integrallash sohasi (D) ning S yuziga teng bo‘ladi:

$$\lim_{\lambda\to 0}\sum_{i=1}^n f(x_i,y_i)\,\Delta s_i=\lim_{\lambda\to 0}\sum_{i=1}^n \Delta s_i=s,$$

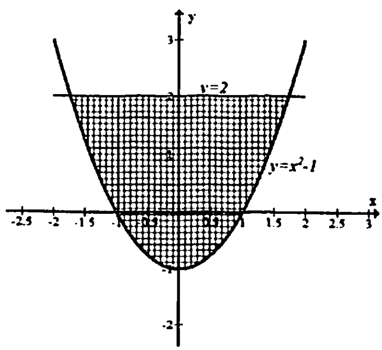
$$\lim_{\lambda\to 0}\sum_{i=1}^{n}\Delta s_i=\iint_D dxdy.$$

$$\iint f(x,y)dxdy=\int_a^b dx\int_{\phi_1(x)}^{\phi_2(x)}dy=\int_a^b[\phi_2(x)-\phi_1(x)]dx=S \qquad (5)$$

bo'ladi.

1-misol. $y=2$ to'g'ri chiziq va $y=x^2-1$ parabola bilan chegaralangan D tekis soha yuzini hisoblang.

Yechish. D sohani Ox va Oy o'qlariga proyeksiyalaymiz; Oy o'qiga proyeksiyalaganda D soha Oy o'qiga simmetrik ekanligini anglash mumkin. Shundan kelib chiqib sohaning o'ng tomoni yuzasini topib uni ikkilantirish yetarli. D sohani Oy o'qiga proyeksiyalaganda esa $[-1,2]$ kesmaga ega bo'lamiz va D soha



54-chizma.

$x=0$, $y=2$ to'g'ri chiziqlar va $y=x^2-1$ yoki $x=\sqrt{y+1}$ chiziqlar bilan chegaralangan sohaning ikkilanganiga teng (54-chizma). Natijada

$$\frac{S}{2}=\int_{-1}^{2}dy\int_{0}^{\sqrt{y+1}}dx=\int_{-1}^{2}x\Big|_0^{\sqrt{y+1}}dy=\int_{-1}^{2}\sqrt{y+1}dy=\frac{2}{3}(y+1)^{\frac{3}{2}}\Big|_{-1}^{2}=2\sqrt{3},$$

bu yerdan $S=4\sqrt{3}.$

yoki $\frac{S}{2}=\int_0^{\sqrt{3}}dx\int_{x^2-1}^{2}dy=\int_0^{\sqrt{3}}(y|_{x^2-1}^{2})dx=\int_0^{\sqrt{3}}(2-x^2+1)dx=$

$$= \int_0^{\sqrt{3}} (3 - x^2)dx = 3x|_0^{\sqrt{3}} - \frac{x^3}{3}\Big|_0^{\sqrt{3}} = 3\sqrt{3} - \frac{3\sqrt{3}}{3} = 2\sqrt{3} \Rightarrow$$

$\Rightarrow S = 4\sqrt{3}$ (kv. birlik).

2-misol. $x = 0, x = 2, y = 0$ to'g'ri chiziqlar va $y = e^x$ egri chizig'i bilan chegaralangan sohaning yuzasi topilsin.

Yechish. Berilgan sohani OX -o'qiga proyeksiyalab, x-ning o'zgarish sohasi $[0; 2]$ kesmani hosil qilamiz. Soha pastdan $y = 0$ va yuqoridan $y = e^x$ chiziqlar bilan chegaralangan. Bu holda izlanayotgan yuza quyidagicha hisoblanadi:

$$S = \iint_D dxdy$$

$$= \int_0^2 dx \int_0^{e^x} dy = \int_1^2 y \ \Big|_0^{e^x} dx = \int_1^2 e^x dx = e^x \ |_1^2 = e^2 - e$$

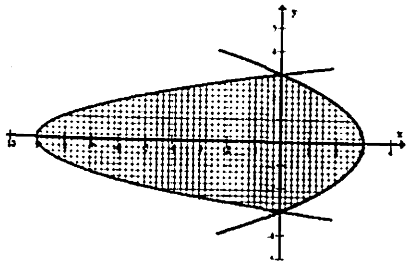
$$= e(e - 1).$$

3-misol. $x = 0, y = 1, y = 3$ to'g'ri chiziqlar va $y = \frac{1}{x}$ giperbola bilan chegaralangan shaklning yuzasi topilsin.

Yechish. Bu sohani OY o'qiga proyeksiyalaymiz, chunki OX -o'qiga proyeksiyalasak, 2 ta ikki karali integralni hisoblashga to'g'ri keladi. Bu yerda ham formuladan, foydalanib, undan karrali integralga o'tsak, $S = \int_1^3 dy \int_0^{\frac{1}{y}} dx = \int_1^3 \frac{1}{y} dy = ln y| \ _1^3 = ln 3 - ln 1 = ln 3$ bo'ladi.

4-Misol. $y^2 = 9 + x$ va $y^2 = 9 - 3x$ parabolalar bilan chegaralangan sohaning yuzasi topilsin.

Yechish. Soha OX–o'qiga nisbatan simmetrik joylashgani uchun, (55-chizma).



55-chizma.

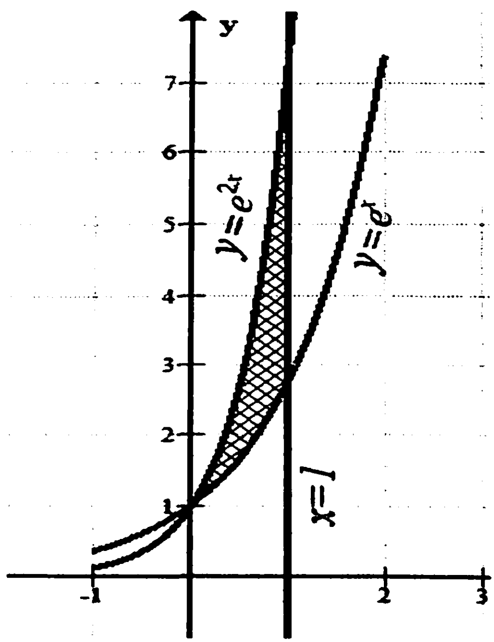
uning yuqori yarmining yuzasini topib, 2 ga ko‘paytiramiz:

$$S = 2\int_0^3 dy\int_{y^2-9}^{3-\frac{1}{3}y^2} dx = 2\int_0^3 x\Big|_{y^2-9}^{3-\frac{1}{3}y^2} dy = 2\int_0^3 (3-\frac{1}{3}y^2 - y^2 + 9)dy = 2\int_0^3 (12-\frac{4y^2}{3})dy = \frac{2}{3}\int_0^3 36-4y^2)dy = \frac{8}{3}(9y-\frac{y^3}{3})\Big|_0^3 = 48$$

kv.birlik.

5-Misol. $y = e^x, y = e^{2x}$ egri chiziqlar va $x = 1$ to‘g‘ri chiziq bilan chegaralangan sohaning yuzasi topilsin.

Yechish. (56-chizma).



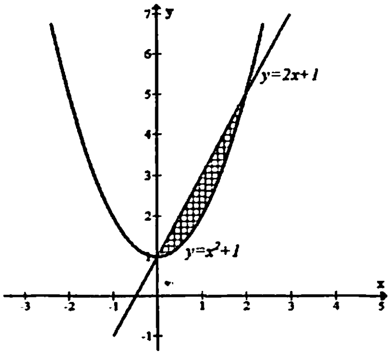
56-chizma.

$$S = \int_0^1 dx \int_{e^x}^{e^{2x}} dy = \int_0^1 (e^{2x} - e^x)\, dx = \left[\frac{1}{2}e^{2x} - e^x\right] \Big|_0^1$$

$$= \frac{1}{2}e^2 - e - \frac{1}{2} + 1 = \frac{1}{2}e^2 - e + \frac{1}{2} = \frac{1}{2}(e-1)^2$$

kvadrat birlik.

6-Misol. $y = 2x + 1$ to‘g‘ri chiziq va $y = x^2 + 1$ parabola bilan chegaralangan soha yuzasini hisoblang



57-chizma.

Yechilishi. $2x + 1 = x^2 + 1,\ 2x - x^2 = 0$ chiziqlarning kesishish nuqtalarini topamiz

$$\begin{cases} y = 2x + 1, \\ y = x^2 + 1, \end{cases} \quad x(2 - x) = 0 \ \Rightarrow \ x_1 = 0,\ x_2 = 2.$$

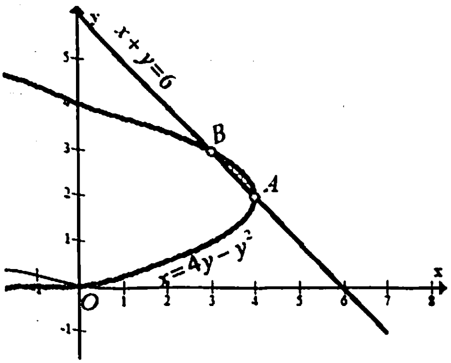
Integrallanuvchi Dsohani hosil qilamiz (57-chizma).U holda

$$S = \iint_D dxdy = \int_0^2 dx \int_{x^2+1}^{2x+1} dy = \int_0^2 dx \cdot y\Big|_{x^2+1}^{2x+1} =$$

$$= \int_0^2 (2x + 1 - x^2 - 1)dx = \left(x^2 - \frac{x^3}{3}\right)\Bigg|_0^2 = \frac{4}{3}.$$

7-misol. $x = 4y - y^2, x + y = 6$ chiziqlar bilan chegaralangan sohaning yuzini toping.

Yechish. Berilgan chiziqlarning kesishish nuqtalarini topamiz (58-chizma). $x = 4y - y^2, x = 6 - y$ dan $4y - y^2 = 6 - y, y^2 - 5y + 6 = 0,$ $y_1 = 2,\ y_2 = 3;\ x_1 = 4, x = 3;\ A(4;2) va B(3;3)$ kesishish



58-chizma.

nuqtalari bo'ladi. Shunday qilib, yuza

$$S = \iint_D dxdy = \int_2^3 dy \int_{6-y}^{4y-y^2} dx = \int_2^3 x \Big|_{6-y}^{4y-y^2} dy$$

$$= \int_2^3 (4y - y^2 - 6 + y)dy =$$

$$= \int_2^3 (5y - y^2 - 6)dy = \left(\frac{5}{2}y^2 - \frac{y^3}{3} - 6y\right)\Big|_2^3 = \frac{1}{6} \text{ (kv. birlik)}$$

Ushbu chiziqlar bilan chegaralangan yuza hisoblansin.

1. $S = \iint_{(P)} dp$, bunda (P) soha $y = \frac{a^2}{x}$, $y = \frac{2a^2}{x}$,

$x > 0, \quad x = 1, \quad x = 2$

2. $S = \iint_{(P)} dp$, bunda (P) soha $x = 0, y = 0, x = 2, y = e^x$

3. $S = \iint_{(P)} dp$, bunda (P) soha $y = 0, y = x, x^2 + y^2 = 2x$

4. $S = \iint_{(P)} dp$, bunda (P) soha $y = \frac{1}{3}x^3$, $y = 4 - \frac{2}{3}x^2$

5. $S = \iint_{(P)} dp$, bunda (P) soha $y = x^2 + 4x$, $y = x + 4$.

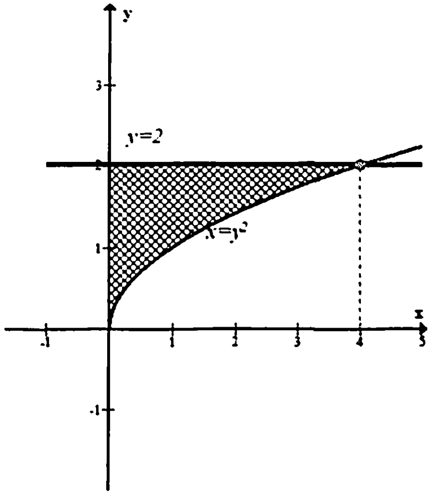
3.10. Tekis figura massasi

*A*gar D soha modda taqsimotining $\rho(x,y)$ sirt zichligiga ega, *xOy* tekislikda yotuvchi qalinligi bir birlik bo'lgan yassi jism(plastinka) bo'lsa, u holda *yassi jismninig massasi*ni quyidagi formula bilan hisoblanadi:

$$m = \iint_D \rho(x, y) dx\, dy. \qquad (4)$$

1- misol. $y = \sqrt{x}$, $x = 0$, $y = 2$ chiziqlar bilan chegaralangan va zichligi $\rho(x, y) = 2y$ bo'lgan D plastinka massasini toping.

Yechilishi. Integrallash sohasini qurib olamiz(1.31-chizma). D soha $0 \le x \le 4$, $\sqrt{x} \le y \le 2$ tengsizliklar bilan berilgan $m = \iint_D \rho(x, y) dx\, dy = \iint_D 2y dx\, dy$ formuladan plastinka massasini topamiz. U holda



59-chizma.

$$m = \iint_D y dx\, dy == \int_0^4 dx \int_{\sqrt{x}}^2 2y dy =$$

$$= \int_0^4 dx\, y^2 \Big|_{\sqrt{x}}^2 = \int_0^4 dx \left(4 - \left(\sqrt{x}\right)^2\right) =$$

$$= \int_0^4 4 - x\; dx = \int_0^4 4dx - \int_0^4 x dx = \left(4x - \frac{x^2}{2}\right)\Big|_0^4 = 16 - 8 = 8$$

3.10.1 Tekis figuraning og'irlik markazi koordinatalarini va statik momentini hisoblash

Yassi jismning Ox va Oy o'qlariga nisbatan *statik moment*lari quyidagi formulalar bo'yicha topiladi:

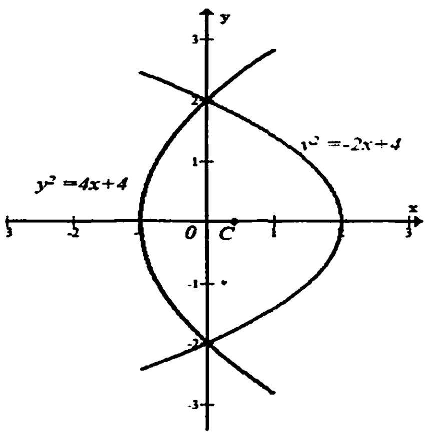
$$M_x = \int_D y\rho(x, y) dx\, dy, \qquad M_y = \int_D x\rho(x, y) dx\, dy \,. \qquad (5)$$

Yassi jism massasi *og'irlik markazi* koordinatalari:

$$x_c = \frac{M_y}{m}, \qquad y_c = \frac{M_x}{m}. \tag{6}$$

1-misol. $y^2 = 4x + 4$, $y^2 = -2x + 4$ chiziqlar bilan chegaralangan sohaning o‘girlik markazini toping.

Yechilishi. ► Berilgan soha Ox o‘qiga simmetrik bo‘lganligi sababli, $y_c = 0$ boladi. x_c ni topamiz(60-chizma). Berilgan soha yuzini hisoblaymiz:



60-chizma.

$$S = \iint_D dxdy =$$

$$= 2\int_0^2 dy \int_{\frac{(y^2-4)}{4}}^{\frac{(4-y^2)}{2}} dx =$$

$$= 2\int_0^2 \left(\frac{4-y^2}{2} - \frac{y^2-4}{4}\right) dy$$

$$= 2\int_0^2 \left(3 - \frac{3y^2}{4}\right) dy =$$

$$6\left[y - \frac{1}{12}y^3\right]_0^2 = 8.$$

U holda,

$x_c = \frac{1}{8}\iint_D xdxdy = \frac{1}{8}2\int_0^2 dy \int_{\frac{(y^2-4)}{4}}^{\frac{(4-y^2)}{2}} xdx = \frac{1}{8}\int_0^2 \left[\frac{1}{4}(4-y^2)^2 - \frac{1}{16}(y^2-4)^2\right] dy = = \frac{1}{8}\int_0^2 \left(3 - \frac{3}{2}y^2 + \frac{3}{16}y^4\right) dy = \frac{1}{8}\left[3y - \frac{y^3}{2} + \frac{3y^5}{80}\right]_0^2 = \frac{2}{5}.$

2-misol. $\mathbf{y^2 = 4x + 4, y^2 = -2x + 4}$ chiziqlar bilan chegaralangan figuraning og‘irlik markazining koordinatlarini toping.

Yechish. Chiziqlar OX o'qiga nisbatan simmetrik bo'lganligi uchun $\bar{y}_c = 0$ $\bar{x}_c$ ni topamiz:

$$S = \iint_D dxdy = 2\int_0^2 dy \int_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}} 2\int_0^2 \left(\frac{4-y^2}{2} - \frac{y^2-4}{4}\right) dy = 2\int_0^2 \left(3 - \frac{3y^2}{4}\right) dy == 6\left[y - \frac{y^3}{12}\right]_0^2 = 6\left(2 - \frac{8}{12}\right) = 8;$$

$$\bar{x}_c = \frac{1}{8}\iint_L xdxdy = \frac{1}{8}2\int_0^2 dy \int_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}} xdxdy =$$

$$\frac{1}{8}\int_0^2 \left[\frac{4-y^2}{4} - \frac{(y^2-4)}{16}\right] dy = \frac{1}{8}\int_0^2 \left(3 - \frac{3}{2}y^2 + \frac{3}{16}y^4\right) dy =$$

$$= \frac{1}{8}\left(3y - \frac{y^3}{2} + \frac{3y^5}{80}\right)_0^2 = \frac{2}{5}.$$

Demak ,$C(\frac{2}{5}; 0)$.

3.11 Tekis figuraning inersiya momentini

D yassi jismning koordinata o'qlariga va koordinata boshiga nisbatan *inersiya moment*lari:

$$I_x = \iint_D y^2 \rho(x,y) dx\, dy, I_y = \iint_D x^2 \rho(x,y) dx\, dy,$$

$$I_0 = I_x + I_y = \iint_D (x^2 + y^2) \rho(x,y) dx\, dy \qquad (7)$$

formulalar bilan hisoblanadi.

1-misol. $\frac{x}{a} + \frac{y}{b} = 1, x = 0, y = 0$ chiziqlar bilan chegaralangan sohaning koordinata boshiga nisbatan inersiya momentini hisoblang.

► $I_0 = \iint_D (x^2+y^2)dxdy = \int_0^a dx \int_0^{\frac{b}{a}\cdot(a-x)} (x^2+y^2)dy == \int_0^a \left[x^2 y + \frac{1}{3}y^3\right]_0^{\frac{b}{a}\cdot(a-x)} dx = \int_0^a \left[\frac{b}{a}x^2(a-x) + \frac{1}{3}\frac{b^3}{a^3}(a-x)^3\right] dx =$

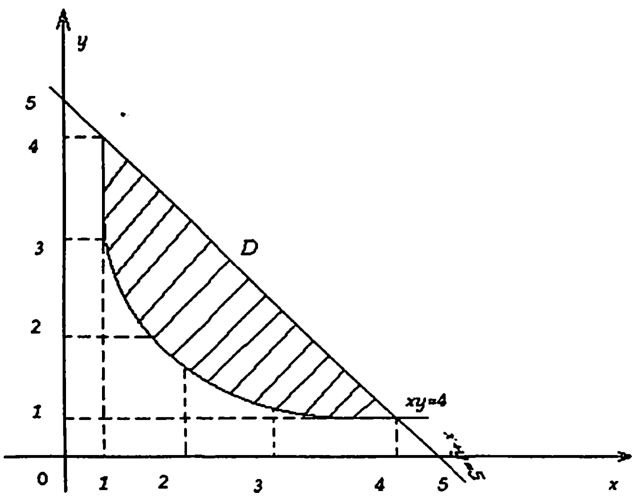
$= \left[\frac{1}{3}bx^3 - \frac{b}{4a}x^4 - \frac{1}{3}\cdot\frac{b^3}{a^3}\frac{1}{4}(a-x)^4\right]_0^a = \frac{ab(a^2+b^2)}{12}.$ ◄

3.12. Ikki karrali integrallar yordamida tekis figuraning yuzini va jismning hajmini hisoblash.

Tekis figuraning yuzini hisoblash

1-misol. $xy=4$ *va* $x+y=5$ *chiziqlar bilan chegaralangan sohaning yuzini toping.*

Yechim. Avvalo berilgan sohani koordinatalar tekisligida tasvirlaymiz.



D sohada x ning eng katta va eng kichik qiymatlarini topish sizga 1-kurs matematik analiz kursidan ma'lum. Buning uchun har ikkala tenglamani birgalikda yechamiz:

$$\begin{cases} xy=4 \\ x+y=5 \end{cases} \Rightarrow \begin{cases} x(5-x)=4 \\ y=5-x \end{cases} \Rightarrow \begin{cases} x^2-5x+4=0 \\ y=5-x \end{cases} \Rightarrow \begin{cases} x_1=4; x_2=1 \\ y=5-x \end{cases}$$

Demak, x o'zgaruvchi $[1,4]$ oraliqdagi qiymatlarni qabul qilar ekan. D sohaning chegaralari $y=\frac{4}{x}$ va $y=5-x$ chiziqlaridir. $a=1$; $b=4$ bo'ladi. (3)

fomulada $f(x,y)=1$, $\varphi_1=\frac{4}{x}$ va $\varphi_2(x)=5-x$ larni o'rniga qo'ysak va D sohaning yuzini S bilan belgilasak, quyidagini hosil qilamiz:

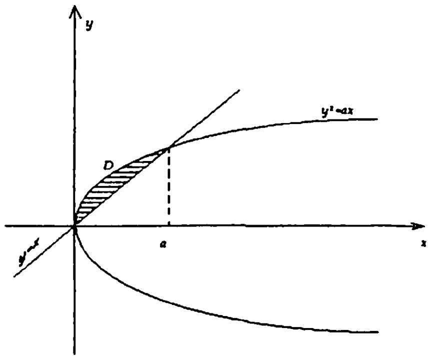
$$S=\iint_D dD=\int_1^4 dx\int_{\frac{4}{x}}^{5-x} dy=\int_1^4 (5-x-\frac{4}{x})dx=(5x-\frac{x^2}{2}-4\ln x)\Big|_1^4=$$

$$=20-8-4\ln 4-5+\frac{1}{2}=7{,}5-4\ln 4\ \ (kv\ birlik).$$

2-misol. $y^2=ax$ *parabola va* $y=a$ *to'g'ri chiziq bilan chegaralangan sohaning yuzini toping,* $(a>0)$.

Yechim. Berilgan chiziqlarning grafigini chizamiz. a ning son qiymati aniq berilmagani uchun parabolaning formal (yuzaki) grafigini chizamiz. To'g'ri chiziq va parabolaning kesishish nuqtalarini topamiz:

$$\begin{cases}y^2=ax\\ y=x\end{cases}\Rightarrow\begin{cases}x^2=ax\\ y=x\end{cases}\Rightarrow\begin{cases}x(x-a)=0\\ y=x\end{cases}\Rightarrow\begin{cases}x_1=0;\ \ x_2=a\\ y=x\end{cases}$$



Demak, x ning chegaralari 0 va a sonlari ekan. Parabolaning yuqori bo'lagida uning tenglamasi $y=\sqrt{ax}$ ko'rinishda bo'ladi. Yuqoridagi shakldan ko'rinadiki, $\varphi_1(x)=x$, $\varphi_2(x)=\sqrt{ax}$. Bularni (3) formulaga qo'ysak va $f(x,y)=1$ ekanini hisobga olsak, quyidagini topamiz:

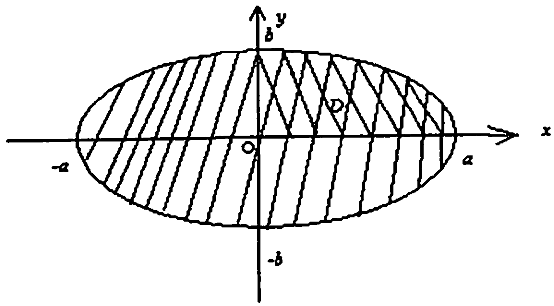
$$S=\iint_D dD=\int_0^a dx\int_x^{\sqrt{ax}} dy=\int_0^a(\sqrt{ax}-x)dx=\sqrt{a}\cdot\int_0^a\sqrt{x}dx-$$

$$-\int_0^a xdx=\sqrt{a}\cdot\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^a-\frac{x^2}{2}\Bigg|_0^a=\frac{2}{3}\sqrt{a}\cdot a\sqrt{a}-\frac{1}{2}a^2=(\frac{2}{3}-\frac{1}{2})a^2=\frac{1}{6}a^2$$

(kv birlik).

3-misol. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *ellips bilan chegaralangan sohaning yuzini toping.*

Yechim. Ellipsning grafigini chizamiz.



Berilgan D sohaning yuzini hisoblash uchun uning koordinatalar tekisligi I choragida joylashgan qismi yuzini hisoblab, natijani 4 ga ko'paytirish kifoya. Demak, $(D) = 4 \cdot (D_1)$

Ellips tenglamasini y ga nisbatan yechamiz

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} \Rightarrow y^2 = \frac{b^2}{a^2}(a^2 - x^2) \Rightarrow y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

Shaklidan ko'rinadiki, D sohaning yuzi uchun $\varphi_1(x) = 0$ va

$$\varphi_2(x) = \frac{b}{a}\sqrt{a^2 - x^2}, \; x \in [0, a].$$

Bularni (3) formulada o'rniga qo'ysak va $f(x, y) = 1$ ekanini hisobga olsak, quyidagini topamiz:

$$S = \iint_D dD = 4\iint_{D_1} dD_1 = 4\int_0^a dx \int_0^{\frac{b}{a}\sqrt{a^2 - x^2}} dy = 4\int_0^a y\Big|_0^{\frac{b}{a}\sqrt{a^2 - x^2}} \cdot dx = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}dx,$$

bu yerda $x = a\sin t$ belgilash kiritamiz:

$$dx = a\cos t dt, \; x = 0 \Rightarrow t = 0, \; x = a \Rightarrow t = \frac{\pi}{2}.$$

Bularni o'rniga qo'ysak,

$$S = \frac{4b}{a}\int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2 t} \cdot a\cos t dt = 4b\int_0^{\frac{\pi}{2}} a\cos^2 t dt = 4ab\int_0^{\frac{\pi}{2}} \frac{1 + \cos 2t}{2} =$$

$$= 2ab\int_0^{\frac{\pi}{2}} (1 + \cos 2t)dt = 2ab\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^{\frac{\pi}{2}} = 2ab\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) = 2ab \cdot \frac{\pi}{2} = \pi ab$$

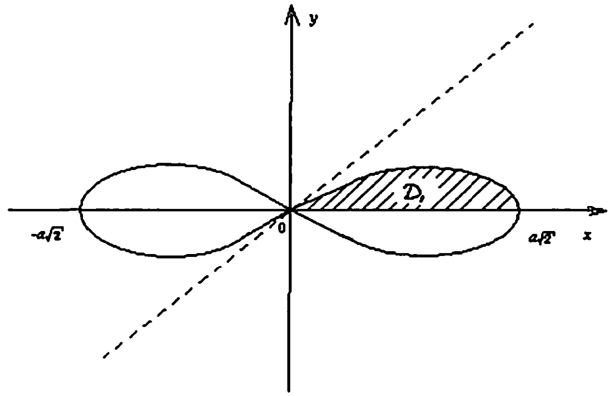
(kv birlik).

4-misol. $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$ *lemniskata bilan chegaralangan sohaning yuzini toping.*

Bu misolni yechishga kirishishdan oldin ma'ruzadan ikki karrali itegrallarni hisoblashda qutb koordinatalarini qo'llanilishi mavzusini takrorlang. Sizga ma'lumki, agar D soha konturining qutb koordinatalaridagi tenglamasi $r = r(\varphi)$ bo‘lsa, (r - qutb o‘qi, φ - qutb burchagi), bu yerda φ o‘zgaruvchi $[\alpha, \beta]$ oraliqda n qiymatlar qabul qilsa, u holda D sohaning yuzi quydagicha aniqlanadi:

$$S = \iint_D dD = \int_\alpha^\beta d\varphi \int_0^{r(\varphi)} rdr$$

Endi 4-misolni yechishga o‘tamiz. Oldin lemniskata grafigini chizamiz:



Lemniskataning qutb koordinatalardagi tenglamasiga o‘tish uchun uning tenglamasidagi x,y lar o‘rniga $x = r\cos\varphi, y = r\sin\varphi$ larni qo‘yamiz.

$$[r^2(\cos^2\varphi + \sin^2\varphi)]^2 = 2a^2r^2(\cos^2\varphi - \sin^2\varphi) \Rightarrow$$

$$\Rightarrow r^4 = 2a^2r^2\cos 2\varphi \Rightarrow r^2 = 2a^2\cos 2\varphi \Rightarrow$$

$$\Rightarrow r = a\sqrt{2\cos 2\varphi} \quad (\varphi \in [-\frac{\pi}{4}, \frac{\pi}{4}] \cup [\frac{3\pi}{4}, \frac{5\pi}{4}])$$

Shaklda ko‘rsatilgan D_1 soha uchun $0 \le \varphi \le \frac{\pi}{4}$ oraliqda qiymatlar qabul qiladi.

$S = 4 \cdot (D_1)$ ekanini hisobga olib, quyidagini topamiz:

$$S = 4\iint_{D_1} dD_1 = 4\int_0^{\frac{\pi}{4}} d\varphi \int_0^{a\sqrt{2\cos 2\varphi}} rdr = 4\int_0^{\frac{\pi}{4}} \frac{r^2}{2}\Bigg|_0^{a\sqrt{2\cos 2\varphi}} d\varphi =$$

$$= 2\int_0^{\frac{\pi}{4}} 2a^2\cos 2\varphi d\varphi = 2a^2\sin 2\varphi\Big|_0^{\pi/4} = 2a^2(\sin\frac{\pi}{2} - 0) = 2a^2$$ (kv birlik).

5-misol. *Egri chiziqli koordinatalar kiritish yordamida quyidagi chiziqlar bilan chegaralangan sohaning yuzini hisoblang:*

$y^2 = 2px, \ y^2 = 2qx, \ x^2 = 2ry, \ x^2 = 2sy.$

Bu yerda $0 < p < q, \ 0 < r < s.$

Yechim. Bu misolni yechishga kirishishdan oldin ma'ruzadan "ikki karrali integrallarda o'zgaruvchilarni almashtirish" mavzusini takrorlang. Bu yerda yangi ξ , η egri chiziqli koordinatalarni quyidagicha kiritamiz:

$$y^2 = 2\xi x \quad (p \le \xi \le q) \quad x^2 = 2\eta y \quad (r \le \eta \le s)$$

Bu tenglamalardagi x,y larni ξ , η lar orqali ifodalaymiz:

$$\left.\begin{aligned} x^2 &= 2\eta y \\ x &= \frac{y^2}{2\xi} \end{aligned}\right\} \Rightarrow \begin{cases} \dfrac{y^4}{4\xi^2} = 2\eta y \\ x = \dfrac{y^2}{2\xi} \end{cases} \Rightarrow \begin{cases} y^4 = 8\xi^2\eta y \\ x = \dfrac{y^2}{2\xi} \end{cases} \Rightarrow \begin{cases} y = 2\sqrt[3]{\xi^2\eta} \\ x = \dfrac{4\sqrt[3]{\xi^4\eta^2}}{2\xi} \end{cases} \Rightarrow x = 2\cdot\sqrt[3]{\xi\eta^2}; \quad y = 2\sqrt[3]{\xi^2\eta}$$

.

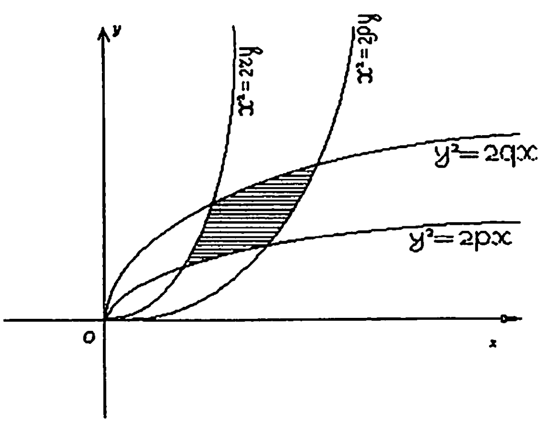
Endi x va y larning ξ , η lar bo'yicha xususiy hosilalarini hisoblaymiz:

$$\frac{\partial x}{\partial \xi} = \frac{2}{3}(\xi\eta^2)^{-\frac{2}{3}}\cdot\eta^2 ; \qquad \frac{\partial x}{\partial \eta} = \frac{2}{3}(\xi\eta^2)^{-\frac{2}{3}}\cdot 2\xi\eta ;$$

$$\frac{\partial y}{\partial \xi} = \frac{2}{3}(\xi^2\eta)^{-\frac{2}{3}}\cdot 2\xi\eta ; \qquad \frac{\partial y}{\partial \eta} = \frac{2}{3}(\xi^2\eta)^{-\frac{2}{3}}\cdot\xi^2 ;$$

Bularni $I(\xi,\eta) = \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{vmatrix}$ yakobianda o'rniga qo'yamiz va determinantning xossasiga ko'ra uning satrlaridagi umumiy ko'paytuvchilarni determinant belgisidan tashqariga chiqaramiz:

$$I(\xi,\eta) = \frac{4}{9}\xi\eta(\xi\eta^2)^{-\frac{2}{3}}\cdot(\xi^2\eta)^{-\frac{2}{3}}\cdot\begin{vmatrix} \eta & 2\xi \\ 2\eta & \xi \end{vmatrix} = \frac{4}{9}\xi\eta(\xi\eta)^{-2}\cdot(-3\xi\eta) = -\frac{4}{3}$$



Bu yerda shuni esda tutish kerakki, yangi ξ,η o'zgaruvchilar bo'yicha integrallash sohasi $\Delta = \{[p,q]\times[r,s]\}$ to'g'ri to'rtburchak bo'ladi.

Demak,

$$S = \iint_D dD = \iint_\Delta |I(\xi,\eta)| d\xi d\eta = \frac{4}{3}\int_p^q d\xi \int_r^s d\eta = \frac{4}{3}(q-p)(s-r).$$

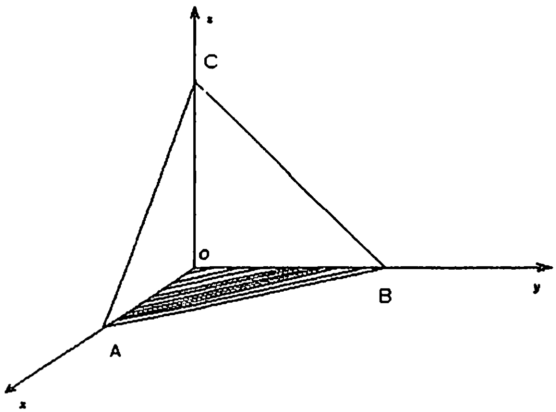
Yuqorida yechilgan misollarda shunday xulosaga kelamiz: D sohaning konturi (chegarasi) silliq yopiq egri chiziq bo'lgan hollarda qutb koordinatalari kiritish, egri chiziqli to'rtburchak bo'lgan hollarda esa egri chiziqli koordinatalar yordamida o'zgaruvchilarni almashtirish maqsadga muvofiqdir.

Jismning hajmini hisoblash

6-misol. *Koordinatalar tekisliklari va* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ *tekisligi bilan chegaralangan jismning hajmini toping.*

Bu misolni yechishga kirishishdan oldin "ikki karrali integrallar yordamida jismning hajmini hisoblash" mavzusini ma'ruzadan takroran o'qing.

Yechim.



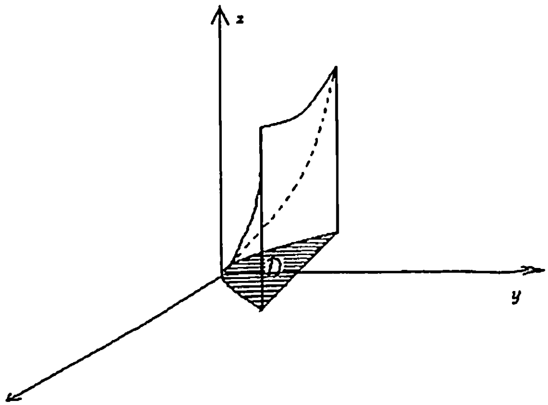
Qaralayotgan jism uchburchakli piramida bo'lib, uning uchta yoqlari koordinata tekisliklarida yotadi.

D sohaning chegaralari $x=0, y=0$ va $\frac{x}{a}+\frac{y}{b}=1$ to'g'ri chiziqlardir.

Demak,

$$D = \{x,y : 0 \le x \le a,\ 0 \le y \le b(1-\frac{x}{a})\}$$

$$z = f(x,y) = c\left(1-\frac{x}{a}-\frac{y}{b}\right).$$



Bularni (3) formulaga qo'ysak va qaralayotgan jism hajmini V bilan belgilasak, quyidagini hosil qilamiz:

$$V = \iint\limits_D f(x,y)dD = \iint\limits_D c\left(1 - \frac{x}{a} - \frac{y}{b}\right)dxdy = \int\limits_0^a dx \int_0^a dx \int\limits_0^a c\left(1 - \frac{x}{a} - \frac{y}{b}\right)dy =$$

$$= c\int\limits_0^a \left(y - \frac{xy}{a} - \frac{y^2}{2b}\right)\Bigg|_0^{b-\frac{bx}{a}} \cdot dx = \int\limits_0^a \left[b\left(1 - \frac{x}{a}\right) - \frac{bx}{a}\left(1 - \frac{x}{a}\right) - \left(\frac{b^2\left(1 - \frac{x}{a}\right)^2}{2b}\right)\right]dx =$$

$$= bc\int\limits_0^a \frac{(1 - \frac{x}{a})^2}{2}dx = -abc \cdot \frac{\left(1 - \frac{x}{a}\right)^3}{6}\Bigg|_0^a = \frac{abc}{6} \text{ (kub birlik).}$$

7-misol. $z = x^2 + y^2$ *aylanma paraboloid,* $y = x^2$ *silindr va* $y = 1$ *tekisliklar bilan chegaralangan jismning hajmini toping.*

Yechim.

D soha quyidagicha bo'ladi:

$$D = \{(x,y): -1 \le x \le 1; x^2 \le y \le 1\}, \quad f(x,y) = x^2 + y^2.$$

Bularni (3) ga qo'yamiz:

$$V \iint\limits_D f(x,y)dD = \iint\limits_D (x^2 + y^2)dxdy = \int\limits_{-1}^{1} dx \int\limits_{x^2}^{1} (x^2 + y^2)dy = \int\limits_{-1}^{1}\left(x^2 y + \frac{y^3}{3}\right)\Bigg|_{x^2}^{1} dx =$$

$$=\int_{-1}^{1}(x^2+\frac{1}{3}-x^4-\frac{x^6}{3})dx=\left(\frac{x^3}{3}+\frac{x}{3}-\frac{x^5}{5}-\frac{x^7}{21}\right)\Bigg|_{-1}^{1}=2\left(\frac{1}{3}+\frac{1}{3}-\frac{1}{5}-\frac{1}{21}\right)=$$

$$=2\cdot\frac{70-21-5}{105}=\frac{88}{105};\ (\textit{kub birlik}).$$

8-misol. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *ellipsoidning hajmini toping.*

Yechim. Berilgan tenglamadan z ni aniqlaymiz:

$$z=\pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}.$$

Bu yerda D sohaning konturi $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellips bo‘ladi.

O‘zgaruvchilarni $x=ar\cos\varphi,\ y=br\sin\varphi$ formulalar yordamida almashtiramiz.

Natijada $D'=\{(\varphi,r):0\le\varphi\le 2\pi;0\le r\le 1\}$

$$\frac{\partial x}{\partial\varphi}=-ar\sin\varphi;\ \frac{\partial x}{dr}=a\cos\varphi$$

$$\frac{\partial y}{\partial\varphi}=br\cos\varphi\ ;\ \frac{\partial y}{\partial r}=b\sin\varphi$$

Yakobian

$$I(\varphi,r)=abr\begin{vmatrix}-\sin\varphi & \cos\varphi\\ \cos\varphi & \sin\varphi\end{vmatrix}=abr(-\sin^2\varphi-\cos^2\varphi)=-abr\ ;\ |I(\varphi,r)|=abr.$$

Ellipsoidning XOY tekisligidan yuqorida joylashgan bo‘lagining hajmini hisoblab, natijani 2 ga ko‘paytirsak, uning hajmi kelib chiqadi.

$$\frac{1}{2}V=\iint_D c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}dxdy=c\iint_{D'}\sqrt{1-r^2\cos^2\varphi-r^2\sin^2\varphi}\cdot|I(\varphi,r)|\cdot d\varphi dr=$$

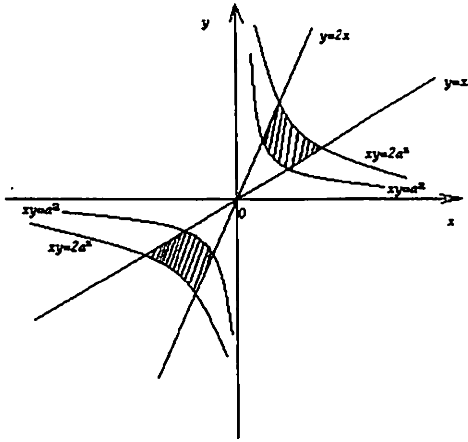
$$=c\int_0^1 dr\int_0^{2\pi}\sqrt{1-r^2}\cdot abrd\varphi=abc\cdot 2\pi\int_0^1\sqrt{1-r^2}r\cdot dr=-\pi abc\cdot\frac{\frac{(1-r^2)^{\frac{3}{2}}}{3}}{2}\Bigg|_0^1=\frac{2}{3}\pi abc$$

(kub birlik).

Bundan $V=\frac{4}{3}\pi abc$ kelib chiqadi.

9-misol. $xy=a^2$, $xy=2a^2$ *silindrlar,* $y=\frac{x}{2}$, $z=0$, $y=2x$ *tekisliklar hamda* $z=\frac{x^2}{p}+\frac{y^2}{q}$ *paraboloid bilan chegaralangan jismning hajmini toping.*

Yechim. Integrallash sohasini XOY koordinatalar tekisligida tasvirlaymiz:



Shaklda ko'rsatilgan sohalar simmetrik bo'lgani uchun va paraboloid tenglamasida

$z = \frac{(-x)^2}{p} + \frac{(-y)^2}{q} = \frac{x^2}{p} + \frac{y^2}{q}$ tenglik o'rinli bo'lgani uchun $x > 0,\ y > 0$ bo'lgan sohada jism hajmini hisoblab natijani 2 ga ko'paytirsak, izlanayotgan jismning hajmi kelib chiqadi.

Egri chiziqli koordinatalarni

$$xy = ua^2,\ 1 \le u \le 2;$$

$$y = vx\ ,\ \frac{1}{2} \le v \le 2$$

tengliklar yordamida kiritamiz. Bu tengliklardan $x,\ y$ o'zgaruvchilarnilarni aniqlaymiz: $x = a\sqrt{\frac{u}{v}}\ ,\ y = a\sqrt{uv}$. Bundan

$$\frac{\partial x}{\partial u} = \frac{a}{2\sqrt{uv}},\ \frac{\partial x}{\partial v} = -\frac{au}{2v\sqrt{uv}}$$

$$\frac{\partial y}{\partial u} = \frac{av}{2\sqrt{uv}},\ \frac{\partial y}{\partial v} = \frac{au}{2\sqrt{uv}}$$

xususiy hosilalarni aniqlaymiz. Bularni $I(u,v)$ yakobianda o'rniga qo'ysak va umumiy ko'paytuvchilarni determinant ishorasidan tashqariga chiqarsak, quyidagi kelib chiqadi:

$$I(u,v) = \frac{a^2}{4uv}\begin{vmatrix} 1 & -\frac{u}{v} \\ v & u \end{vmatrix} = \frac{a^2}{4uv}\cdot 2u = \frac{a^2}{2v}.$$

Natijada

$$\frac{1}{2}V = \iint_{D} f(x,y)dD = \iint_{D'} f(x(u,v),y(u,v))|I(u,v)|dudv =$$

$$= \int_1^2 dv \int_1^2 \left(\frac{a^2 u}{pv} + \frac{a^2 uv}{q}\right)\frac{a^2}{2v}du = \frac{a^4}{2}\int_1^2 udu \int_1^2 \left(\frac{1}{pv} + \frac{v}{q}\right)\frac{1}{v}dv =$$

$$= \frac{a^4}{4}u^2\Big|_1^2 \cdot \int_1^2 \left(\frac{1}{pv^2} + \frac{1}{q}\right)dv = \frac{3a^4}{4}\left(-\frac{1}{pv} + \frac{v}{q}\right)\Big|_1^2 =$$

$$= \frac{3a^4}{4}\left(-\frac{1}{2p} + \frac{2}{q} + \frac{2}{p} - \frac{1}{2q}\right) = \frac{3a}{4}\left(\frac{3}{2p} + \frac{3}{2q}\right) = \frac{9a^4}{8}\left(\frac{1}{p} + \frac{1}{q}\right) = \frac{9a^4(p+q)}{8pq}$$ (kub birlik).

Bundan

$$V = \frac{9a^4(p+q)}{4pq}$$

kelib chiqadi.

3.13. Ikki karrali itegrallarning mexanikada qo'llanilishi

Ikki karrali integrallarni jismning massasini, o'qqa va tekislikka nisbatan statik va inersiya momentlarini, shuningdek og'irlik markazlarini hisoblashda qo'llanilishi.

Umumiy tushunchalar. Bu qism bo'yicha misollar yechish uchun ikki karrali va uch karrali integrallarni mexanikada qo'llanilishi mavzuni mukammal takrorlab chiqing. Endi ba'zi muhum formulalar bilan tanishamiz.

Agar (P) biror tekis figura (masalan, plastinka) bo'lib, uning $M(x,y)$ nuqtalaridagi zichligi $\rho(M) = \rho(x,y)$ bo'lsa, u holda (P) figuraning (palstinkaning) massasi

$$m = \iint_P \rho(x,y)dp \qquad (1)$$

formula bilan aniqlanadi . Agar (P) figura bir jinsli bo'lsa , ya'ni hamma nuqtalarida zichligi bir xil bo'lsa , u holda $\rho(x,y) = a(const)$ bo'ladi, (1) formula esa ,

$$m = a\iint_P dp \qquad (2)$$

ko'rinishni oladi .

OX va OY koordinata o‘qlariga nisbatan statik momentlar quyidagi formulalar bilan aniqlanadi.

$$M_x = \iint_P y\rho(x,y)dp \quad M_y = \iint_P x^2\rho(x,y)dp \qquad (3)$$

Inersiya momentlari esa,

$$I_x = \iint_P y^2\rho(x,y)dp, \quad I_y = \iint_P x^2\rho(x,y)dp, \qquad (4)$$

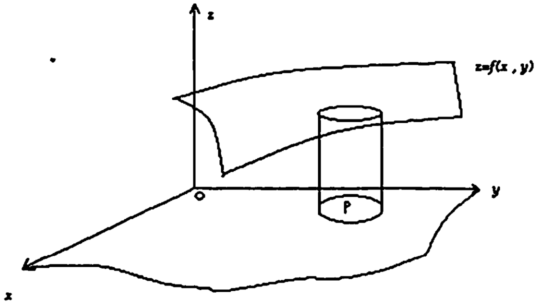
formulalar yordamida topiladi. Qaralayotgan jismning og‘irlik markazining koordinatalari quyidagicha topiladi:

$$x_0 = \frac{\iint_P x\rho(x,y)dp}{m}; \qquad y_0 = \frac{\iint_P y\rho(x,y)dp}{m} \qquad (5)$$

Agar jism bir jinsli bo‘lsa, ya’ni $\rho(x,y) = const$ bo‘lsa, (5) formulalar soddalashadi va

$$x_0 = \frac{\iint_P xdp}{p}; \qquad y_0 = \frac{\iint_P ydp}{p} \qquad (6)$$

ko‘rinishini oladi. Bu yerda p berilgan figuraning yuzi.



Yasovchilari Oz o‘qiga parallel, ostki asosii xOy tekisligida va ustki asosi esa $z = f(x,y)$ sirt bilan chegaralangan silindrik g‘o‘lani qaraymiz.

G‘o‘la bir jinsli bo‘lsin, ya’ni $\rho(x,y) = const$. Soddalik uchun $\rho(x,y) = 1$ deb olamiz. Shu g‘o‘lani *xoy, zox* va *yoz* koordinata tekisliklariga nisbatan statik momentlari deb quyidagi formulalar bilan aniqlanuvchi kattaliklarga aytiladi:

$$M_{xy} = \frac{1}{2}\iint_P z^2dp; \quad M_{xz} = \iint_P yzdp; \quad M_{yz} = \iint_P xzdp \qquad (7)$$

Bular yordamida g‘o‘laning og‘irlik markazining koordinatalarini osongina aniqlay olamiz.

$$x_0 = \frac{M_{yz}}{V} = \frac{\iint_P xzdp}{V}; \quad y_0 = \frac{M_{xz}}{V} = \frac{\iint_P yzdp}{V}; \quad z_0 = \frac{M_{xy}}{V} = \frac{\frac{1}{2}\iint_P z^2dp}{V}; \qquad (8)$$

Shunga o'xshash silindrik g'o'laning koordinata tekisliklariga nisbatan inersiya momentlari quyidagicha aniqlanadi:

$$I_{zx} = \iint_P y^2 z dp, \quad I_{zy} = \iint_P x^2 z dp \quad (9)$$

Oz o'qiga nisbatan inersiya momenti esa

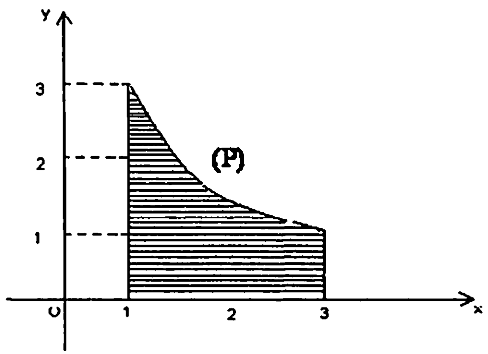
$$I_z = I_{zx} + I_{zy} = \iint_P (x^2 + y^2) z dp \quad (10)$$

formula bilan hisoblanadi. Agar g'o'laning yasovchilari Ox yoki Oy o'qiga parallel bo'lsa, u holda (7), (8), (9) va (10) formulalarda z o'zgaruvchi x yoki y bilan almashtiriladi. Agar qaralayotgan jismning zichligi xOy tekisligiga parallel qismda yotuvchi nuqtalari uchun $\rho(x,y)$ ga $(\rho(x,y) \neq 1)$ teng bo'lib, Oz o'qiga parallel ustunchalarida o'zgarmas bo'lsa, u holda (7), (8),(9) va (10) formulalarda integral ostidagi ifoda $\rho(x,y)$ ga ko'paytiriladi.

Ikki karrali integralning mexanik tatbiqlariga misollar.

1-misol. (P) *tekis figura egri chiziqli trapetsiya ko'rinishida bo'lib,* $x=1, x=3$ *va* $y=0$ *to'g'ri chiziqlar,* $y=\frac{3}{x}$ *giperbola bilan chegaralangan bo'lsin. Uning hamma nuqtalarida zichligi* $\rho(x,y)=1$ *bo'lsa shu tekis figuraning massasini va koordinata o'qlariga nisbatan statik momentlarni toping.*

Yechim.



$P = \{(x,y) \in R^2 : 1 \le x \le 3;\ 0 \le y \le \frac{3}{x}\}$, (1) formulada $\rho(x,y)=1$ qo'ysak, figuraning massasi kelib chiqadi.

$$m = \iint_P dp = \int_1^3 dx \int_0^{\frac{3}{x}} dy = \int_1^3 \frac{3}{x} dx = 3\ln x\Big|_1^3 = 3\ln 3$$

Shunga o'xshash (3) formulalarda ham $\rho(x,y)=1$ qo'ysak

$$M_x = \iint_P y\,dp = \int_1^3 dx \int_0^{\frac{3}{x}} y\,dy = \int_1^3 \frac{y^2}{2}\Big|_0^{\frac{3}{x}} dx = \int_1^3 \frac{9}{2x^2}dx = -\frac{9}{2x}\Big|_1^3 = -\frac{9}{6}+\frac{9}{2} = \frac{9}{2}+\frac{3}{2} = 3$$

$$M_y = \iint_P x\,dp = \int_1^3 dx \int_0^{\frac{3}{x}} x\,dy = \int_1^3 x\cdot\frac{3}{x}dx = 3\cdot 2 = 6$$

Ox va Oy o‘qlariga nisbatan statik momentlar kelib chiqadi.

2-misol. $z=0, z=ky \ (k>0)$ *tekisliklar* $x^2+y^2=a^2$ *slindr bilan chegaralangan bir jinsli silindrik g‘o‘la kesimi og‘irlik markazining koordinatalarini toping* $(y \geq 0)$.

Yechim. Qaralayotgan jism bir jinsli bo‘lgani uchun zichligi $\rho(x,y,z)=1$ deb olamiz. $y \geq 0$ bo‘lgani uchun $z = ky \geq 0$ bo‘ladi.

Demak, $p = \{(x,y) \in R^2 : -a \leq x \leq a;\ 0 \leq y \leq \sqrt{a^2-x^2}\}$ -bu slindrik g‘o‘laning xOy tekisligida yotuvchi asosi. (7) formulalarga ko‘ra quyidagilarni topamiz:

$$M_{xy} = \frac{1}{2}\iint_P z^2 dp = \frac{1}{2}\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} k^2y^2dy = \frac{k^2}{2}\int_{-a}^{a}\frac{(a^2-x^2)^{\frac{3}{2}}}{3}dx = \frac{\pi}{16}k^2a^2$$

$$M_{xz} = \iint_P yz\,dp = k\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} y^2dy = \frac{k}{3}\int_{-a}^{a}(a^2-x^2)^{\frac{3}{2}}dx = \frac{\pi}{8}ka^4$$

$$M_{yz} = \iint_P xz\,dp = k\int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} xy\,dy = k\int_{-a}^{a}\frac{xy^2}{2}\Big|_0^{\sqrt{a^2-x^2}} dx = \frac{k}{2}\int_{-a}^{a}(a^2x-x^3)dx = 0$$

Berilgan jisimning hajmi V bo‘lsa, u holda

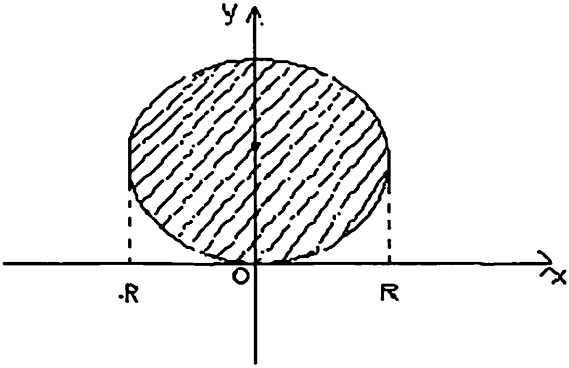
$$V = \iint_P z\,dp = \int_{-a}^{a} dx \int_0^{\sqrt{a^2-x^2}} ky\,dy = \frac{k}{2}\int_{-a}^{a}(a^2-x^2)dx = \frac{k}{2}\left(a^2x-\frac{x^3}{3}\right)\Big|_{-a}^{a} = k\left(a^3-\frac{a^3}{3}\right) = \frac{2}{3}ka^3$$

Bularni (8) formulalarga qo‘ysak, og‘rlik markazining koordinatalari kelib chiqadi:

$$x_0 = \frac{M_{yz}}{V} = 0; \quad y_0 = \frac{M_{xz}}{V} = \frac{\frac{\pi}{3}ka^4}{\frac{2}{3}ka^3} = \frac{3\pi}{16}a; \quad z_0 = \frac{M_{xy}}{V} = \frac{\frac{\pi}{16}k^2a^4}{\frac{2}{3}ka^3} = \frac{3\pi}{32}ka.$$

3-misol. *Tekis figura R radusli doira shaklda bo‘lsa, uning urunmasiga nisbatan inersiya momentini toping.*

Yechim. Bu yerda shu narsaga e’tibor qilish kerakki, doira aylanasiga urinmalar soni cheksiz ko‘p, lekin shu urinmalarga nisbatan inersiya momentlari hamma urinmalar uchun bir xil bo‘ladi. Shuning uchun Ox o‘qiga koordinatalar boshida urinuvchi doiraning shu Ox o‘qiga nisbatan inersiya momentini aniqlasak, masala yechilgan bo‘ladi.



Aylana tenglamasi $x^2+(y-R)^2=R^2$, bundan $R-\sqrt{R^2-x^2}\le y\le R+\sqrt{R^2-x^2}$

$$P=\{(x,y)\in R^2:-R\le x\le R;\ R-\sqrt{R^2-x^2}\le y\le R+\sqrt{R^2-x^2}\}$$

(4) formulaga ko'ra quyidagini topamiz $(\rho(x,y)=1)$:

$$I_x=\iint_P y^2dP=\int_{-R}^{R}dx\int_{R-\sqrt{R^2-x^2}}^{R+\sqrt{R^2-x^2}}y^2dy=\frac{1}{3}\int_{-R}^{R}\left[\left(R+\sqrt{R^2-x^2}\right)^3-\left(R-\sqrt{R^2-x^2}\right)^3\right]dx=$$

$$-3R(R^2-x^2)+(R^2-x^2)^{\frac{3}{2}}dx=\frac{2}{3}\int_0^R\left[6R^2\sqrt{R^2-x^2}+2(R^2-x^2)^{\frac{3}{2}}\right]dx$$

$x=R\sin t$ belgilash kiritamiz.

$$dx=R\cos t\,dt\ ;\ x=0\Rightarrow t=0;\ x=R\Rightarrow t=\frac{\pi}{2}$$

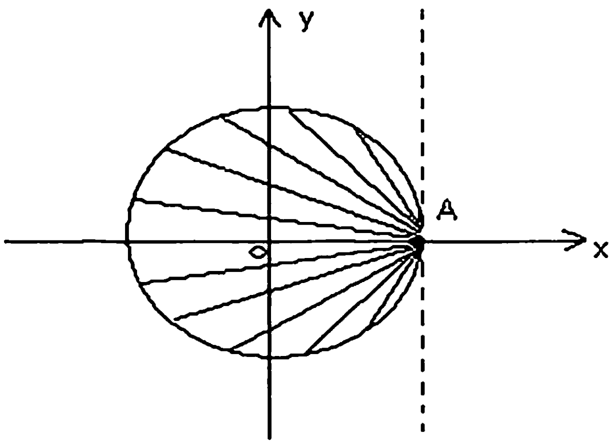
Bularni o'rniga qo'ysak,

$$I_x=\frac{4}{3}\int_0^{\frac{\pi}{2}}(3R^2\cdot R\cos t+R^3\cos^3 t)R\cos t\,dt=\frac{4}{3}R^4\int_0^{\frac{\pi}{2}}(3\cos^2 t+\cos^4 t)dt=$$

$$\frac{4}{3}R^2\int_0^{\frac{\pi}{2}}\left[3\cdot\frac{1+\cos 2t}{2}+\left(\frac{1+\cos 2t}{2}\right)^2\right]dt=\pi R^4+\frac{1}{3}R^4\int_0^{\frac{\pi}{2}}\left(1+\frac{1+\cos 4t}{2}\right)dt=$$

$$=\pi R^4+\frac{1}{3}R^4\cdot\frac{3}{2}\cdot\frac{\pi}{2}=\pi R^4+\frac{\pi R^4}{4}=\frac{5}{4}\pi R\ .$$

4-misol. *$x^2+y^2\le a^2$ doiraviy plastinkaning har bir $M(x,y)$ nuqtasidagi zichligi shu nuqtadan $A(a,0)$ nuqtagacha bo'lgan masofaga proporsional bo'lib, plastinkaning markazida a ga teng bo'lsa, uning og'irlik markazi koordinatlarini toping.*



$P = \{(x, y) \in R^2 : -a \le x \le a;\ -\sqrt{R^2 - x^2} \le y \le \sqrt{R^2 - x^2}\}.$

Shartga ko'ra $M(x,y)$ nuqtadagi zichligi $\rho(M) = |MA| \cdot k$, demak, $\rho(x,y) = k\sqrt{(x-a)^2 + y^2}$, $\rho(0,0) = a$ bo'lgani uchun

$k \cdot \sqrt{(0-a)^2 + 0^2} = a \Rightarrow ka = a \Rightarrow k = 1$

Bundan $\rho(x,y) = k\sqrt{(x-a)^2 + y^2}$ kelib chiqadi.

Plastinkaning massasini topamiz.

$$m = \iint_P \rho(x,y)dp = \int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{(x-a)^2 + y^2}\,dy$$

$$\left.\begin{array}{l} x - a = r\cos\varphi \\ y = r\sin\varphi \end{array}\right\}$$

Bu yerda x, y larni qiymatlarini platinka aylanasining tenglamasiga qo'ysak, r ning φ orqali ifodasi kelib chiqadi.

$$(a + r\cos\varphi)^2 + r^2\sin^2\varphi = a^2 \Rightarrow a^2 + 2ar\cos\varphi + r^2(\cos^2\varphi + \sin^2\varphi) = a^2$$

$$\Rightarrow r = -2a\cos\varphi, \ \left(\frac{\pi}{2} \le \varphi \le \frac{3\pi}{2}\right).$$

Qutb boshi *A(a,0)* nuqtada yotadi (shaklga qarang).

Natijada,

$$m = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\varphi \int_0^{-2a\cos\varphi} r^2 dr = \frac{1}{3}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (-8a^3)\cos^3\varphi d\varphi = -\frac{8a^3}{3}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1 - \sin^2\varphi)d(\sin\varphi) =$$

$$= -\frac{8a}{3}\left(\sin\varphi - \frac{\sin^3\varphi}{3}\right)\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -\frac{8a^2}{3}\left(-1-1+\frac{1}{3}+\frac{1}{3}\right) = -\frac{8a^2}{3}\left(-\frac{1}{3}\right) = \frac{32a^2}{9}$$

Endi o‘qlarga (Ox, Oy) nisbatan statik momentlarni aniqlaymiz.

$$M_y = \iint_P y\rho(x,y)dp = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\varphi \int_0^{-2a\cos\varphi} r^3 \sin\varphi dr = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{16a^4\cos^4\varphi}{4}\sin\varphi d\varphi =$$

$$= -4a^4 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^4\varphi d(\cos\varphi) = -4a^4 \frac{\cos^5\varphi}{5}\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = 0$$

$$M_y = \iint_P y\rho(x,y)dp = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\varphi \int_0^{-2a\cos\varphi} (r\cos\varphi + a)r^2 dr = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{r^4}{4}\cos\varphi + a\cdot\frac{r^3}{3}\right)\Bigg|_0^{-2a\cos\varphi} d\varphi =$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(4a^4\cos^5\varphi - \frac{a}{3}8a^3\cos^3\varphi\right)d\varphi = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[(4a^4(1-\sin^2\varphi)^2 d\varphi - \frac{8}{3}a^4(1-\sin^2\varphi)\right]d(\sin\varphi) =$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[4a^4(1-2\sin^2\varphi + \sin^4\varphi) - \frac{8}{3}a^4(1-\sin^2\varphi)\right]d(\sin\varphi) =$$

$$= \left[4a^4\left(\sin\varphi - 2\frac{\sin^3\varphi}{3} + \frac{\sin^5\varphi}{5}\right) - \frac{8}{3}a^4\left(\sin\varphi - \frac{\sin^3\varphi}{3}\right)\right]\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = 4a^4\left(-2+\frac{4}{3}-\frac{2}{5}\right) -$$

$$-\frac{8}{3}a^4\left(-2+\frac{2}{3}\right) = 4a^4\frac{-30+20-6}{15} - \frac{8}{3}a^4\left(-\frac{4}{3}\right) =$$

$$= -\frac{64}{15}a^4 + \frac{32}{9}a^4 = \frac{-192+160}{45}a^4 = -\frac{32}{15}a^4$$

Bularni (5) formulalarga qo‘yib, og‘irlk markazining koordinatalarini topamiz:

$$x_0 = \frac{M_y}{m} = -\frac{32}{45}a^4 : \frac{32}{9}a^3 = -\frac{9}{5}.$$

IV BOB. UCH KARRAL INTEGRALLAR

4.1. Uch karrali integral

Uch karrali integral ham ikki karrali integral kabi aniqlanadi. Fazoda hajmi V ga teng bo'lgan jism berilgan bo'lsin. Bu jismning (sohaning) har bir nuqtasida uzluksiz $u = f(P)$ yoki $u = f(x, y, z)$ funksiya aniqlangan bo'lsin.

Sohani ixtiyoriy ravishda umumiy ichki nuqtaga ega bo'lmagan va hajmlari $\Delta V_1, \Delta V_2, \dots, \Delta V_n$ bo'lgan n ta bo'lakka bo'lamiz, bunda $V = \sum_{i=1}^{n} \Delta V_i$. Har bir bo'lakda ixtiyoriy $P_i(x_i, y_i, z_i)$ nuqta tanlab, bu nuqtada $f(P_i) = f(x_i, y_i, z_i)$ ni hisoblaymiz va

$f(x_i, y_i, z_i)\Delta V_i$ ko'paytmani tuzamiz. Bu ko'paytmalardan

$$I_n = \sum_{n=1}^{r} f(x_i, y_i, z_i)\Delta V_i \qquad (1)$$

yig'indisini hosil qilamiz. Bu yig'indiga $u = f(x, y, z)$ *funksiya uchun* V *sohada integral yig'indi* deyiladi. $n \to \infty$ da bo'laklar diametrlarining eng kattasi nolga intiladi, ya'ni $\max\limits_{d_i \to 0} d_i \to 0$.

1-ta'rif. Agar (1) integral yig'indining $\max d_i \to 0$ gi chekli limiti V sohani bo'laklarga bo'lish usuliga va bu bo'laklarda $P_i(x_i, y_i, z_i)$ nuqtani tanlash usuliga bog'liq bo'lmagan holda mavjud bo'lsa, u holda bu limitga $f(x, y, z)$ funksiyadan V soha bo'yicha olingan ***uch karrali integral*** deyiladi va $\iiint\limits_V f(x, y, z)dV$ kabi belgilanadi.

Demak,

$$\iiint\limits_V f(x, y, z)dV = \lim_{\max d_i \to \infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i . \qquad (2)$$

Uch karrali integral uchun ham ikki karrali integraldagidek mavjudlik teoremasi o'rinli bo'ladi.

Uch karrali integralni ikki karrali integralga o'xshash quyidagicha belgilash mumkin:

$$\iiint\limits_V f(x, y, z)dxdydz .$$

Agar V sohada $f(x, y, z) \equiv 1$ bo'lsa, u holda uch karrali integral bu sohaning $\bar{V}$ hajmiga teng bo'ladi, ya'ni

$$V = \iiint\limits_V dxdydz . \qquad (3)$$

Bu ifoda uch karrali integralning *geometrik ma'nosi*ni anglatadi.

Agar $\gamma_1 = \gamma(x,y,z)$ funksiya V sohada massa taqsimotining zichligi bo'lsa, u holda uch o'lchovli integral V hajmdagi modda massasini beradi:

$$m = \iiint_V \gamma(x,y,z)dxdydz \qquad (4)$$

Bu ifoda uch karrali integralning *mexanik ma'nosi*ni bildiradi.

Uch karrali integral ikki karrali integral ega bo'lgan xossalarga ega. Shu sababli ikki karrali integralda keltirilgan xossalar uch karrali integral uchun to'laligicha ko'chiriladi.

1°. $\iiint_V kf(x,y,z)dV = k\iiint_V f(x,y,z)dV$.

2°. $\iiint_V (f(x,y,z) \pm g(x,y,z))dV = \iiint_V f(x,y,z)dV \pm \iint_V g(x,y,z)dV$.

3°. $\iiint_V f(x,y,z)dV = \iiint_{V_1} f(x,y,z)dV_1 + \iint_{V_2} f(x,y,z)dV_2 + ... + \iint_{V_n} f(x,y,z)dV_n$,

bunda V soha $V_1, V_2, ..., V_n$ o'zaro kesishmaydigan sohalardan tashkil topgan.

4°. Agar V sohada $f(x,y,z) \geq 0$ $(f(x,y,z) \leq 0)$ bo'lsa, u holda

$$\iiint_V f(x,y,z)dV \geq 0 \quad \left(\iiint_V f(x,y,z)dV \leq 0\right).$$

5°. Agar V sohada $f(x,y,z) \geq \varphi(x,y,z)$ $(f(x,y,z) \leq \varphi(x,y,z))$ bo'lsa, u holda

$$\iiint_V f(x,y,z)dV \geq \iiint_V \varphi(x,y,z)dV \quad \left(\iiint_V f(x,y,z)dV \leq \iiint_V \varphi(x,y,z)dV\right).$$

6°. $\iiint_V f(x,y,z)dV = f(x_0,y_0,z_0)V$, bu yerda $P_i(x_i,y_i,z_i)$ nuqta V sohada yotadi.

7°. Agar V sohada $m \leq f(x,y,z) \leq M$ bo'lsa, u holda

$$m\overline{V} \leq \iiint_V f(x,y,z)dV \leq M\overline{V}.$$

4.2. Uch karrali integrallar yordamida jismning hajmini hisoblash

Ushbu mustaqil ishni bajarishdan maqsad. Uch karrali integrallarni hisoblashga doir berilgan nazariy tushunchalardan foydalanib, shu integrallarni geometrik jismlarning hajmlarini hisoblashda qo'llay bilish. Uch karrali integrallar yordamida turli xil sirtlar bilan chegaralangan jismlarning hajmlarini hisoblay bilish.

Umumiy tushunchalar. Bu qism bo'yicha misollar yechishga kirishishdan oldin $f(x,y,z)$ funksiyadan V soha bo'yicha olingan uch karrali integrallarning mavjudlik shartlari va uni hisoblash usullarini ma'ruzadan takrorlab o'qing. Bu yerda V uch o'lchovli soha (jism) bo'lib, u silliq yoki bo‘lakli-silliq sirt bilan chegaralangan. Agar V soha quyidan $z=f_1(x,y)$ yuqoridan $z=f_2(x,y)$ sirtlar bilan chegaralangan bo‘lsa, bundan tashqari V sohada x,y lar orasida $\varphi_1(x)\le y\le\varphi_2(x)$ tengsizlik o‘rinli bo‘lib, x esa shu sohada $[a,b]$ oraliqda qiymatlar qabul qilsa, u holda agar $f(x,y,z)$ funksiyadan V soha bo‘yicha olingan uch karrali integral mavjud va quyidagi tenglik o‘rinli bo‘ladi:

$$\iiint\limits_V f(x,y,z)dv=\int\limits_a^b dx\int\limits_{\varphi_1(x)}^{\varphi_2(x)}dy\int\limits_{f_1(x,y)}^{f_2(x,y)}f(x,y,z)dz \quad (5)$$

agar $f(x,y,z)=1$ bo‘lsa (4) tenglik V sohaning hajmini aniqlaydi:

$$\iiint\limits_V dv=\int\limits_a^b dx\int\limits_{\varphi_1(x)}^{\varphi_2(x)}dy\int\limits_{f_1(x,y)}^{f_2(x,y)}dz \quad (6)$$

(6) tenglik o‘ng tomonini z bo‘yicha integrallasak,

$$\iiint\limits_V dv=\int\limits_a^b dx\int\limits_{\varphi_1(x)}^{\varphi_2(x)}\left(f_2(x,y)-f_1(x,y)\right)dy$$

kelib chiqadi. Bu tenglikning o‘ng tomonini bizga tanish bo‘lgan ikki karrali integralning takroriy integrallar orqali ifodalanishidir.

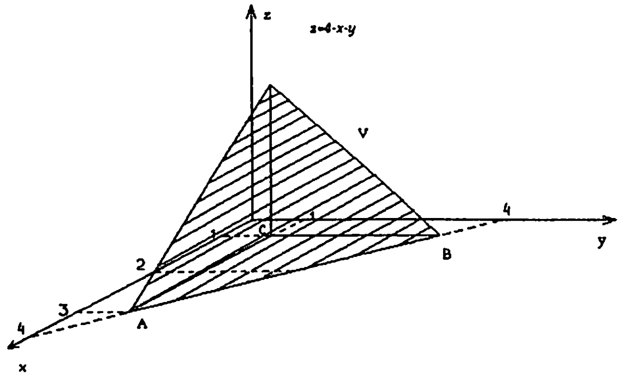
1. Uch karrali integrallarni hisoblashda sferik va silindrik koordinatalardan foydalanish.

2. Uch karrali integrallarda o‘zgaruvchilarni almashtirish.

Endi tipik misollarni yechishga o‘tamiz.

1-misol. $x-1=0,\ y-1=0,\ z=0$ *va* $x+y+z=4$ *tekisliklar bilan chegaralangan jismning (piramidaning) hajmini toping.*

Yechim.



$x+y+z=4$ tenglamada $z=0$ qo‘ysak, $x+y=4$ bo‘ladi. Bu XOY tekislikda (AB) to‘g‘ri chiziq tenglamasidir. $x+y=4$ tenglamada $y=1$

qo'ysak, A nuqtaning absissasi kelib chiqadi $x+1=4 \Rightarrow x=3$. Demak, v sohada: $1 \le x \le 3;\ 1 \le y \le 4-x;\ 0 \le z \le 4-x-y$. Bularni (5) formulaga qo'ysak,

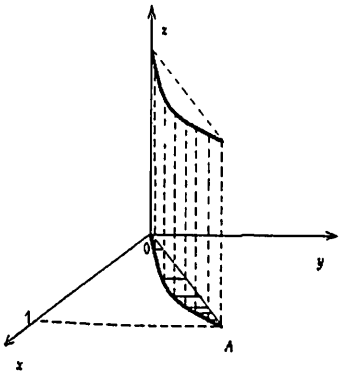
$$\iiint_V dv = \int_1^3 dx \int_1^{4-x} dy \int_0^{4-x-y} dz = \int_1^3 dx \int_1^{4-x} dy \int_1^{4-x-y} dz = \int_1^3 dx \int_1^{4-x} (4-x-y)dy =$$

$$= \int_1^3 \left(4y - xy - \frac{y^2}{2}\right)\Bigg|_1^{4-x} dx = \int_1^3 \left[4(4-x) - x(4-x) - \frac{(4-x)^2}{2} - 4 + x + \frac{1}{2}\right]dx =$$

$$= \int_1^3 (16 - 4x - 4x + x^2 - 8 + 4x - \frac{1}{2}x^2 + x - 3{,}5)dx = \int_1^3 \left(\frac{1}{2}x^2 - 3x + 4{,}5\right)dx =$$

$$= \left(\frac{x^3}{6} - \frac{3x^2}{2} + 4{,}5x\right)\Bigg|_1^3 = \frac{27}{6} - \frac{27}{2} + 13{,}5 - \frac{1}{6} + \frac{3}{2} - 4{,}5 = 13{,}5 - 12 - \frac{1}{6} = 1{,}5 - \frac{1}{6} = \frac{4}{3}$$

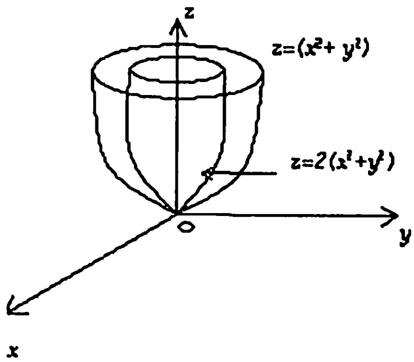
kub birlik bo'ladi.



2-misol.

$z = x^2 + y^2, z = 2(x^2 + y^2), y = x, y = x^2$ *sirtlar bilan chegaralangan jismning hajmini toping.*

Yechim. Berilgan sirtlarning dastlabki ikkitasi aylanma paraboloid bo'lib, uchinchisi Oz o'qidan o'tuvchi tekislikdir, $y = x^2$ esa yasovchilari Oz o'qiga parallel bo'lgan parabolik silindr ekani bizga ma'lum.



Demak, 1-shaklda tasvirlangan jismning $z = x^2 + y^2$ va $z = 2(x^2 + y^2)$ aylanma paraboloidlar orasiga olingan bo'lagining hajmini hisoblash talab etiladi. $\begin{cases} y = x \\ y = x^2 \end{cases}$ sistemadan x ni topamiz:

$$\begin{cases} y = x \\ y = x^2 \end{cases} \Rightarrow \begin{cases} x^2 = x \\ y = x^2 \end{cases} \Rightarrow \begin{cases} x^2 - x = 0 \\ y = x^2 \end{cases} \Rightarrow \begin{cases} x_1 = 0, x_2 = 1 \\ y = x^2 \end{cases}$$

Bundan $V = \{x, y, z : 0 \le x \le 1;\ x^2 \le y \le x;\ x^2 + y^2 \le z \le 2(x^2 + y^2)\}$. Bularni (6) formulaga qo'ysak, quydagi kelib chiqadi:

$$V = \iiint_V dv = \int_0^1 dx \int_{x^2}^{x} dy \int_{x^2+y^2}^{2(x^2+y^2)} dz = \int_0^1 dx \int_{x^2}^{x} (x^2 + y^2) dy =$$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3} \right)\Bigg|_{x^2}^{x} dx = \int_0^1 \left(x^3 + \frac{x^3}{3} - x^4 - \frac{x^6}{3} \right) dx = \int_0^1 \left(\frac{4}{3}x^3 - x^4 - \frac{x^6}{3} \right) dx = \left(\frac{x^4}{3} - \frac{x^5}{5} - \frac{x^7}{21} \right)\Bigg|_0^1 =$$

$$= \frac{1}{3} - \frac{1}{5} - \frac{1}{21} = \frac{35 - 21 - 5}{105} = \frac{9}{105} = \frac{3}{55} \text{ (kub birlik).}$$

3-misol. $(x^2 + y^2 + z^2)^3 = a^3 xyz$ *sirt bilan chegaralangan jismning hajmini toping.*

Yechim. Tenglikning chap tomoni kvadratlar yig'indisi bo'lgani uchun manfiy bo'la olmaydi. Demak, tenglik o'rinli bo'lishi uchun xyz ko'paytma manfiy bo'lmasligi kerak. $xyz \Rightarrow$

$\Rightarrow$ 1) $x \ge 0, y \ge 0, z \ge 0$;   2) $x \ge 0, y \le 0, z \le 0$

3) $x \le 0, y \ge 0, z \le 0$;   4) $x \le 0, y \le 0, z \ge 0$

Berilgan jism kordinatalar sistemasining 4ta oktantida yotar ekan. Agar $M(x, y, z)$ qaralayotgan jismning ixtiyoriy nuqtasi bo'lsa, unga OX o'qiga nisbatan simmetrik $M_1(x, -y, -z)$ nuqta, OY o'qiga nisbatan simmetrik $M_2(-x, y, -z)$ nuqta va OZ o'qiga nisbatan simmetrik $M_3(-x, -y, z)$ nuqtalar mavjuddir. Bundan berilgan jismning yuqorida ko'rsatilgan 4 ta oktantdagi bo'laklarining kattaliklari bir xil ekani kelib chiqadi. Shuning uchun jismning 1 oktantdagi bo'lagining hajmini hisoblab, natijani 4 ga ko'paytirsak, berilgan jismning hajmi kelib chiqadi. Sferik koordinatalar kiritamiz:

$$x = r \sin\psi \cos\varphi; \quad y = r \sin\psi \sin\varphi; \quad z = r \cos\psi$$

1-oktantda $0 \le \psi \le \frac{\pi}{2}$ va $0 \le \varphi \le \frac{\pi}{2}$ bo'ladi. Yakobian $|I(r, \varphi, \psi)| = r^2 \sin\psi$ va sferik koordinatalarni berilgan tenglamaga qo'ysak, quyidagiga ega bo'lamiz:

$$\left[r^2 \sin^2\psi \left(\cos^2\varphi + \sin^2\varphi \right) + r^2 \cos^2\psi \right]^3 = a^3 r^3 \sin^2\psi \cos\varphi \cos\psi \sin\psi \Rightarrow$$

$$\Rightarrow r^3 = a^3 \sin^2\psi \cos\varphi \cos\psi \sin\varphi \Rightarrow \quad r = a\sqrt[3]{\sin^2\varphi \cos\varphi \cos\psi \sin\psi}$$

Demak,

$$V = \iiint_V dV = 4\int_0^{\frac{\pi}{2}} d\psi \int_0^{\frac{\pi}{2}} d\varphi \int_0^{a\sqrt[3]{\sin^2\psi \cos\varphi \cos\psi \sin\varphi}} r^2 \sin\psi dr =$$

$$= \frac{4}{3}\int_0^{\frac{\pi}{2}} d\psi \int_0^{\frac{\pi}{2}} a^3 \sin^3\psi \cos\varphi \cos\psi \sin\varphi \, d\varphi = \frac{4}{3} a^3 \int_0^{\frac{\pi}{2}} \cos\varphi \sin\varphi \, d\varphi \cdot$$

$$\cdot \int_0^{\frac{\pi}{2}} \sin^3\psi \cos\psi d\psi = \frac{4}{3} a^3 \left[\frac{\sin^2\varphi}{2}\Bigg|_0^{\frac{\pi}{2}} \cdot \frac{\sin^4\varphi}{4}\Bigg|_0^{\frac{\pi}{2}} \right] = \frac{4}{3} a^3 \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{a^3}{6} \text{ (kub birlik).}$$

4-misol $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ *sirt bilan chegaralangan jismning hajmini toping.*

Yechim. Tenglikning har ikkala tomoni kvadratlar yig'indisi bo'lgani uchun berilgan jismning hamma oktantlardagi bo'laklarining kattaliklari bir hil bo'ladi.

Shuning uchun jismning 1 oktantdagi bo'lagining hajmini hisoblab , natijani 8 ga ko'paytirsak , berilgan jismning hajmi kelib chiqadi .Umumlashgan sferik koordinatalardan foydalanamiz,

$$x = ar \sin\psi \cos\varphi \; ; y = br \sin\psi \sin\varphi ;$$

$$z = cr \cos\varphi \quad \left(0 \le \varphi \le \frac{\pi}{2} \; ; \; 0 \le \psi \le \frac{\pi}{2}\right)$$

Yakobian $|I(r,\varphi,\psi)| = abcr^2 \sin\varphi$

x , y , z lar uchun sferik koordinatalarni berilgan tenglamaga qo'yib quyidagini topamiz:

$$\left[r^2 \sin^2\varphi(\cos^2\psi + \sin^2\psi) + r^2\cos^2\varphi\right]^2 = r^2 \sin^2\varphi(\cos^2\psi + \sin^2\psi) \Rightarrow$$

$$\Rightarrow \quad r^4 = r^2 \sin^2\varphi \Rightarrow r^2 \sin^2\varphi \Rightarrow r = \sin\varphi$$

Bundan $V = \iiint_V dV = 8\int_0^{\frac{\pi}{2}} d\psi \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\sin\varphi} abcr^2 \sin\varphi dr = 8abc$ (kub birlik).

5-misol. $(x+2y+z)^2 + (y+2z)^2 + (2x+z)^2 = R^2$ *ellipsoidning hajmini toping* .

Yechim. O'zgaruvchilarni $\begin{cases} x + 2y + z = u \\ y + 2z = \vartheta \\ 2x + z = \omega \end{cases}$ formulalar yordamida almashtiramiz .Natijada ellipsoidning tenglamasi $u^2 + v^2 + \omega^2 = R^2$ ko'rinishni oladi, bu esa $u\vartheta\omega$ koordinatalar sistemasida sfera tenglamasidir. Yuqoridagi almashtirishlar orqali V soha (ellipsoid) V' sohaga (sharga) o'tadi .

$$V' = \left\{(u,\vartheta,\omega) \in R^3 : -R \le u \le R ; -\sqrt{R^2 - u^2} \le \vartheta \le \sqrt{R^2 - u^2} ; \; |\omega| \le \sqrt{R^2 - u^2 - \vartheta^2}\right\}$$

Endi tenglamalar sistemasini yechib x, y, z larni aniqlaymiz:

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1 + 8 - 2 = 7;$$

$$\Delta_x = \begin{vmatrix} u & 2 & 1 \\ \vartheta & 1 & 2 \\ \omega & 0 & 1 \end{vmatrix} = u - 2\vartheta + 3\omega;$$

$$\Delta_y = \begin{vmatrix} 1 & u & 1 \\ 0 & \vartheta & 2 \\ 2 & \omega & 1 \end{vmatrix} = \vartheta + 4u - 2\vartheta - 2\omega = 4u - \vartheta - 2\omega$$

$$\Delta_z = \begin{vmatrix} 1 & 2 & u \\ 0 & 1 & \vartheta \\ 2 & 0 & \omega \end{vmatrix} = \omega + 4\vartheta - 2u.$$

Bundan

$$x = \frac{\Delta x}{\Delta} = \frac{1}{7}(u - 2\vartheta + 3\omega),$$

$$y = \frac{\Delta y}{\Delta} = \frac{1}{7}(4u - \vartheta - 2\omega),$$

$$z = \frac{\Delta z}{\Delta} = \frac{1}{7}(\omega + 4\vartheta - 2u),$$

kelib chqadi.

$$\frac{\partial x}{\partial u} = \frac{1}{7}; \qquad \frac{\partial x}{\partial \vartheta} = -\frac{2}{7}; \qquad \frac{\partial x}{\partial \omega} = \frac{3}{7};$$

$$\frac{\partial y}{\partial u} = \frac{4}{7}; \qquad \frac{\partial y}{\partial \vartheta} = -\frac{1}{7}; \qquad \frac{\partial y}{\partial \omega} = -\frac{2}{7};$$

$$\frac{\partial z}{\partial u} = -\frac{2}{7}; \qquad \frac{\partial z}{\partial \vartheta} = \frac{4}{7}; \qquad \frac{\partial z}{\partial \omega} = \frac{1}{7}.$$

Yakobian

$$I(u,\vartheta,\omega)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial \vartheta} & \frac{\partial x}{\partial \omega}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial \vartheta} & \frac{\partial y}{\partial \omega}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial \vartheta} & \frac{\partial z}{\partial \omega}\end{vmatrix}=\begin{vmatrix}\frac{1}{7} & -\frac{2}{7} & \frac{3}{7}\\ \frac{4}{7} & -\frac{1}{7} & -\frac{2}{7}\\ -\frac{2}{7} & \frac{4}{7} & \frac{1}{7}\end{vmatrix}=\frac{1}{7^3}\begin{vmatrix}1 & -2 & 3\\ 4 & -1 & -2\\ -2 & 4 & 1\end{vmatrix}=$$

$$\frac{1}{7^3}(-1-8+48-6+8+8)=\frac{1}{7^3}\cdot 49=\frac{1}{7}$$

$$V=\iiint\limits_V dV=\iiint\limits_V |I(u,\vartheta,\omega)|du d\vartheta d\omega=\frac{8}{7}\int_0^R du\int_0^{\sqrt{R^2-u^2}} dv\int_0^{\sqrt{R^2-u^2-v^2}} d\omega=\frac{8}{7}\int_0^R du\int_0^{\sqrt{R^2-u^2}}\sqrt{R^2-u^2-\vartheta^2}d\vartheta;$$

Lekin $\int_0^a\sqrt{a^2-x^2}dx=\frac{\pi a^2}{4}$ ekani bizga ma'lum .

Bu yerda a ni $\sqrt{a^2-u^2}$ ga almashtirsak,

$\int_0^{\sqrt{R^2-u^2}}\sqrt{R^2-u^2-\vartheta^2}d\vartheta=\frac{\pi}{4}(R^2-u^2)$ kelib chaqdi .

Natijada $V=\frac{8}{7}\int_0^R\frac{\pi}{4}(R^2-u^2)du=\frac{2\pi}{7}\left(R^2u-\frac{u^3}{3}\right)\Bigg|_0^R=\frac{2\pi}{7}\left(R^3-\frac{R^3}{3}\right)=\frac{2\pi}{7}\cdot\frac{2R^3}{3}=\frac{4\pi R^3}{21}$

(kub birlik).

Bu yerda ellipsoidning hajmini hisoblashda uning birinchi oktantdagi bo'lagining hajmini hisoblab, natijani 8 ga ko'paytirdik.

4.3. Uch karrali integrallarning mexanikaga tatbiqlari

Jismning hajmi V bo'lib, ixtiyoriy $M(x,y,z)$ nuqtaning zichligi $\rho(M)=\rho(x,y,z)$ bo'lsa, uning massasi

$$m=\iiint\limits_V \rho(x,y,z)dV \quad (7)$$

formula bilan aniqlanadi.

Koordinata tekisliklariga nisbatan statik momentlari

$$M_{xy}=\iiint\limits_V z\rho(x,y,z)dV;\quad M_{xz}=\iiint\limits_V y\rho(x,y,z)dV;\quad M_{yz}=\iiint\limits_V x\rho(x,y,z)dV;\quad (8)$$

Og'irlik markazining koordinatalari

$$x_0=\frac{\iiint\limits_V x\rho(x,y,z)dV}{m};\quad y_0=\frac{\iiint\limits_V y\rho(x,y,z)dV}{m};\quad z_0=\frac{\iiint\limits_V z\rho(x,y,z)dV}{m}\quad (9)$$

Koordinata o'qlariga nisbatan inersiya momentlari

$$I_x = \iiint_V (y^2 + z^2)\rho(x,y,z)dV; \qquad I_y = \iiint_V (z^2 + x^2)\rho(x,y,z)dV;$$

$$I_z = \iiint_V (x^2 + y^2)\rho(x,y,z)dV; \qquad (10)$$

formula bilan aniqlanadi.

Koordinata tekisliklariga nisbatan inersiya momentlari esa quyidagicha topiladi:

$$I_{xy} = \iiint_V z^2\rho(x,y,z)dV; \qquad I_{xz} = \iiint_V y^2\rho(x,y,z)dV; \qquad I_{zy} = \iiint_V x^2\rho(x,y,z)dV;$$

Qaralayotgan jism massasi $A(\xi,\eta,\zeta)$ nuqtani Nyuton qonuni bo'yicha tortish kuchini $\bar{F}$ bilan belgilasak, $\bar{F}$ ning o'qlardagi proyeksiyalari quyidagi formula bilan aniqlanadi: (A nuqta massasi m ga teng bo'lsa)

$$F_x = k\iiint_V m\frac{x-\xi}{r^3}\rho(x,y,z)dV; \qquad F_y = k\iiint_V m\frac{y-\eta}{r^3}\rho(x,y,z)dV;$$

$$F_z = k\iiint_V m\frac{z-\zeta}{r^3}\rho(x,y,z)dV; \quad (11)$$

Bu yerda

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}; \qquad \bar{F} = \{F_x, F_y, F_z\}$$

k- Nyuton koeffitsenti (gravitatsion doimiylik).

Sunga o'xshash berilgan jismning A nuqtaga potensialini aniqlaymiz:

$$W = \iiint_V \frac{\rho(x,y,z)dV}{r} \quad (12)$$

Bu yerda ikkita holga e'tibor berish kerak:

1) Agar $A(\xi,\eta,\zeta)$ nuqta berilgan jismdan tashqarida yotsa, u holda (11), (12) integrallar xosmas integrallar hisoblanadi, lekin bu holda ham (11), (12) integrallar mavjuddir. Endi tipik misollar yechishga o'tamiz.

1-misol. *Birlik kub* $(0 \le x \le 1;\ 0 \le y \le 1;\ 0 \le z \le 1)$ *har bir* $M(x,y,z)$ *nuqtasidagi zichligi* $\rho(x,y,z) = x + y + z$ *formula bilan berilgan.Massasini hisoblang.*

Yechim. (II) formulaga ko'ra

$$m = \iiint_V \rho(x,y,z)dV = \int_0^1 dx\int_0^1 dy\int_0^1 (x+y+z)dz = \int_0^1 dx\int_0^1\left(xz + yz + \frac{z^2}{2}\right)\Bigg|_0^1 dy =$$

$$= \int_0^1 dx\int_0^1\left(x + y + \frac{1}{2}\right)dy = \int_0^1\left(xy + \frac{y^2}{2} + \frac{1}{2}y\right)\Bigg|_0^1 dx = \int_0^1 (x+1)dx = \left(\frac{x^2}{2} + x\right)\Bigg|_0^1 = \frac{1}{2} + 1 = 1{,}5$$

2-misol. $z = x^2 + y^2$ paraboloid $x + y = a,\ x = 0;\ y = 0;\ z = 0;$ tekisliklar bilan chegaralangan bir jinsli og'irlik markazining koordinatalarini topining.

Yechim. Berilgan jism bir jinsli bo‘lgani uchun $\rho=(x,y,z)=1$ deb olamiz. Bu holda jism massasi son jihatdan uning hajmiga teng, ya’ni $m=V$ bo‘ladi V soha quyidagicha bo‘ladi:

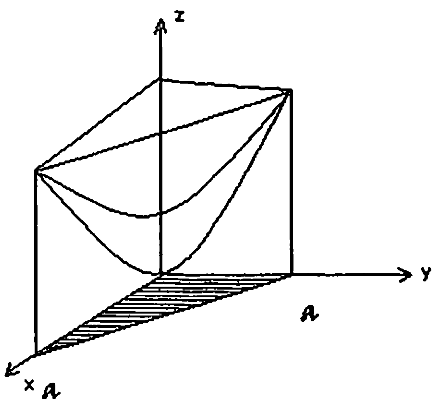
$$V=\{(x,y,z)\in R^3 : 0\le x\le a;\ 0\le y\le a-x, 0\le z\le x^2+y^2\}$$

jism massasi

$$m=\iiint_V dV=\int_0^a dx\int_0^{a-x}dy\int_0^{x^2+y^2}dz=\int_0^a dx\int_0^{a-x}(x^2+y^2)dy=\int_0^a dx\int_0^{a-x}(x^2+y^2)dy=$$

$$\int_0^a\left(x^2y+\frac{y^3}{3}\right)\Bigg|_0^{a-x}dx=\int_0^a\left(ax^2-x^3+\frac{(a-x)^3}{3}\right)dx=$$

$$=\left(a\cdot\frac{x^3}{3}-\frac{x^4}{4}-\frac{(a-x)^4}{12}\right)\Bigg|_0^a=\frac{a^4}{3}-\frac{a^4}{4}+\frac{a^4}{12}=\frac{a^4}{6}$$



Endi (12) formulalar yordamida koordinata tekisliklariga nisbatan statik momentlarini topamiz:

$$M_{xy}=\iiint_V z dV=\int_0^a dx\int_0^{a-x}dy\int_0^{x^2+y^2}zdz=\int_0^a dx\int_0^{a-x}\frac{(x^2+y^2)^2}{2}dy=$$

$$=\int_0^a dx\int_0^{a-x}\frac{1}{2}(x^4+2x^2y^2+y^4)dy=\frac{1}{2}\int_0^a\left(x^4y+2x^2\cdot\frac{y^3}{3}+\frac{y^5}{5}\right)\Bigg|_0^{a-x}dx=$$

$$=\frac{1}{2}\int_0^a\left[ax^4-x^5+\frac{2}{3}x^2\left(a^3-3a^2x+3ax^2-x^3\right)+\frac{1}{5}(a-x)^5\right]dx=\frac{1}{2}\int_0^a\left[ax^4-x^5+\frac{2}{3}a^3x^2-\right.$$

$$\left.-2a^2x^3+2ax^4-\frac{2}{3}x^5+\frac{1}{5}(a-x)^5\right]dx=\frac{1}{2}\left[a\cdot\frac{x^5}{5}-\frac{x^6}{6}+\frac{2}{3}a^3-\frac{x^3}{3}-2a^2\cdot\frac{x^4}{4}+2a\cdot\frac{x^5}{5}-\frac{2}{3}\cdot\frac{x^6}{6}-\right.$$

$$\left.-\frac{1}{5}\cdot\frac{(a-x)^6}{6}\right]\Bigg|_0^a=\frac{1}{2}a^6\left(\frac{1}{5}-\frac{1}{6}+\frac{2}{9}-\frac{1}{2}+\frac{2}{5}-\frac{1}{9}+\frac{1}{30}\right)=\frac{1}{2}a^3\left(\frac{1}{15}+\frac{1}{9}-\frac{1}{10}\right)=\frac{7}{180}a^6.$$

$$M_{xz} = \iiint_V y\,dV = \int_0^a dx \int_0^{a-x} dy \int_0^{x^2+y^2} y\,dz = \int_0^a dx \int_0^{a-x} (x^2 y + y^3)dy = \int_0^a \left(\frac{x^2 y^2}{2} + \frac{y^4}{4} \right)\Bigg|_0^{a-x} dx =$$

$$= \int_0^a \left[\frac{1}{2} a^2 x^2 - ax^3 + \frac{1}{2} x^4 (a-x)^4 \right] dx = \left(\frac{a^2}{2} \cdot \frac{x^3}{3} - a \cdot \frac{x^4}{4} + \frac{1}{2} \cdot \frac{x^5}{5} - \frac{1}{4} \cdot \frac{(a-x)^5}{5} \right)\Bigg|_0^a =$$

$$= a^5 \left(\frac{1}{6} - \frac{1}{4} + \frac{1}{10} + \frac{1}{20} \right) = a^5 \left(\frac{3}{20} - \frac{1}{12} \right) = \frac{9-5}{60} a^5 = \frac{1}{15} a^5;$$

$$M_{yz} = \iiint_V x\,dV = \int_0^a dx \int_0^{a-x} dy \int_0^{x^2+y^2} x\,dz = \int_0^a x\,dx \int_0^{a-x} (x^2 + y^2)dy = \int_0^a x \left(x^2 y + \frac{y^3}{3} \right)\Bigg|_0^{a-x} dx =$$

$$= \int_0^a \left[ax^3 - x^4 + \frac{1}{3}(a^3 x - 3a^2 x^2 + 3ax^3 - x^4) \right] dx = \left(a \cdot \frac{x^4}{4} - \frac{x^5}{5} + \frac{1}{3} a^3 \cdot \frac{x^2}{2} - \frac{1}{3} a^2 x^3 + a \cdot \frac{x^4}{4} - \frac{x^5}{15} \right)\Bigg|_0^a =$$

$$= a^5 \left(\frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{3} + \frac{1}{4} - \frac{1}{15} \right) = a^5 \left(\frac{1}{2} - \frac{4}{15} - \frac{1}{6} \right) = a^5 \left(\frac{1}{3} - \frac{4}{15} \right) = \frac{1}{15} a^5;$$

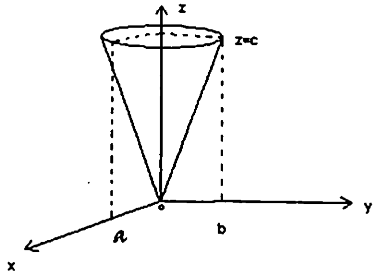
Demak, $M_{zx} = M_{yz} = \frac{1}{15} a^5$, $M_{xy} = \frac{7}{180} a^6$

Bularni (13) formulaga qo'ysak, og'irlik markazining koordinatalari kelib chiqadi:

$$x_0 = y_0 = \frac{a^5}{15} : \frac{a^4}{6} = \frac{2}{5} a; \qquad z_0 = \frac{7a^6}{180} : \frac{a^4}{6} = \frac{7}{30} a^2.$$

3-misol. $\left(\frac{z}{c}\right)^2 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$ konusning har bir $M(x,y,z)$ nuqtasidagi zichligi shu nuqtadan xoy tekisligigacha bo'lgan masofaga proporsional bo'lib, proporsionallik koffitsienti k ga teng. Shu konusning $z = c$ $(c > 0)$ tekislik bilan kesilgan bo'lagining massaasi va og'irlik markazining koordinatalarini toping.

Yechim.



Shartga ko'ra $\rho \neq (x,y,z) = kz$

$$V = \left\{ x,y,z : -a \le x \le a;\ -b\sqrt{1-\frac{x^2}{a^2}} \le y \le b\sqrt{1-\frac{x^2}{a^2}};\ c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}} \le z \le c \right\}$$

Masalani yechishda umumlashgan silindrik koordinatalardan foydalanamiz: $x = ar\cos\varphi,\quad y = br\sin\varphi,\quad z = z$.

Bularni yuqoridagi tenglamaga qo'ysak, $z = cz$ kelib chiqadi. Natijada yangi koordinatalar uchun integrallash sohasi quyidagicha bo'ladi:

$V' = \{(\varphi, r, z) : 0 \le \varphi \le 2\pi;\ 0 \le r \le 1;\ cr \le z \le c\}$, $I(\varphi, r, z) = abr$ jism massasi

$$m = \iiint_{V'} kz dV' = k\int_0^{2\pi} d\varphi \int_0^1 dr \int_{cr}^{c} z|I(\varphi,r,z)|dz = abk\int_0^{2\pi} d\varphi \cdot \int_0^1 dr \int_{cr}^{c} rzdz =$$

$$= \frac{abk}{2}\int_0^{2\pi} d\varphi \int_0^1 r(c^2 - c^2r^2)dr = = \frac{abc^2k}{2}\int_0^{2\pi}\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^1 d\varphi = \frac{abc^2k}{2\cdot 4}\cdot 2\pi = \frac{\pi kabc^2}{4};$$

(12) formulalarga ko'ra quyidagini topamiz:

$$M_{xy} = \iiint_{V'} z\rho(\varphi, r, z)dV' = k\iiint_{V'} z^2 dV' = k\int_0^{2\pi} d\varphi \int_0^1 dr \int_{cr}^{c} abcz^2 dz = kab\cdot 2\pi\int_0^1 r\left(\frac{c^3}{3} - \frac{c^3r^3}{3}\right)dr =$$

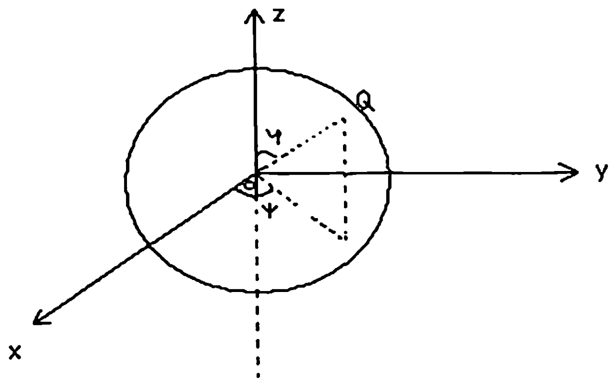
$$= \frac{2\pi kabc^3}{3}\int_0^1 (r - r^4)dr = \frac{2\pi kabc^3}{3}\left(\frac{r^2}{2} - \frac{r^5}{5}\right)\Bigg|_0^1 = \frac{2\pi kabc^3}{3}\cdot\frac{3}{10} = \frac{1}{5}\pi kabc^2.$$

Berilgan jismning zichligi *x,y* larga bog'liq emas, demak *y* gorizontal kesimda o'zgarmasdir, bundan tashqari Oz jismning simmetriya o'qidir, shuning uchun og'rlik markazi *Oz* o'qida yotadi, demak $x_0 = y_0 = 0$ bo'ladi.

$$z_0 = \frac{M_{xy}}{m} = \frac{4}{5}c; \qquad \left(0, 0, \frac{4}{5}c\right)$$

4-misol. Radiusi R va massasi M ga teng bo'lgan sharning har bir nuqtasidagi zichligi shu nuqtadan shar markazigacha bo'lgan masofaga proporsional bo'lsa, shu sharning diametriga nisbatan inersiya momentini toping.

Yechim



Bu yerda shu narsaga e'tibor qilish kerakki, sharning diametrlari cheksiz ko'p, lekin ularning hammasiga nisbatan inertsiya momentlari bir xil, ya'ni o'zgarmas- bo'ladi. Sharni koordinatalar sistemasida shunday joylashtiramizki, uning markazi koordinatalar boshi bo'lsin. Agar sharning koordinata o'qlaridan biriga (masalan Oz ga) nisbatan inertsiya momentini topsak, masala yechilgan bo'ladi. Shartga ko'ra ixtiyoriy $Q(x,y,z)$ nuqtasidagi zichligi $\rho(Q)=\rho(x,y,z)=k\sqrt{x^2+y^2+z^2}$ bo'ladi. Bu yerda proporsionallik koeffisienti k ni quydagicha shartdan aniqlaymiz:

$$M=\iiint_{V}\rho(x,y,z)dV=k\iiint_{V}\sqrt{x^2+y^2+z^2}dV$$

$$\left.\begin{aligned} x&=r\sin\psi\cos\varphi\\ y&=r\sin\psi\sin\varphi\\ z&=r\cos\psi \end{aligned}\right\}$$ sferik koordinatalarga o'tamiz

Yakobian $|I(\varphi,\psi,r)|=r^2\sin\psi$ ekani bizga ma'lum. Bizning masalamizda

$V'=\{(\varphi,\psi,r): 0\le\varphi\le\pi;\ 0\le\varphi\le 2\pi;\ 0\le r\le R\}$

Bularni o'rniga qo'yib, yuqoridagi integralni hisoblaymiz.

$$M=k\int_0^{\pi}d\psi\int_0^{2\pi}d\varphi\int_0^{R}r^3\sin\psi dr=k\cdot 2\pi\int_0^{\pi}\frac{R^4}{4}\sin\psi d\psi=\frac{k\pi R^4}{2}\cdot(-\cos\psi)\Big|_0^{\pi}=k\pi R^4.$$

Bundan $k=\frac{M}{\pi R^4}$ kelib chiqadi. (14) formulaga ko'ra I_z quydagiga teng bo'ladi:

$$I_z = \iiint_V \left(x^2 + y^2\right) \cdot \rho(x, y, z) dV = \iiint_{V^*} r^2 \sin^2 \psi \cdot k \cdot r \left| I(\varphi, \psi, r) \right| dV =$$

$$= k \cdot \int_0^{\pi} d\psi \int_0^{2\pi} d\varphi \int_0^{R} r^5 \sin^3 \psi \, dr =$$

$$= -2\pi k \cdot \frac{R^6}{6} \int_0^{\pi} (1 - \cos^2 \psi) d \cos \psi = -\frac{\pi k R^6}{3} \left(\cos \psi - \frac{\cos^3 \psi}{3} \right) \Bigg|_0^{\pi} = -\frac{\pi k R^6}{3} \left(-2 + \frac{2}{3} \right) = \frac{4 \pi k R^6}{9};$$

Bu yerda k o'rniga yuqorida topilgan qiymatini qo'ysak, diametrga yoki baribir Oz o'qiga nisbatan inertsiya momenti kelib chqadi.

$$I_z = \frac{4\pi R^6}{9} \cdot \frac{M}{\pi R^4} = \frac{4}{9} M R^2$$

Hosilalar jadvali

№ t/r	Elementar funksiyalarning hosilalalari	Murakkab funksiyalarning hosilalalri
1.	$c' = 0; x' = 1;$	
2.	$(x^a)' = ax^{a-1};$	$(u^a)' = au^{a-1}u';$
3.	$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$	$(\sqrt{u})' = \frac{1}{2\sqrt{u}}u';$
4.	$(a^x)' = a^x \ln a;$	$(a^u)' = a^u \ln au';$
5.	$(e^x)' = e^x;$	$(e^u)' = e^u u';$
6.	$(\log_a x)' = \frac{1}{x \ln a};$	$(\log_a u)' = \frac{1}{u \ln a}u';$
7.	$(\ln x)' = \frac{1}{x};$	$(\ln u)' = \frac{1}{u}u';$
8.	$(\sin x)' = \cos x;$	$(\sin u)' = \cos u \cdot u';$
9.	$(\cos x)' = -\sin x;$	$(\cos u)' = -\sin u \cdot u';$
10.	$(tgx)' = \frac{1}{\cos^2 x};$	$(tgu)' = \frac{1}{\cos^2 u} \cdot u';$
11.	$(ctgx)' = -\frac{1}{\sin^2 x};$	$(ctgu)' = -\frac{1}{\sin^2 u} \cdot u'$
12.	$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}};$	$(\arcsin u)' = \frac{1}{\sqrt{1-u^2}} \cdot u';$
13.	$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}};$	$(\arccos u)' = -\frac{1}{\sqrt{1-u^2}} \cdot u';$
14.	$(arctgx)' = \frac{1}{1+x^2};$	$(arctgu)' = \frac{1}{1+u^2} \cdot u';$
15.	$(arcctgx)' = -\frac{1}{1+x^2};$	$(arcctgu)' = -\frac{1}{1+u^2} \cdot u'.$

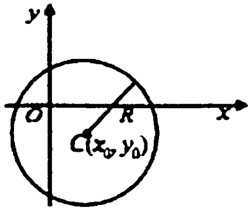
Aniqmas integrallar jadvali

1.	$\int du = u + C$;	**2.**	$\int 0\,du = C$;
3.	$\int u^{\alpha} du = \frac{u^{\alpha+1}}{\alpha+1} + C$;	**4.**	$\int \frac{du}{u^2} = -\frac{1}{u} + C$, $\int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C$;
5.	$\int a^u du = \frac{a^u}{\ln a} + C, (a > 0, a \neq 1)$	**6.**	$\int e^u du = e^u + C$;
7.	$\int \sin u\,du = -\cos u + C$;	**8.**	$\int \cos u\,du = \sin u + C$;
9.	$\int \frac{du}{\cos^2 u} = \operatorname{tg} u + C$;	**10.**	$\int \frac{du}{\sin^2 u} = -\operatorname{ctg} u + C$;
11.	$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \operatorname{arctg} \frac{u}{a} + C$;	**12.**	$\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln\left\|\frac{u+a}{u-a}\right\| + C$;
13.	$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln\left\|\frac{u-a}{u+a}\right\| + C$;	**14.**	$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$;
15.	$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln\left\|u + \sqrt{u^2 + a^2}\right\| + C$	**16.**	$\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left\|u + \sqrt{u^2 - a^2}\right\| + C$
17.	$\int \frac{du}{\sin u} = \ln \left\| \operatorname{tg} \frac{u}{2} \right\| + C$;	**18.**	$\int \frac{du}{\cos u} = \ln\left\|\operatorname{tg}\left(\frac{u}{2} + \frac{\pi}{4}\right)\right\| + C$;
19.	$\int \operatorname{tg} u\,du = -\ln\left\|\cos u\right\| + C$;	**20.**	$\int \operatorname{ctg} u\,du = \ln\left\|\sin u\right\| + C$;

Ikkinchi tartibli egri chiziqlar

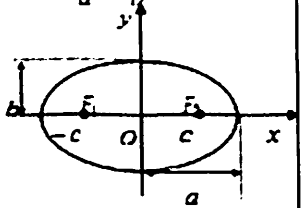
Aylana

$(x-x_0)^2+(y-y_0)^2=R^2$



Ellips

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



a - katta yarim o'q

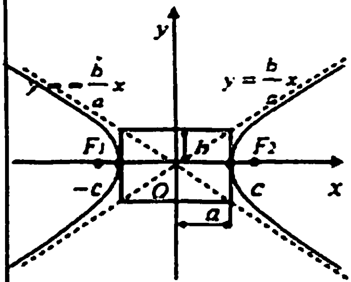
b - kichik yarim o'q

$F_1(-c,0)$, $F_2(c,0)$ - fokuslar

$c^2=a^2-b^2$

Giperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$



a - haqiqiy yarim o'q

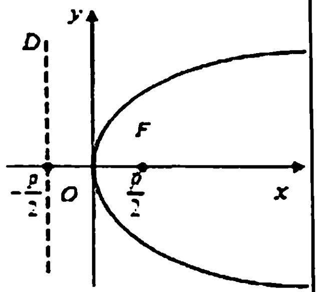
b - mavhum yarim o'q

$F_1(-c,0)$, $F_2(c,0)$ - fokuslar

$c^2=a^2+b^2$

Parabola

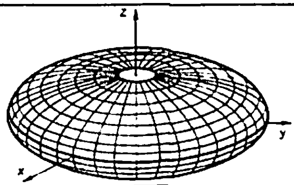
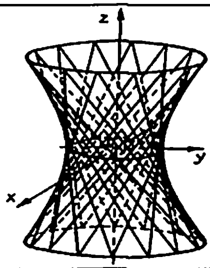
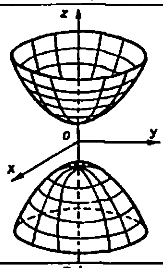
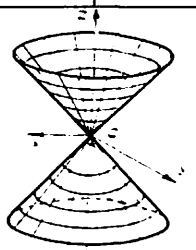
$y^2=2px\ (p>0)$

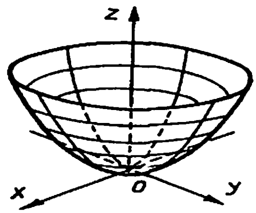
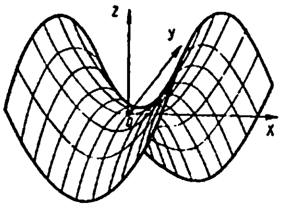
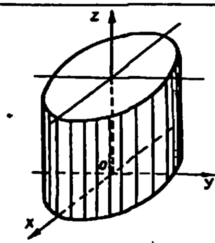
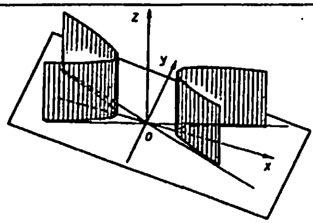
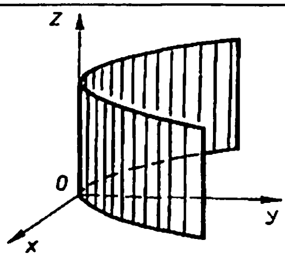


$D: x=-\frac{p}{2}$ - direktrisa

$F(\frac{p}{2},0)$ - fokus

Ikkinchi tartibli sirtlar

№ t/r	Rasm	Sirtlarning nomlanishi va ularning kanonik ko‘rinishi
1		Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
2		Bir pallali giperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$
3		Ikki pallali giperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$
4		Ikkinchi tartibli konus $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$

5		Elliptik paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$
6		Giperbologik paraboloid $z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$
7		Elliptik silindr $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
8		Giperbologik silindr $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
9		Parabologik silindr $x^2=2py$

TESTLAR

1. Funksiyani differensiallash.

1. Differensiallash qoidasi qayerda xato ko'rsatilgan ?

A) $(Cu)'=Cu'$ (C-const.). B) $(u\pm v)'= u'\pm v'$. C) $(u\cdot v)'= u'v+uv'$.

D) $\left(\frac{u}{v}\right)' = \frac{u'v+uv'}{v^2}$. E) $(f(u))'=f'(u)u'$.

2. Diffrentsiallanuvchi u va v funksiyalar u/v nisbatining hosilasini hisoblash formulasi to'g'ri yozilgan javobni ko'rsating.

A) $\frac{u'v+uv'}{v}$. B) $\frac{u'v+uv'}{v^2}$. C) $\frac{u'v-uv'}{v^2}$.

D) $\frac{u'v-uv'}{v}$ E) $\frac{u'v'-uv}{v^2}$.

3. $y=x^2/\sin x$ funksiyaning y' hosilasini hisoblang.

A) $y'= x^2/\cos x$. B) $y'= 2x/\sin x$. C) $y'= 2x/\cos x$.

D) $y'= x(2\sin x +x\cos x)/\sin^2 x$. E) $y'= x(2\sin x- x\cos x)/ \sin^2 x$.

4. Diffrentsiallanuvchi u va v funksiyalar $u\cdot v$ ko'paytmasining hosilasini hisoblash formulasi qayerda to'g'ri yozilgan ?

A) $u'v'$. B) $u'v'+uv$. C) $u'v+uv'$. D) $u'v-uv'$. E) $u'v'-uv$.

5. $y=x^2\sin x$ funksiyaning y' hosilasini hisoblang.

A) $y'= x^2\cos x$. B) $y'= x(x\sin x-2\cos x)$. C) $y'= 2x\sin x$.

D) $y'= x(x\cos x+2\sin x)$. E) $y'= x(x\cos x-2\sin x)$.

6. Agar $y=f(x)$, $u=u(x)$ differensiallanuvchi funksiyalar bo'lsa, $y=f(u)$ murakkab funksiya hosilasini hisoblash formulasini ko'rsating.

A) $y'=f'(u)$. B) $y'=f(u')$. C) $y'=f'(u)$.

D) $y'=f'(u)u'$. E) $y'=f(u')u'$.

7. $f(x)=\sin x$, $u(x)=\ln x$ funksiyalar bo'yicha tuzilgan $y=f(u)=\sin\ln x$ murakkab funksiya hosilasini hisoblang.

A) $y'=\cos\ln x$. B) $y'=\sin(1/x)$. C) $y'=(\sin\ln x)/x$.

D) $y'=(\cos\ln x)/\ln x$. E) $y'=(\cos\ln x)/x$.

8. $y=\sin\arcsin x$ $(-1 \le x \le 1)$ murakkab funksiya hosilasini hisoblang.

A) $y'=\cos\arcsin x$. B) $y'=1$. C) $y'= \sin\arccos x$.

D) $y'= \cos\arccos x$. E) $y'=x$.

9. $y=\cos(x^2+1)$ funksiyaning y' hosilasini hisoblang.

A) $y'= \sin(x^2+1)$. B) $y'= -\sin(x^2+1)$. C) $y'= \sin 2x$.

D) $y'=2x\sin(x^2+1)$. E) $y'=-2x\sin(x^2+1)$.

10. $y = e^{x^4}$ funksiyaning hosilasi to'g'ri yozilgan javobni toping.

A) $x^4 \cdot e^{x^4-1}$. B) $4x^3 \cdot e^{x^4}$. C) $4x^3 \cdot e^{x^4} \cdot lg\, e$.

D) $4x^3 \cdot e^{x^4-1} \cdot lg\, e$. E) e^{x^4}.

11. $y=f(x)$ differensiallanuvchi va qat'iy monoton funksiya bo'lsa, unga teskari $y=f^{-1}(x)$ funksiya hosilasini qanday shartda topib bo'lmaydi ?

A) $f'(x)>0$. B) $f'(x)<0$. C) $f'(x)\neq 0$. D) $f'(x)=0$.

E) keltirilgan barcha shartlarda topib bo'ladi.

12. $y=2x+5$ funksiyaga teskari funksiya hosilasini toping.

A) 2. B) 5. C) 1/2. D) 1/5. E) 0.

13. Giperbolik sinus deb ataladigan sh$x=(e^x-e^{-x})/2$ funksiyaga teskari arcsh$x$ funksiyaning hosilasini toping.

A) (arcsh$x$)' $=(e^x+e^{-x})/2$. B) (arcsh$x$)' $=2/(e^x+e^{-x})$. C) (arcsh$x$)' $=\sqrt{1+x^2}$.

D) (arcsh$x$)' $=1/\sqrt{1+x^2}$. E) (arcsh$x$)'$=1/\sqrt{1-x^2}$.

14. Giperbolik cosinus deb ataladigan ch$x=(e^x+e^{-x})/2$ funksiyaga teskari arccsh$x$ funksiyaning hosilasini toping.

A) (arcch$x$)' $=(e^x-e^{-x})/2$. B) (arcch$x$)' $=2/(e^x-e^{-x})$. C) (arcch$x$)' $=\sqrt{x^2+1}$.

D) (arcch$x$)' $=1/\sqrt{x^2+1}$. E) (arcch$x$)'$=1/\sqrt{x^2-1}$.

15. $x=\varphi(t)$, $y=\psi(t)$ differensillanuvchi funksiyalar orqali parametrik ko'rinishda berilgan $y=y(x)$ funksiyaning $y'=y'(x)$ hosilasini hisoblash formulasini toping.

A) $y' = (\frac{\phi(t)}{\psi(t)})'$. B) $y' = (\frac{\psi(t)}{\phi(t)})'$. C) $y' = \frac{\phi'(t)}{\psi'(t)}$. D) $y' = \frac{\psi'(t)}{\phi'(t)}$.

E) to'g'ri javob keltirilmagan.

16. $x=\sin t$, $y=\cos t$ funksiyalar orqali parametrik ko'rinishda berilgan $y=y(x)$ funksiyaning $y'=y'(x)$ hosilasini toping.

A) $y' = -\text{ctg}t$. B) $y' = -\text{tg}t$. C) $y' = 1/\cos^2 t$.

D) $y' = -1/\sin^2 t$. E) to'g'ri javob keltirilmagan.

17. $y=a^x$ ko'rsatkichli funksiya hosilasini toping.

A) $y'=a^x$. B) $y'=xa^{x-1}$. C) $y'=a^x\ln a$. D) $y'=a^x\log_a e$.

E) to'g'ri javob ko'rsatilmagan.

18. $y=\log_a x$ funksiya hosilasini toping.

A) $y' = \frac{1}{x}$. B) $y' = \frac{1}{x}\ln a$. C) $y' = \frac{1}{x}a$. D) $y' = \frac{1}{x\ln a}$.

E) to'g'ri javob ko'rsatilmagan.

19. Qaysi trigonometrik funksiya hosilasi xato ko'rsatilgan ?

A) $(\sin x)'=\cos x$. B) $(\cos x)'=\sin x$. C) $(\text{tg}x)' = \frac{1}{\cos^2 x}$.

D) $(\mathrm{ctg}x)' = -\frac{1}{sin^2 x}$. E) barcha hosilalar to‘g‘ri ko‘rsatilgan.

20. Qaysi teskari trigonometrik funksiyaning hosilasi to‘g‘ri ko‘rsatilgan ?

A) $(arcsin\, x)' = 1/(1 - x^2)$. B) $(arccos\, x)' = 1/\sqrt{x^2 - 1}$.

C) $(\mathrm{arctg}x)' = 1/\sqrt{x^2 + 1}$. D) $(\mathrm{arcctg}x)' = -1/\sqrt{x^2 + 1}$.

E) barcha hosilalar xato ko‘rsatilgan.

2. Boshlang'ich funksiya va aniqmas integral.

1. Quyidagi shartlarning qaysi birida $F(x)$ berilgan $f(x)$ funksiyaning boshlang'ich funksiyasi deyiladi ?

A) $F(x)=f(x)+C$ (C-const). B) $\lim\limits_{t\to x} F(t) = f(x)$.

C) $F'(x) = f(x)$. D) F''. E) $F(x) = f'(x)$.

2. Quyidagilardan qaysi biri $f(x)=\ln x$ uchun boshlang'ich funksiya bo'ladi?

A) $\frac{1}{x}$. B) $x\ln x$. C) $x\ln x+x$. D) $x\ln x-x$. E) $\frac{1}{x}\ln x - x$.

3. Quyidagilardan qaysi birining boshlang'ich funksiyasi $F(x)=x\cos x$ bo'ladi?

A) $x^2 \sin x$. B) $-\sin x$. C) $\cos x - x\sin x$.

D) $\sin x+x\cos x$. E) $x(\sin x+\cos x)$.

4. ***Teoremani to'ldiring:*** Agar $F(x)$ biror $f(x)$ funksiya uchun boshlang'ich funksiya bo'lsa, unda ixtiyoriy C o'zgarmas soni uchun ... funksiya ham $f(x)$ uchun boshlang'ich funksiya bo'ladi.

A) $C\cdot F(x)$. B) $C-F(x)$. C) $C+F(x)$. D) $C/F(x)$. E) $F(x+C)$.

5. Agar $F_1(x)$ va $F_2(x)$ berilgan $f(x)$ funksiya uchun boshlang'ich funksiyalar bo'lsa, unda biror C o'zgarmas soni uchun quyidagi tengliklardan qaysi biri o'rinli bo'ladi?

A) $F_1(x)F_2(x)=C$. B) $F_1(x)/F_2(x)=C$. C) $F_1(x)+F_2(x)=C$.

D) $F_1(x)-F_2(x)=C$. E) $F_1(x)\pm F_2(x)=C$.

6. Agar $F(x)$ biror $f(x)$ funksiya uchun boshlang'ich funksiya bo'lsa, unda ta'rif bo'yicha $\int f(x)dx$ aniqmas integral qanday aniqlanadi ?

A) $C\cdot F(x)$. B) $C-F(x)$. C) $C+F(x)$. D) $C/F(x)$. E) $F(x+C)$.

7. Aniqmas integralning geometrik ma'nosi qayerda to'g'ri va to'liq ko'rsatilgan?

A) Qandaydir to'g'ri chiziq.

B) Qandaydir egri chiziq.

C) Qandaydir chiziqlar sinfi.

D) OX o'qi bo'yicha o'zaro parallel chiziqlar sinfi.

E) OY o'qi bo'yicha o'zaro parallel chiziqlar sinfi.

8. Qayerda aniqmas integralning xossasi xato ko'rsatilgan ?

A) $(\int f(x)dx)' = f(x)$. B) $d(\int f(x)dx) = f(x)dx$.

C) $\int dF(x) = F(x) + C$. D) $\int F'(x)dx = F(x) + C$.

E) Barcha xossalar to'g'ri ko'rsatilgan.

9. Aniqmas integral uchun qaysi tenglik bajarilmaydi ?

A) $\int [f(x) + g(x)]\, dx = \int f(x)dx + \int g(x)dx$.

B)$\int f(x)g(x)dx = \int f(x)dx \cdot \int g(x)dx$.

C)$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$.

D)$\int kf(x)dx = k\int f(x)dx \quad (k - const)$.

E) keltirilgan barcha tengliklar bajariladi.

10. Agarda $F(x)$ va $G(x)$ mos ravishda $f(x)$ va $g(x)$ funksiyalar uchun boshlang'ich funksiyalar , a va b ixtiyoriy o'zgarmas sonlar bo'lsa, $\int[af(x) + bg(x)]\,dx$ aniqmas integral javobi qayerda to'g'ri ko'rsatilgan?

A) $(a+C)F(x)+(b+C)G(x)$. B) $aF(x)+bG(x)+C$.

C) $aF(x)-bG(x)+C$. D) $F(ax)+G(bx)+C$.

E) $(a+C)F(x)-(b+C)G(x)$.

11. Quyidagi tengliklardan qaysi biri integralning chiziqlilik xossasini ifodalamaydi ?

A) $\int[f(x) + g(x)]\,dx = \int f(x)dx + \int g(x)dx$.

B) $\int kf(x)dx = k\int f(x)dx \quad (k - const)$.

C) $\int[f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$.

D) $\int[A \cdot f(x) + B \cdot g(x)]\,dx = A\int f(x)dx + B\int g(x)dx$.

E) Barcha tengliklar integralni chiziqlilik xossasini ifodalaydi.

12. Qaysi darajali funksiyaning aniqmas integrali noto'g'ri yozilgan ?

A) $\int \sqrt{x}dx = \frac{2x\sqrt{x}}{3} + C$. B) $\int \frac{1}{x^2}dx = -\frac{1}{x} + C$. C)$\int \frac{1}{\sqrt{x}}dx = 2\sqrt{x} + C$.

D) $\int \sqrt[3]{x}\,dx = \frac{3x\sqrt[3]{x}}{4} + C$. E) $\int \frac{1}{x}dx = -\frac{1}{x^2} + C$.

13. Integrallar jadvalidan keltirilgan quyidagi tengliklardan qaysi biri xato yozilgan ?

A)$\int \cos x\,dx = \sin x + C$. B) $\int \sin x\,dx = -\cos x + C$.

C)$\int a^x\,dx = a^x + C$. D) $\int \mathrm{tg}x dx = -\ln|\cos x| + C$. E)$\int \frac{dx}{x} = \ln|x| + C$.

14. Integrallar jadvalidan keltirilgan quyidagi tengliklardan qaysi biri to'g'ri yozilgan?

A)$\int \frac{dx}{\cos^2 x} = \mathrm{ctg}x + C$. B) $\int \frac{dx}{\sin^2 x} = -\mathrm{tg}x + C$. C)$\int e^x dx = \frac{e^x}{\lg e} + C$.

D) $\int \frac{dx}{1+x^2} = \arcsin x + C$. E) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$.

15. Qaysi trigonometrik funksiya aniqmas integralining javobi noto'g'ri ko'rsatilgan?

A) $\int \cos x\,dx = \sin x + C$. B) $\int \sin x\,dx = -\cos x + C$.

C) $\int \mathrm{tg}x dx = \ln|\cos x| + C$. D) $\int \mathrm{ctg}x dx = \ln|\sin x| + C$.

E) barcha aniqmas integrallarning javobi to'g'ri ko'rsatilgan.

16. Teskari trigonometrik funksiyalar bilan bog'liq qaysi aniqmas integral javobi xato ifodalangan?

A) $\int \frac{dx}{1+x^2} = \text{arctg}x + C$. B) $\int \frac{dx}{\sqrt{1-x^2}} = arcsin\, x + C$.

C) $\int \frac{dx}{1+x^2} = -\text{arcctg}x + C$. D) $\int \frac{dx}{\sqrt{1-x^2}} = -\, arccos\, x + C$.

E) barcha aniqmas integrallar to'g'ri ifodalangan.

17. $\int (3\cos x - 2\sin x)dx$ aniqmas integralni hisoblang.

A) $-3\sin x - 2\cos x + C$. B) $3\sin x - 2\cos x + C$. C) $3\sin x + 2\cos x + C$.

D) $-3\sin x + 2\cos x + C$. E) $-\sin 3x + \cos 2x + C$.

18. $\int (\frac{5}{x} - 2x + 1)dx$ aniqmas integralni hisoblang.

A) $-\frac{5}{x^2} - 2 + C$. B) $5x - x^2 + C$. C) $-5x - 2x^2 + x + C$.

D) $5\, ln|x| - x^2 + x + C$. E) $5\, ln|x| - x^2 + C$.

19. Agar $f(x)$ funksiya uchun $F(x)$ boshlang'ich funksiya bo'lsa, unda $f(ax+b)$ uchun quyidagilardan qaysi biri boshlang'ich funksiya bo'ladi?

A) $F(ax+b)$. B) $aF(ax+b)$. C) $(a+b)F(ax+b)$.

D) $\frac{1}{a+b}F(ax+b)$. E) $\frac{1}{a}F(ax+b)$.

20. $\int cos(10x + 7)dx$ aniqmas integral javobi qayerda to'g'ri ifodalangan ?

A) $\sin(10x+7)+C$. B) $10\sin(10x+7)+C$. C) $7\sin(10x+7)+C$.

D) $10^{-1}\sin(10x+7)+C$. E) $7^{-1}\sin(10x+7)+C$.

3. Aniqmas integralni hisoblash

1. Aniqmas integralni hisoblashning qaysi usuli mavjud emas ?
A) ko'paytirish usuli. B) o'zgaruvchini almashtirish usuli.
C) differensial ostiga kiritish usuli. D) yoyish usuli.
E) bo'laklab integrallash usuli.

2. Qaysi tenglik aniqmas integralni yoyish usulida hisoblashni ifodalaydi?
A) $\int f(x)dx = \int udv = uv - \int vdu$. B) $\int f(x)dx = \int f(\phi(t))\phi'(t)dt$.
C) $\int f(x)dx = \int \sum_{k=1}^{n} a_k f_k(x)\, dx = \sum_{k=1}^{n} a_k \int f_k(x)dx$.
D) $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$.
E) $\int f(x)dx = \int f[\phi(t)]\phi'(t)dt$.

3. $\int \frac{2x^2-3x+5}{x^2}dx$ integralni yoyish usulida hisoblang.
A) $2 - \frac{3}{x} + \frac{5}{x^2} + C$. B) $2x - 3\ln|x| + \frac{5}{x^2} + C$. C) $2x - \frac{3}{x} - \frac{5}{3x^3} + C$.
D) $2x - 3\ln|x| - \frac{5}{x} + C$. E) $2x + \frac{3}{x^2} - \frac{5}{x} + C$.

4. Quyidagi integrallardan qaysi biriga differensial ostiga kiritish usulini qo'llab bo'lmaydi?
A) $\int f(x)f'(x)dx$. B) $\int \frac{f'(x)dx}{f(x)}$. C) $\int [f(x) \pm f'(x)]dx$.
D) $\int \sqrt{f(x)}f'(x)dx$.
E) barcha integrallarga differensial ostiga kiritish usulini qo'llab bo'ladi.

5. Quyidagi integrallardan qaysi biri differensial ostiga kiritish usulida xato hisoblangan?
A) $\int f(x)f'(x)dx = \frac{1}{2}f^2(x) + C$. B) $\int cos[f(x)]f'(x)dx = sin[f(x)] + C$.
C) $\int sin[f(x)]f'(x)dx = cos[f(x)] + C$. D) $\int \frac{f'(x)dx}{f(x)} = ln|f(x)| + C$.
E) keltirilgan barcha integrallar to'g'ri hisoblangan.

6. Qaysi tenglik aniqmas integralni o'zgaruvchilarni almashtirish usulida hisoblashni ifodalaydi ?
A) $\int f(x)dx = \int udv = uv - \int vdu$.
B) $\int f(x)dx = \int f(\phi(t))\phi'(t)dt$.
C) $\int f(x)dx = \int \sum_{k=1}^{n} a_k f_k(x)\, dx = \sum_{k=1}^{n} a_k \int f_k(x)dx$.
D) $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$.

E) $\int f(x)dx = \int f[\phi(t)]\phi'(t)dt$.

7. $\int f(x)dx$ aniqmas integralda $x=\varphi(t)$ almashtirma bajarilganda u qanday ko'rinishga keladi ?

A) $\int f[\phi(t)]\phi(t)dt$. B) $\int f[\phi'(t)]\phi'(t)dt$. C) $\int f[\phi(t)]dt$.

D) $\int f[\phi(t)]\phi'(t)dt$. E) $\int f[\phi'(t)]\phi(t)dt$.

8. $\int \frac{x^4 dx}{\sqrt{x^{10}-1}}$ integral qaysi almashtirma orqali jadval integraliga keltiriladi ?

A) $t=x^2$. B) $t=x^3$. C) $t=x^4$. D) $t=x^5$. E) $t=x^6$.

9. $\int \frac{\sin x dx}{\sqrt{\cos x}}$ integral qaysi almashtirma orqali jadval integraliga keltiriladi ?

A) $t=\sin x$. B) $t=\cos x$. C) $t = \sqrt{\cos x}$. D) $t=\operatorname{tg} x$. E) $t=\operatorname{ctg} x$.

10. $\int \frac{\cos x dx}{\sqrt{\sin x}}$ integral qaysi almashtirma orqali jadval integraliga keltiriladi ?

A) $t=\sin x$. B) $t=\cos x$. C) $t = \sqrt{\sin x}$. D) $t=\operatorname{tg} x$. E) $t=\operatorname{ctg} x$.

11. Qaysi tenglik bo'laklab integrallash usulini ifodalaydi ?

A) $\int f(x)dx = \int u dv = uv - \int v du$. B) $\int f(x)dx = \int f(\phi(t))\phi'(t)dt$.

C) $\int f(x)dx = \int \sum_{k=1}^{n} a_k f_k(x)\, dx = \sum_{k=1}^{n} a_k \int f_k(x)dx$.

D) $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$.

E) $\int f(x)dx = \int f[\phi(t)]\phi'(t)dt$.

12. Aniqmas integralni bo'laklab integrallash formulasini ko'rsating.

A) $\int u dv = \int v du$. B) $\int u dv = u - \int v du$. C) $\int u dv = uv - \int v du$.

D) $\int u dv = uv + \int v du$. E) $\int u dv = u + \int v du$.

13. $\int x^2 \ln x\, dx$ integralni hisoblash uchun integral ostidagi ifodani qanday bo'laklash maqsadga muvofiq bo'ladi?

A) $u=x$, $dv=x\ln x dx$. B) $u=x^2$, $dv=\ln x dx$. C) $u=\ln x$, $dv=x^2dx$.

D) $u=x\ln x$, $dv=xdx$. E) $u=x^2\ln x$, $dv=dx$.

14. $\int xe^x dx$ integralni hisoblash uchun integral ostidagi ifodani qanday bo'laklash kerak ?

A) $u=x$, $dv=e^x dx$. B) $u=e^x$, $dv=xdx$. C) $u=xe^x$, $dv=dx$.

D) $u=1$, $dv=x e^x dx$. E) $u = \sqrt{x}$, $dv = \sqrt{x} e^x dx$.

15. Ushbu integrallardan qaysi birini bo'laklab integrallash usulida hisoblab bo'lmaydi ?

A) $\int x^n a^x dx$. B) $\int x^n \ln x\, dx$. C) $\int x^n \sin x\, dx$. D) $\int x^n \cos x dx$.

E) keltirilgan barcha integrallarni bo'laklab integrallash mumkin.

16. $\int \frac{dx}{\sqrt{x^2+bx+c}}$ $(b,c\neq 0)$ kvadrat uchhadli integral qaysi almashtirma orqali jadval integraliga keltiriladi?

A) $t=x+c/2$. B) $t=x+b/2$. C) $t=x-c/2$. D) $t=x-b/2$. E) $t=x^2+bx+c$.

17. $\int \frac{dx}{\sqrt{x^2+4x-5}}$ integralni hisoblang.

A) $\frac{1}{2}\ln\left|\frac{x-1}{x+5}\right| + C$. B) $\ln\left|x+2+\sqrt{x^2+4x-5}\right| + C$. C) $\arcsin\frac{x+2}{3} + C$.

D) $\arccos\frac{x+2}{3} + C$. E) $2\sqrt{x^2+4x-5} + C$.

18. $\int \frac{dx}{x^2+bx+c}$ $(b,c\neq 0)$ kvadrat uchhadli integral qaysi almashtirma orqali jadval integraliga keltiriladi?

A) $t=x+c/2$. B) $t=x+b/2$. C) $t=x-c/2$. D) $t=x-b/2$. E) $t=x^2+bx+c$.

19. $\int \frac{dx}{x^2+4x-5}$ integralni hisoblang.

A) $\frac{1}{6}\ln\left|\frac{x-1}{x+5}\right| + C$. B) $\frac{1}{6}\ln\left|\frac{x+5}{x-1}\right| + C$. C) $\frac{1}{6}\ln\left|\frac{x+5}{x+1}\right| + C$.

D) $\frac{1}{6}\ln\left|\frac{x-5}{x-1}\right| + C$. E) $\ln\left|x^2+4x-5\right| + C$.

20. Quyidagi integrallardan qaysi biri elementar funksiyalar orqali ifodalanmaydi?

A) $\int \frac{\ln x}{x}dx$. B) $\int \frac{\sin\ln x}{x}dx$. C) $\int \frac{\cos\ln x}{x}dx$. D) $\int \frac{\sin x}{x}dx$;

E) keltirilgan barcha integrallar elementar funksiyalar orqali ifodalanadi.

4. Ratsional funksiyalar va ularni integrallash

1. Quyidagi yig'indilardan qaysi biri ko'phadni ifodalaydi ?
A)$\sum_{k=0}^{n} a_k \sin kx$. B)$\sum_{k=0}^{n} a_k \sin^k x$. C)$\sum_{k=0}^{n} a_k x^k$. D)$\sum_{k=0}^{n} a_k x^{-k}$. E)$\sum_{k=0}^{n} a_k x_k$.
2. $P(x)=(x^2+2x-10)^2(3x+5)$ ko'phadning darajasini aniqlang:
A) 1. B) 2. C) 3. D) 4. E) 5.
3. Agar $P_n(x)$ va $Q_m(x)$ ko'phadlar bo'lsa, unda quyidagi funksiyalardan qaysi biri ratsional funksiya deyiladi ?
A) $P_n(x)+Q_m(x)$. B) $P_n(x)-Q_m(x)$. C) $P_n(x)/Q_m(x)$.
D) $P_n(x)\cdot Q_m(x)$. E) $P_n[Q_m(x)]$.
4. Qaysi shartda $R(x)=P_n(x)/Q_m(x)$ to'g'ri ratsional funksiya deyiladi ?
A)$m \geq n$. B) $m \leq n$. C) $m>n$; D) $m<n$. E) $m \neq n$.
5. Qaysi holda $R(x)=P_n(x)/Q_m(x)$ noto'g'ri ratsional funksiya bo'lmaydi?
A)$m \geq n$. B) $m \leq n$. C) $m=n$. D) $m>n$. E) $m=n-1$.
6. Quyidagilardan qaysi biri to'g'ri ratsional funksiya bo'ladi ?
A) $\frac{x^2+x+1}{x+1}$. B) $\frac{(x+1)^3}{x^2+x+1}$. C) $\frac{x^2+x+1}{x^3+1}$.
D) $\frac{x^2+x+1}{(x+1)^2}$. E) $\frac{x^2+x+1}{x^2+1}$.
7. I tur eng sodda ratsional funksiyani ko'rsating.
A) $\frac{Ax+B}{x-a}$. B) $\frac{Ax+B}{(x-a)^k}$. C) $\frac{A}{(x-a)^k}, k \geq 2$. D) $\frac{A}{x-a}$. E) $\frac{x+b}{x-a}$.
8. II tur eng sodda ratsional funksiyani ko'rsating.
A) $\frac{Ax+B}{x^2+px+q}$. B) $\frac{Ax+B}{(x-a)^k}$. C) $\frac{A}{x-a}$.
D) $\frac{A}{x^2+px+q}$. E) $\frac{A}{(x-a)^k}, k \geq 2$.
9. III tur eng sodda ratsional funksiyani ko'rsating.
A) $\frac{Ax+B}{x^2+px+q}$. B) $\frac{Ax+B}{(x^2+px+q)^k}, k \geq 2$. C) $\frac{Ax+B}{(x-a)^k}, k \geq 3$.
D) $\frac{Ax^2+Bx+C}{x^2+px+q}$. E) $\frac{Ax^2+Bx+C}{(x^2+px+q)^k}$.
10. IV tur eng sodda ratsional funksiyani ko'rsating.
A) $\frac{Ax+B}{x^2+px+q}$. B) $\frac{Ax+B}{(x^2+px+q)^k}, k \geq 2$. C) $\frac{Ax+B}{(x-a)^k}, k \geq 3$.
D) $\frac{Ax^2+Bx+C}{x^2+px+q}$. E) $\frac{Ax^2+Bx+C}{(x^2+px+q)^k}$.
11. I tur eng sodda ratsional funksiya integrali qaysi elementar funksiyalar orqali ifodalanadi ?
A) faqat logarifmik funksiya. B) logarifmik va arktangens funksiyalar.

C) faqat ratsional funksiya. D) ratsional va arktangens funksiyalar.
E) faqat arktangens funksiyalar.

12. II tur eng sodda ratsional funksiya integrali qaysi elementar funksiyalar orqali ifodalanadi ?
A) faqat logarifmik funksiya. B) logarifmik va arktangens funksiyalar.
C) faqat ratsional funksiya. D) ratsional va arktangens funksiyalar.
E) logarifmik va ratsional funksiyalar.

13. III tur eng sodda ratsional funksiya integrali qaysi elementar funksiyalar orqali ifodalanadi ?
A) faqat logarifmik funksiya. B) logarifmik va arktangens funksiyalar.
C) faqat ratsional funksiya. D) ratsional va arktangens funksiyalar.
E) logarifmik va ratsional funksiyalar.

14. IV tur eng sodda ratsional funksiya integrali qaysi elementar funksiyalar orqali ifodalanadi ?
A) faqat logarifmik funksiya. B) logarifmik va arktangens funksiyalar.
C) faqat ratsional funksiya. D) ratsional va arktangens funksiyalar.
E) logarifmik va ratsional funksiyalar.

15. Agar $R(x)=Q_m(x)/P_n(x)$ to‘g‘ri ratsional funksiya maxraji $P_n(x)$ faqat oddiy haqiqiy ildizlarga ega bo‘lsa, uning yoyilmasi qaysi turdagi eng sodda funksiyalardan iborat bo‘ladi ?
A) faqat I va II turdagi. B) faqat I turdagi.
C) faqat II turdagi. D) I, II,III turdagi. E) barcha turdagi.

16. $R(x)=\frac{3x^2+5x-1}{(x-5)(x-3)(x+1)}$ ratsional funksiyaning yoyilmasi qanday ko‘rinishda bo‘ladi ?
A) $\frac{A_1x+B_1}{x-5}+\frac{A_2x+B_2}{x-3}+\frac{A_3x+B_3}{x+1}$. B) $\frac{A_1}{x-5}+\frac{A_2}{x-3}+\frac{A_3}{x+1}$.
C) $\frac{A_1}{(x-5)(x-3)}+\frac{A_2}{(x-5)(x+1)}+\frac{A_3}{(x-3)(x+1)}$.
D) $\frac{A_1x+B_1}{(x-5)(x-3)}+\frac{A_2x+B_2}{(x-5)(x+1)}+\frac{A_3x+B_3}{(x-3)(x+1)}$.
E) to‘g‘ri javob keltirilmagan.

17. $\int\frac{3x+5}{x^2+2x-3}dx$ integralni hisoblang.
A) $\frac{2}{x-1}+\frac{1}{x+3}+C$. B) $\frac{1}{x-1}-\frac{2}{x+3}+C$. C) $2\,ln\left|\frac{x+3}{x-1}\right|+C$.
D) $2\ln|x-1|+\ln|x+3|+C$. E) $\ln|x-1|-2\ln|x+3|+C$.

18. Agar $R(x)=Q_m(x)/P_n(x)$ to‘g‘ri ratsional funksiya maxraji $P_n(x)$ faqat karrali haqiqiy ildizlarga ega bo‘lsa, uning yoyilmasi qaysi turdagi eng sodda funksiyalardan iborat bo‘ladi ?
A) faqat I va II turdagi . B) faqat I turdagi.
C) faqat II turdagi. D) I, II,III turdagi. E) barcha turdagi.

19. Agar $R(x)=Q_m(x)/P_n(x)$ to'g'ri ratsional funksiya maxraji $P_n(x)$ faqat oddiy kompleks ildizlarga ega bo'lsa, uning yoyilmasi qaysi turdagi eng sodda funksiyalardan iborat bo'ladi ?

A) faqat I turdagi. B) faqat II turdagi. C) faqat III turdagi.

D) faqat IV turdagi. E) faqat III va IV turdagi.

20. $R(x) = \frac{3x^3+2x^2+5x-1}{(x^2+x+1)(x^2+2x+5)}$ ratsional funksiyaning yoyilmasi qanday ko'rinishda bo'ladi ?

A) $\frac{A_1}{x^2+x+1} + \frac{A_2}{x^2+2x+5}$. B) $\frac{A_1x}{x^2+x+1} + \frac{A_2x}{x^2+2x+5}$.

C) $\frac{A_1x+B_1}{x^2+x+1} + \frac{A_2x+B_2}{x^2+2x+5}$. D) $\frac{A_1x^2+B_1x+C_1}{x^2+x+1} + \frac{A_2x^2+B_2x+C_2}{x^2+2x+5}$.

E) to'g'ri javob keltirilmagan.

5. Ayrim irratsional ifodali integrallarni hisoblash

1. Binomial integralning umumiy ko'rinishi qayerda to'g'ri ifodalangan ?

A) $\int x(a+bx)^p dx$. B) $\int x^r(a+bx)^p dx$. C) $\int x^r(a+bx^s)^p dx$.
D) $\int x^r(a+bx^s)dx$. E) $\int x^r(ax^t+bx^s)^p dx$.

2. Qaysi shartda $\int x^r(a+bx^s)^p dx$ binomial integral albatta elementar funksiyalar orqali ifodalanadi ?

A) p–butun son. B) r–butun son. C) s–butun son.
D) $r+s$–butun son. E) keltirilgan barcha hollarda.

3. Qaysi shartda $\int x^r(a+bx^s)^p dx$ binomial integral elementar funksiyalarda integrallanuvchi bo'lmasligi mumkin ?

A) $\frac{r+1}{s}+p$–butun son. B) $\frac{r+1}{s}$–butun son. C) s–butun son.
D) p–butun son. E) keltirilgan barcha hollarda integrallanuvchi bo'ladi.

4. $\int x^r(a+bx^s)^p dx$ binomial integralda $(r+1)/s$– butun son, $p=k/m$ bo'lsa , qaysi almashtirma orqali undan ratsional funksiyali integralga o'tiladi ?

A) $a+bx^s=t$. B) $a+bx^s=t^k$. C) $a+bx^s=t^m$.
D) $a+bx=t^m$. E) $ax^{-s}+b=t^m$.

5. $\int x^r(a+bx^s)^p dx$ binomial integralda p– butun son, $r=k/m$ va $s=q/m$ bo'lsa , qaysi almashtirma orqali undan ratsional funksiyali integralga o'tiladi ?

A) $x=t^r$. B) $x=t^m$. C) $x=t^k$. D) $x=t^q$. E) $x=t^{kq}$.

6. $\int x^r(a+bx^s)^p dx$ binomial integralda $p+(r+1)/s$– butun son, $p=k/m$ bo'lsa , qaysi almashtirma orqali undan ratsional funksiyali integralga o'tiladi ?

A) $a+bx^s=t$. B) $a+bx^s=t^k$. C) $a+bx^s=t^m$.
D) $a+bx=t^m$. E) $ax^{-s}+b=t^m$.

7. $\int x^{2/3}(a+bx^{3/4})^2 dx$ binomial integraldan ratsional funksiyali integralga qaysi almashtirma orqali o'tiladi ?

A) $a+bx^{3/4}=t$. B) $a+bx^{3/4}=t^3$. C) $x=t^{12}$. D) $x=t^3$. E) $x=t^4$.

8. $\int x^{2/3}(a+bx^{5/6})^{1/2}dx$ binomial integraldan ratsional funksiyali integralga qaysi almashtirma orqali o'tiladi ?

A) $a+bx^{5/6}=t$. B) $a+bx^{5/6}=t^2$. C) $ax^{-5/6}+b=t$.
D) $ax^{-5/6}+b=t^2$. E) $x=t^6$.

9. $\int x^{2/3}(a+bx^{3/4})^{7/9}dx$ binomial integraldan ratsional funksiyali integralga qaysi almashtirma orqali o'tiladi ?

A) $a+bx^{3/4}=t$. B) $a+bx^{3/4}=t^9$. C) $ax^{-3/4}+b=t$.
D) $ax^{-3/4}+b=t^9$. E) $x=t^{12}$.

10. $\int R(x,x^{m/n},\dots,x^{r/s})dx$ irratsional funksiyali integral qanday almashtirma orqali ratsional funksiyali integralga keltiriladi ?

A)$x=t^n$. B) $x=t^s$. C) $x=t^k, k=EKUK(n,\dots,s)$.
D) $x=t^k,\ k=EKUB(n,\dots,s)$. E) $x=t^{ns}$.

11. $\int R(x,x^{2/3},x^{3/4},x^{1/6})dx$ integral qanday almashtirma orqali ratsional funksiyali integralga keltiriladi ?

A)$x=t^3$. B) $x=t^4$. C) $x=t^6$. D) $x=t^{12}$. E) $x=t^{13}$.

12. $\int \frac{1+\sqrt[4]{x}}{1+\sqrt[3]{x}}dx$ integralni hisoblash qaysi ko'rinishdagi ratsional funksiyali integralga keltiriladi?

A) $\int \frac{1+t^3}{1+t^4}t^{12}dt$. B) $\int \frac{1+t^4}{1+t^3}t^{12}dt$. C) $\int \frac{1+t^3}{1+t^4}t^{11}dt$. D)$\int \frac{1+t^4}{1+t^3}t^{11}\,dt$.
E) to'g'ri javob keltirilmagan.

13. $\int R\left[x,(\frac{ax+b}{cx+d})^{\frac{m}{n}},\dots,(\frac{ax+b}{cx+d})^{\frac{r}{s}}\right]dx$ irratsional funksiyali integral qanday almashtirma orqali ratsional funksiyali integralga keltiriladi ?

A)$\frac{ax+b}{cx+d}=t^n$. B) $\frac{ax+b}{cx+d}=t^s$. C) $\frac{ax+b}{cx+d}=t^{ns}$.

D) $\frac{ax+b}{cx+d}=t^k,\quad k=EKUK\ (n,\cdots,s)$. E) $\frac{ax+b}{cx+d}=t^k,\quad k=EKUB\ (n,\cdots,s)$.

14. $\int R\left(x,\sqrt{\frac{ax+b}{cx+d}},\sqrt[3]{\frac{ax+b}{cx+d}}\right)dx$ irratsional funksiyali integral qanday almashtirma orqali ratsional funksiyali integralga keltiriladi ?

A)$\frac{ax+b}{cx+d}=t^2$. B) $\frac{ax+b}{cx+d}=t^3$. C) $\frac{ax+b}{cx+d}=t^6$. D) $\frac{ax+b}{cx+d}=t^5$. E) $x=t^6$.

15. $\int\frac{1+\sqrt{6x+1}}{1+\sqrt[3]{6x+1}}dx$ integralni hisoblash qaysi ratsional funksiyali integralga keltiriladi?

A) $\int\frac{1+t^2}{1+t^3}t^5dt$. B) $\int\frac{1+t^3}{1+t^2}t^5dt$. C) $\int\frac{1+t^2}{1+t^3}dt$. D) $\int\frac{1+t^3}{1+t^2}dt$.

E) to‘g‘ri javob keltirilmagan.

16. $\int R(x,\sqrt{ax^2+bx+c})dx$, $a>0$, irratsional funksiyali integralni ratsional funksiyali integralga keltiruvchi Eylerning I almashtirmasini ko‘rsating.

A) $\sqrt{ax^2+bx+c}=t$. B) $\sqrt{ax^2+bx+c}=x\sqrt{a}-t$.

C) $\sqrt{ax^2+bx+c}=xt+a$. D) $\sqrt{ax^2+bx+c}=(x-a)t$.

E) to‘g‘ri javob keltirilmagan.

17. $\int R(x,\sqrt{ax^2+bx+c})dx$, $c>0$, irratsional funksiyali integralni ratsional funksiyali integralga keltiruvchi Eylerning II almashtirmasini ko‘rsating.

A) $\sqrt{ax^2+bx+c}=t+c$. B) $\sqrt{ax^2+bx+c}=\sqrt{c}x+t$.

C) $\sqrt{ax^2+bx+c}=xt+\sqrt{c}$. D) $\sqrt{ax^2+bx+c}=(x-c)t$.

E) $\sqrt{ax^2+bx+c}=\sqrt{ax^2+bc}+ct$.

18. ax^2+bx+c kvadrat uchxad α va β haqiqiy ildizlarga ega bo‘lsa, $\int R(x,\sqrt{ax^2+bx+c})dx$ irratsional funksiyali integralni ratsional funksiyali integralga keltiruvchi Eylerning III almashtirmasini ko‘rsating.

A) $\sqrt{ax^2+bx+c}=xt-\alpha$. B) $\sqrt{ax^2+bx+c}=xt-\beta$.

C) $\sqrt{ax^2+bx+c}=xt-\alpha\beta$. D) $\sqrt{ax^2+bx+c}=(x-\alpha)t$.

E) $\sqrt{ax^2+bx+c}=(x-\alpha-\beta)t$.

19. $\int R(x,\sqrt{x^2+3x-5})dx$ irratsional funksiyali integralni ratsional funksiyali integralga keltiruvchi Eylerning I almashtirmasini ko‘rsating.

A) $\sqrt{x^2+3x-5}=t$. B) $\sqrt{x^2+3x-5}=x-t$. C) $\sqrt{x^2+3x-5}=xt+1$.

D) $\sqrt{x^2+3x-5}=(x-1)t$. E) to‘g‘ri javob keltirilmagan.

20. $\int R(x,\sqrt{9-2x^2+3x})dx$ irratsional funksiyali integralni ratsional funksiyali integralga keltiruvchi Eylerning I almashtirmasini ko'rsating.

A) $\sqrt{9-2x^2+3x}=t$. B) $\sqrt{9-2x^2+3x}=x-t$.

C) $\sqrt{9-2x^2+3x}=xt+3$. D) $\sqrt{9-2x^2+3x}=(x-3)t$.

E) to'g'ri javob keltirilmagan.

6. Ayrim trigonometrik ifodali integrallarni hisoblash

1. Trigonometrik funksiyali ifodalarni ratsional funksiyaga ketiruvchi universal almashtirmani ko‘rsating.

A) $\sin x=t$. B) $\cos x=t$. C) $\mathrm{tg}x=t$. D) $\mathrm{ctg}x=t$. E) $\mathrm{tg}(x/2)=t$.

2. $t=\mathrm{tg}\frac{x}{2}$ universal almashtirma qatnashgan quyidagi tengliklardan qaysi biri noto‘g‘ri ?

A) $\sin x=\frac{2t}{1+t^2}$. B) $\cos x=\frac{1-t^2}{1+t^2}$. C) $dx=\frac{2dt}{1+t^2}$. D) $x=2\mathrm{arctg}\, t$.

E) keltirilgan barcha tengliklar to‘g‘ri.

3. $\int R(\cos x)\sin x dx$ ko‘rinishdagi integrallarni hisoblash uchun qaysi almashtirmadan foydalaniladi?

A) $\cos x=t$. B) $\sin x=t$. C) $\mathrm{tg}x=t$. D) $\mathrm{ctg}x=t$. E) $\mathrm{tg}2x=t$.

4. Trigonometrik ifodali $\int(1-\cos^4 x)\sin x dx$ integralni hisoblang.

A) $\cos x-\sin^4 x+C$. B) $\sin x-\frac{\cos^5 x}{5}+C$. C) $-\cos x+\frac{\cos^5 x}{5}+C$.

D) $\sin x-\frac{\sin^5 x}{5}+C$. E) $-\cos x+\frac{\sin^5 x}{5}+C$.

5. $\int R(\sin x)\cos x dx$ ko‘rinishdagi integral qanday almashtirma yordamida hisoblanishi mumkin ?

A) $\cos x=t$. B) $\sin x=t$. C) $\mathrm{tg}x=t$. D) $\mathrm{ctg}x=t$. E) $\mathrm{tg}2x=t$.

6. $\int\frac{\cos x}{1-\sin x}dx$ integral javobi qayerda to‘g‘ri ko‘rsatilgan ?

A) $\frac{\sin x}{1-\cos x}+C$. B) $\frac{\sin x}{1-\sin x}+C$. C) $\frac{1+\sin x}{1-\sin x}+C$.

D) $-\ln|1-\sin x|+C$. E) $\ln|1-\cos x|+C$.

7. Quyidagi almashtirmalarning qaysi biridan $\int R(\mathrm{tg}x)dx$ ko‘rinishdagi integralni ratsional funksiyali integralga keltirishda foydalanib bo‘lmaydi?

A) $t=\mathrm{tg}x$. B) $t=\sin x$. C) $t=\mathrm{ctg}x$. D) $t=\mathrm{tg}\frac{x}{2}$.

E) ko‘rsatilgan barcha almashtirmalardan foydalanib bo‘ladi .

8. $\int \sin^{2m+1} x \cos^{n} x dx$ integralni hisoblash uchun qaysi almashtirmadan foydalanish qulay ?

A) $\sin x = t$. B) $\cos x = t$. C) $\mathrm{tg} x = t$. D) $\mathrm{ctg} x = t$. E) $\sin^2 x = t$.

9. $\int \sin^{3} x \cos^{5} x dx$ integralni hisoblang.

A) $\frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + C$. B) $\frac{\sin^8 x}{8} - \frac{\cos^6 x}{6} + C$. C) $\frac{\cos^8 x}{8} - \frac{\sin^6 x}{6} + C$.

D) $\frac{\sin^8 x}{8} - \frac{\sin^6 x}{6} + C$. E) $\frac{\sin^8 x}{8} + \frac{\sin^6 x}{6} + C$.

10. $\int \sin^{m} x \cos^{2n+1} x dx$ integralni hisoblash uchun qaysi almashtirmadan foydalanish qulay ?

A) $\sin x = t$. B) $\cos x = t$. C) $\mathrm{tg} x = t$. D) $\mathrm{ctg} x = t$. E) $\cos^2 x = t$.

11. • $\int \sin^{5} x \cos^{3} x dx$ integralni hisoblang.

A) $\frac{\cos^6 x}{6} - \frac{\cos^8 x}{8} + C$. B) $\frac{\sin^6 x}{6} - \frac{\cos^8 x}{8} + C$. C) $\frac{\cos^6 x}{6} - \frac{\sin^8 x}{8} + C$.

D) $\frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C$. E) $\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + C$.

12. $\int \sin^{-2m} x \cos^{2n} x dx$ integralni hisoblash uchun qaysi almashtirmadan foydalanish qulay ?

A) $\sin x = t$. B) $\cos x = t$. C) $\mathrm{tg} x = t$. D) $\cos^2 x = t$. E) $\sin^2 x = t$.

13. $\int \frac{\cos^2 x dx}{\sin^4 x}$ integralni hisoblang.

A) $-\frac{\sin^3 x}{3} + C$. B) $-\frac{\cos^3 x}{3} + C$. C) $-\frac{\mathrm{ctg}^3 x}{3} + C$.

D) $-\frac{\mathrm{tg}^3 x}{3} + C$. E) $-\frac{5\cos^3 x}{3\sin^5 x} + C$.

14. $\int \sin^{2m} x \cos^{-2n} x dx$ integralni hisoblash uchun qaysi almashtirmadan foydalanish qulay ?

A) $\sin x = t$. B) $\cos x = t$. C) $\mathrm{tg} x = t$. D) $\cos^2 x = t$. E) $\sin^2 x = t$.

15. $\int \frac{\sin^2 x dx}{\cos^4 x}$ integralni hisoblang.

A) $\frac{\sin^3 x}{3} + C$. B) $\frac{\cos^3 x}{3} + C$. C) $\frac{\mathrm{ctg}^3 x}{3} + C$.

D) $\frac{\mathrm{tg}^3 x}{3} + C$. E) $\frac{5\sin^3 x}{3\cos^5 x} + C$.

16. $\int \sin^{-2m} x \cos^{-2n} x dx$ integralni hisoblash uchun qaysi almashtirmadan foydalanish qulay ?

A) $\sin x = t$. B) $\cos x = t$. C) $\mathrm{tg} x = t$. D) $\cos^2 x = t$. E) $\sin^2 x = t$.

17. $\int \frac{dx}{\sin^2 x \cos^2 x}$ integralni hisoblang.

A) $\frac{1}{\sin x \cos x} + C$. B) $-\frac{1}{\sin x \cos x} + C$. C) $\frac{1}{\sin x} + \frac{1}{\cos x} + C$.

D) $\mathrm{tg} x + \mathrm{ctg} x + C$. E) $\mathrm{tg} x - \mathrm{ctg} x + C$.

18. $\int \sin mx \cos nx dx$, $m \neq n$, integral qaysi usulda hisoblanadi ?

A) o'zgaruvchilarni almashtirish. B) bo'laklab integrallash.

C) yoyish. D) universal almashtirmadan foydalanish.

E) aniq bir usulni ko'rsatib bo'lmaydi.

19. $-10\int \sin 3x \cos 2x dx$ integralni hisoblang.

A) $\cos 5x + 5\cos x + C$. B) $\sin 5x + 5\cos x + C$. C) $\cos 5x + 5\sin x + C$.

D) $\sin 5x + 5\sin x + C$. E) $-\cos 3x \cdot \sin 2x + C$.

20. $\int \cos mx \cos nx dx$, $m \neq n$, integral qaysi usulda hisoblanadi ?

A) o'zgaruvchilarni almashtirish. B) bo'laklab integrallash.

C) yoyish. D) universal almashtirmadan foydalanish.

E) aniq bir usulni ko'rsatib bo'lmaydi.

7. Ikki karrali integral va uning xossalari

1. Ikki o'zgaruvchili $z=f(x,y)$ funksiyadan yopiq D soha bo'yicha integral yig'indini tuzishda quyidagilardan qaysi biri bajarilmaydi ?

A) D sohada $z=f(x,y)$ funksiyaning eng katta va eng kichik qiymati aniqlanadi .

B) D soha qandaydir chiziqlar bilan ΔD_i ($i=1,2, ..., n$) kichik sohachalarga ajratiladi .

C) har bir ΔD_i ($i=1,2, ..., n$) sohachadan ixtiyoriy bir $M(x_i,y_i)$ nuqta tanlanadi va unda funksiya qiymati $f(x_i,y_i)$ hisoblanadi .

D) funksiyaning hisoblangan $f(x_i,y_i)$, $i=1,2, ..., n$, qiymatlari ΔD_i sohacha yuzasi ΔS_i ga ko'paytirilib, $f(x_i,y_i)\Delta S_i$ ko'paytmalar yig'indisi topiladi .

E) ko'rsatilgan barcha amallar bajariladi .

2. $V_n = \sum_{i=1}^{n} f(x_i,y_i)\Delta S_i$ integral yig'indi orqali $I = \iint_D f(x,y)dxdy$ ikki karrali integral qanday aniqlanadi ?

A) $I=\max V_n$. B) $I=\min V_n$. C) $I = \lim_{n\to\infty} V_n$.

D) $I = \sum_{k=1}^{n} V_k$. E) $I = \lim_{n\to\infty} \sum_{k=1}^{n} V_k$.

3. ***Teoremaning shartini ko'rsating:*** Agar D yopiq sohada $z=f(x,y)$ funksiya ··· bo'lsa , unda $I = \iint_D f(x,y)dxdy$ ikki karrali integral mavjud bo'ladi .

A) monoton . B) uzluksiz . C) chegaralangan . D) davriy .

E) to'g'ri javob keltirilmagan .

4. Ikki karrali integralning geometrik ma'nosi qayerda to'g'ri ko'rsatilgan ?

A) tekis shakl yuzi . B) egri chiziq yoyi uzunligi . C) silindrik jism hajmi . D) aylanma jism hajmi . E) aylanma jism sirti .

5. Agar integrallash sohasi D tomonlari $a=4$ va $b=5$ bo'lgan to'g'ri to'rtburchak bo'lsa, ikki karrali $I = \iint_D dxdy$ integral qiymati nimaga teng ?

A) $I=8$. B) $I=9$. C) $I=10$. D) $I=18$. E) $I=20$.

6. Agar integrallash sohasi D radiusi R=4 bo'lgan doira bo'lsa, ikki karrali $I = \iint_D 2dxdy$ integral qiymati nimaga teng ?

A) $I=8\pi^2$. B) $I=16\pi^2$. C) $I=32\pi^2$. D) $I=20\pi$. E) $I=16\pi$.

7. Ikki karrali integralning mexanik ma'nosi qayerda to'g'ri ko'rsatilgan ?

A) notekis harakatda bosib o'tilgan masofa .
B) bir jinsli bo'lmagan sterjen massasi .
C) bir jinsli bo'lmagan plastinka massasi .
D) o'zgaruvchi kuch bajargan ish .
E) bir jinsli bo'lmagan jism massasi .

8. Ikki karrali integral xossasi qayerda xato ko'rsatilgan ?

A) $\iint_D f_1(x,y)\cdot f_2(x,y)dxdy = \iint_D f_1(x,y)dxdy \cdot \iint_D f_2(x,y)dxdy$.

B) $\iint_D (f_1(x,y) + f_2(x,y))dxdy = \iint_D f_1(x,y)dxdy + \iint_D f_2(x,y)dxdy$.

S) $\iint_D (f_1(x,y) - f_2(x,y))dxdy = \iint_D f_1(x,y)dxdy - \iint_D f_2(x,y)dxdy$.

D) $\iint_D Cf(x,y)dxdy = C\iint_D f(x,y)dxdy \ , C - const$.

E) barcha xossalat to'g'ri ifodalangan .

9. Agar $\iint_D f(x,y)dxdy = 5$ bo'lsa, $\iint_D (-3)f(x,y)dxdy$ integral qiymatini toping.

A) −3 . B) 15 . C) −15 . D) 0 .
E) bu integral qiymatini aniq ko'rsatib bo'lmaydi .

10. Agar $\iint_D f(x,y)dxdy = 5, \quad \iint_D g(x,y)dxdy = -2$ bo'lsa, $\iint_D f(x,y)g(x,y)dxdy$ integral qiymati nimaga teng ?

A) −10 . B) 10 . C) −2 . D) 5 .
E) bu integral qiymatini aniq ko'rsatib bo'lmaydi .

11. $\iint_D f(x,y)dxdy = 5, \quad \iint_D g(x,y)dxdy = -2$ bo'lsa, $\iint_D [f(x,y) + g(x,y)]dxdy$ integral qiymati nimaga teng ?

A) −7 . B) 7 . C) −3 . D) 3

E) bu integral qiymatini aniq ko'rsatib bo'lmaydi .

12. $\iint\limits_D f(x,y)dxdy = 5, \quad \iint\limits_D [f(x,y)+g(x,y)]dxdy = 4$ bo'lsa, $\iint\limits_D g(x,y)dxdy$ integral qiymati nimaga teng ?

A) −1 . B) 1 . C) −9 . D) 9 .
E) bu integral qiymatini aniq ko'rsatib bo'lmaydi .

13. $\iint\limits_D [f(x,y)-g(x,y)]dxdy = 2, \quad \iint\limits_D [f(x,y)+g(x,y)]dxdy = 4$ bo'lsa, $\iint\limits_D f(x,y)dxdy$ integral qiymati nimaga teng ?

A) 1 . B) 2 . C) 3 . D) 4 .
E) bu integral qiymatini aniq ko'rsatib bo'lmaydi .

14. $\iint\limits_D [f(x,y)-g(x,y)]dxdy = 2, \quad \iint\limits_D [f(x,y)+g(x,y)]dxdy = 4$ bo'lsa, $\iint\limits_D g(x,y)dxdy$ integral qiymati nimaga teng ?

A) 1 . B) 2 . C) 3 . D) 4 .
E) bu integral qiymatini aniq ko'rsatib bo'lmaydi .

15. $\iint\limits_D [f(x,y)-g(x,y)]dxdy = 2, \quad \iint\limits_D [f(x,y)+g(x,y)]dxdy = 4$ bo'lsa, $\iint\limits_D [2f(x,y)+3g(x,y)]dxdy$ integral qiymati nimaga teng ?

A) 5 . B) 7 . C) 9 . D) 11 .
E) bu integral qiymatini aniq ko'rsatib bo'lmaydi .

16. Integrallash sohasi $D=\{(x,y): 1\le x\le 3, \quad 2\le y\le 5\}$ to'g'ri to'rtburchakdan iborat bo'lib, unda $z=f(x,y)$ funksiya uzluksiz va $2\le f(x,y)\le 4$ shart bajarilsin. Bu holda $\iint\limits_D f(x,y)dxdy$ ikki karrali integral qiymati qaysi kesmada joylashgan?

A) [6 , 12] . B) [8 , 16] . C) [10 , 20] .
D) [12 , 24] . E) [14 , 28] .

17. Integrallash sohasi D radiusi R=2 bo'lgan yopiq doiradan iborat bo'lib, unda $z=f(x,y)$ funksiya uzluksiz va $3\le f(x,y)\le 4$ shart bajarilsin.

Bu holda $\iint\limits_D f(x,y)dxdy$ ikki karrali integral qiymati qaysi kesmada joylashgan?

A) $[6\pi^2, 10\pi^2]$. B) $[8\pi^2, 12\pi^2]$. C) $[10\pi^2, 14\pi^2]$.
D) $[12\pi^2, 16\pi^2]$. E) $[14\pi^2, 18\pi^2]$.

18. Qayerda ikki karrali integralning additivlik xossasi ifodalangan ?

A) $\iint\limits_D [f(x,y) + g(x,y)]dxdy = \iint\limits_D f(x,y)dxdy + \iint\limits_D g(x,y)dxdy$.

B) $\iint\limits_D [f(x,y) \pm g(x,y)]dxdy = \iint\limits_D f(x,y)dxdy \pm \iint\limits_D g(x,y)dxdy$.

C) $\iint\limits_{D_1+D_2} f(x,y)dxdy = \iint\limits_{D_1} f(x,y)dxdy + \iint\limits_{D_2} f(x,y)dxdy$.

D) $\iint\limits_D f(x,y)dxdy = f(x_0,y_0)\cdot S(D)$.

E) Bu yerda integralning additivlik xossasi keltirilmagan .

19. Agar D_1 va D_2 o'zaro kesishmaydigan, radiuslari $R_1=2$ va $R_2=3$ bo'lgan doiralardan iborat va $D=D_1+D_2$ bo'lsa, ikki karrali $I = \int\limits_D dxdy$ integral qiymatini toping.

A) $I=4\pi^2$. B) $I=9\pi^2$. C) $I=13\pi^2$. D) $I=16\pi^2$. E) $I=20\pi^2$.

20. Qayerda ikki karrali integralning o'rta qiymati haqidagi teoremaning tasdig'i ko'rsatilgan?

A) $\iint\limits_D [f(x,y) + g(x,y)]dxdy = \iint\limits_D f(x,y)dxdy + \iint\limits_D g(x,y)dxdy$.

B) $\iint\limits_D [f(x,y) \pm g(x,y)]dxdy = \iint\limits_D f(x,y)dxdy \pm \iint\limits_D g(x,y)dxdy$.

C) $\iint\limits_{D_1+D_2} f(x,y)dxdy = \iint\limits_{D_1} f(x,y)dxdy + \iint\limits_{D_2} f(x,y)dxdy$.

D) $\iint\limits_D f(x,y)dxdy = f(x_0,y_0)\cdot S$.

E) Bu yerda o'rta qiymat haqidagi teoremaning tasdig'i keltirilmagan .

8. Ikki karrali integralni hisoblash. Ikki karrali integrallar

1. Tekislikdagi D soha OY koordinata o'qi bo'yicha to'g'ri soha bo'lishi uchun quyidagi shartlardan qaysi biri talab etilmaydi ?

A) D soha chap tomondan $x=a$ va o'ng tomondan $x=b$ $(a<b)$ vertikal to'g'ri chiziqlar bilan chegaralangan .

B) D soha $[a,b]$ kesmada uzluksiz bo'lgan $y=\varphi_1(x)$ va $y=\varphi_2(x)$ $[\varphi_1(x)\leq\varphi_2(x)]$ funksiyalarning grafiklari bilan chegaralangan .

C) D sohaning ichki nuqtasidan o'tuvchi va OY o'qiga parallel har qanday to'g'ri chiziq soha chegarasini faqat ikkita huqtada kesib o'tadi .

D)D sohani ichki nuqtasidan o'tuvchi va OX koordinata o'qiga parallel bo'lgan har qanday L to'g'ri chiziq soha chegarasini faqat ikkita huqtada kesib o'tadi .

E) keltirilgan barcha shartlar talab etiladi .

2. Tekislikdagi D soha OX koordinata o'qi bo'yicha to'g'ri soha bo'lishi uchun quyidagi shartlardan qaysi biri talab etilmaydi ?

A) D soha quyidan $y=c$ va yuqoridna $y=d$ $(c<d)$ gorizontal to'g'ri chiziqlar bilan chegaralangan .

B) D soha $[a,b]$ kesmada uzluksiz bo'lgan $x=\psi_1(y)$ va $x=\psi_2(y)$ $[\psi_1(y)\leq\psi_2(y)]$ funksiyalarning grafiklari bilan chegaralangan .

C) D sohaning ichki nuqtasidan o'tuvchi va OY o'qiga parallel har qanday to'g'ri chiziq soha chegarasini faqat ikkita huqtada kesib o'tadi .

D) D sohani ichki nuqtasidan o'tuvchi va OX koordinata o'qiga parallel bo'lgan har qanday L to'g'ri chiziq soha chegarasini faqat ikkita huqtada kesib o'tadi .

E) keltirilgan barcha shartlar talab etiladi .

3. Quyidagi sohalardan qaysi biri OY o'qi bo'yicha to'g'ri soha bo'lmaydi ?

A) aylana bilan chegaralangan soha . B) ellips bilan chegaralangan soha .

C) to'g'ri to'rtburchak bilan chegaralangan soha .

D) ikkita konsentrik aylana bilan chegaralangan soha .

E) keltirilgan barcha sohalar OY o'qi bo'yicha to'g'ri soha bo'ladi .

4. Quyidagi sohalardan qaysi biri OX o'qi bo'yicha to'g'ri soha bo'lmaydi ?

A) aylana bilan chegaralangan soha . B) ellips bilan chegaralangan soha .

C) to'g'ri to'rtburchak bilan chegaralangan soha .

D) ikkita konsentrik aylana bilan chegaralangan soha .

E) keltirilgan barcha sohalar OX o'qi bo'yicha to'g'ri soha bo'ladi .

5. Quyidagidan qaysi biri ikki karrali integral bo'lmaydi ?

A) $I = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \right) dx$. B) $I = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right) dy$.

C) $\int_a^b \left(\int_c^d f(x,y)dy \right) dx$. D) $\int_c^d \left(\int_a^b f(x,y)dx \right) dy$.

E) keltirilgan integrallarning hammasi ikki karrali integral bo'ladi .

6. $\int_0^1 \left(\int_0^x (x+y)dy \right) dx$ ikki karrali integral qiymatini toping.

A) 0 . B) 0.5 . C) 1 . D) −1 . E) −0.5 .

7. $\int_0^1 \left(\int_0^y (x-y)dx \right) dy$ ikki karrali integral qiymatini toping.

A) 0 . B) 1 . C) 1/6 . D) −1 . E) −1/6 .

8. $\int_0^1 \left(\int_1^2 (x+y)dy \right) dx$ ikki karrali integral qiymatini toping.

A) 2 . B) 1.5 . C) 1 . D) 0.5 . E) 0 .

9. $\int_0^1 \left(\int_1^2 (x-y)dx \right) dy$ ikki karrali integral qiymatini toping.

A) 2 . B) 1.5 . C) 1 . D) 0.5 . E) 0 .

10. OX koordinata o'qi bo'yicha to'g'ri D soha uchun $\iint_D f(x,y)dxdy$ ikki karrali integralni hisoblash formulasi qayerda to'g'ri ko'rsatilgan ?

A) $\iint_D f(x,y)dxdy = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right) dy$.

B) $\iint_D f(x,y)dxdy = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \right) dx$.

C) $\iint\limits_D f(x,y)dxdy = \int\limits_{\psi_1(y)}^{\psi_2(y)}\left(\int\limits_a^b f(x,y)dx\right)dy$

D) $\iint\limits_D f(x,y)dxdy = \int\limits_{\varphi_1(x)}^{\varphi_2(x)}\left(\int\limits_c^d f(x,y)dy\right)dx$.

E) $\iint\limits_D f(x,y)dxdy = \int\limits_{\varphi_1(x)}^{\varphi_2(x)}\left(\int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx\right)dy$.

11. OX koordinata o'qi bo'yicha to'g'ri D soha uchun $\iint\limits_D f(x,y)dxdy$ ikki karrali integralni hisoblash formulasi qayerda to'g'ri ko'rsatilgan ?

A) $\iint\limits_D f(x,y)dxdy = \int\limits_c^d\left(\int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx\right)dy$.

B) $\iint\limits_D f(x,y)dxdy = \int\limits_a^b\left(\int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right)dx$.

C) $\iint\limits_D f(x,y)dxdy = \int\limits_{\psi_1(y)}^{\psi_2(y)}\left(\int\limits_a^b f(x,y)dx\right)dy$.

D) $\iint\limits_D f(x,y)dxdy = \int\limits_{\varphi_1(x)}^{\varphi_2(x)}\left(\int\limits_c^d f(x,y)dy\right)dx$.

E) $\iint\limits_D f(x,y)dxdy = \int\limits_{\varphi_1(x)}^{\varphi_2(x)}\left(\int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx\right)dy$.

12. $x=0$ va $x=1$ vertikal to'g'ri chiziqlar, $y=x^3$ va $y=x^2$ egri chiziqlar bilan chegaralangan D soha bo'yicha $\iint\limits_D xy\,dxdy$ ikki karrali integral qiymatini hisoblang .

A) 1/8 . B) 1/18 . C) 1/28 . D) 1/38 . E) 1/48 .

13. $0\le x\le 2$, $0\le y\le 1$ to'g'ri to'rtburchakdan iborat D soha bo'yicha $\iint\limits_D (x+y)dxdy$ ikki karrali integral qiymatini hisoblang .

A) 1 . B) 2 . C) 3 . D) 4 . E) 5 .

14. $0\le x\le \pi/2$, $0\le y\le \pi/2$ kvadratdan iborat D soha bo'yicha $\iint\limits_D (\sin x + \cos y)dxdy$ ikki karrali integral qiymatini hisoblang .

A) $\pi/4$. B) $\pi/2$. C) $3\pi/4$. D) π . E) $5\pi/4$.

15. $0 \le x \le \pi/2$, $0 \le y \le \pi/2$ kvadratdan iborat D soha bo'yicha $\iint\limits_D \sin x \cos y dxdy$ ikki karrali integral qiymatini hisoblang .

A) π . B) 0 . C) 1 . D) $-\pi$. E) -1 .

16. $\varphi(u,v)$ va $\psi(u,v)$ funksiyalarning yakobiani $I(u,v)$ qanday aniqlanadi ?

A) $I(u,v) = \frac{\partial\varphi}{\partial u} \cdot \frac{\partial\varphi}{\partial v} - \frac{\partial\psi}{\partial u} \cdot \frac{\partial\psi}{\partial v}$. B) $I(u,v) = \frac{\partial\varphi}{\partial u} \cdot \frac{\partial\psi}{\partial v} - \frac{\partial\varphi}{\partial v} \cdot \frac{\partial\psi}{\partial u}$.

C) $I(u,v) = \frac{\partial\varphi}{\partial u} \cdot \frac{\partial\psi}{\partial u} - \frac{\partial\varphi}{\partial v} \cdot \frac{\partial\psi}{\partial v}$. D) $I(u,v) = \frac{\partial\varphi}{\partial v} \cdot \frac{\partial\psi}{\partial v} - \frac{\partial\varphi}{\partial u} \cdot \frac{\partial\psi}{\partial u}$.

E) $I(u,v) = \frac{\partial\psi}{\partial u} \cdot \frac{\partial\psi}{\partial v} - \frac{\partial\varphi}{\partial u} \cdot \frac{\partial\varphi}{\partial v}$.

17. Quyidagi deferminantlarning qaysi biri $\varphi(u,v)$ va $\psi(u,v)$ funksiyalarning $I(u,v)$ yakobianini ifodalamaydi ?

A) $\begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} \end{vmatrix}$. B) $-\begin{vmatrix} \frac{\partial\psi}{\partial u} & \frac{\partial\varphi}{\partial v} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\varphi}{\partial v} \end{vmatrix}$.

C) $\begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\psi}{\partial u} \\ \frac{\partial\varphi}{\partial v} & \frac{\partial\psi}{\partial v} \end{vmatrix}$. D) $\begin{vmatrix} \frac{\partial\psi}{\partial v} & \frac{\partial\psi}{\partial u} \\ \frac{\partial\varphi}{\partial v} & \frac{\partial\varphi}{\partial u} \end{vmatrix}$.

E) barcha deferminantlar $I(u,v)$ yakobianni ifodalaydi .

18. $\varphi(u,v)=u+v$ va $\psi(u,v)=u-v$ funksiyalarning $I(u,v)$ yakobianini toping.

A) $I(u,v)=uv$. B) $I(u,v)=u^2v$. C) $I(u,v)=uv^2$. D) -2 . E) 1 .

19. $\varphi(u,v)=u^2-v^2$ va $\psi(u,v)=u^2+v^2$ funksiyalarning $I(u,v)$ yakobianini toping.

A) $I(u,v)=4uv$. B) $I(u,v)=4u^2v$. C) $I(u,v)=4uv^2$. D) -4 . E) 4 .

20. $x=\rho\cos\theta$, $y=\rho\sin\theta$ qutb koordinatalarining $I(\rho,\theta)$ yakobianini toping.

A) $I(\rho,\theta)=\rho\theta$. B) $I(\rho,\theta)= -\rho\theta$. C) $I(\rho,\theta)=-\theta$.
D) $I(\rho,\theta)= \rho$. E) $I(\rho,\theta)= \theta$.

9. Ikki karrali integralning amaliy tatbiqlari

1. $z=f(x,y)>0$ funksiya bilan aniqlangan σ sirtning XOY koordinata tekisligidagi proyeksiyasi chegarasi L chiziqdan iborat D yopiq soha bo'lsin. Unda σ sirt bilan chegaralangan va yasovchisi L bo'lgan to'gri silindrik jism hajmi V qaysi formula bilan hisoblanadi ?

A) $V=\iint_D f''_{xy}(x,y)dxdy$. B) $V=\iint_D f(x,y)dxdy$. C) $V=\pi\iint_D f^2(x,y)dxdy$.

D) $V=\pi\iint_D [f''_{xy}(x,y)]^2 dxdy$. E) to'g'ri javob keltirilmagan .

2. Jism $z=f(x,y)$ va $z=g(x,y)$ $[f(x,y)\le g(x,y)]$ funksiyalar bilan aniqlangan σ(f) va σ(g) sirtlar bilan chegaralangan, σ(f) va σ(g) sirtlarning XOY koordinata tekisligidagi proyeksiyalari ustma-ust tushib, biror yopiq D sohadan iborat bo'lsa, jism hajmi V qaysi formula formula bilan hisoblanadi?

A) $V=2\pi\iint_D [g^2(x,y)-f^2(x,y)]dxdy$. B) $V=2\pi\iint_D [g(x,y)-f(x,y)]dxdy$.

C) $V=2\pi\iint_D [g(x,y)+f(x,y)]dxdy$. D) $V=\iint_D [g(x,y)-f(x,y)]dxdy$.

E) $V=\iint_D [g(x,y)+f(x,y)]dxdy$.

3. XOY koordinata tekisligida yotuvchi yopiq D soha ko'rinishidagi yassi geometrik shakl yuzasi S qaysi formula bilan hisoblanadi?

A) $S=\iint_D xdxdy$. B) $S=\iint_D ydxdy$. C) $S=\iint_D xydxdy$.

D) $S=\iint_D \sqrt{x^2+y^2}dxdy$. E) $S=\iint_D dxdy$.

4. $y=\varphi(x)$, $y=\psi(x)$ $[\varphi(x)\le\psi(x)]$ va $x=a$, $x=b$ chiziqlar bilan chegaralangan D yopiq sohaning S yuzasi hisoblash formulasi qayerda to'g'ri ifodalangan?

A) $S=\int_a^b\{\int_{\varphi(x)}^{\psi(x)}dx\}dy$. B) $S=\int_a^b\{\int_{\varphi(x)}^{\psi(x)}dy\}dx$. C) $S=\int_{\varphi(x)}^{\psi(x)}\{\int_a^b dx\}dy$.

D) $S=\int_{\varphi(x)}^{\psi(x)}\{\int_a^b dy\}dx$. E) to'g'ri javob keltirilmagan .

5. Quyidagi ikki karrali integrallardan qaysi biri $y=\varphi(x)$, $y=\psi(x)$ $[\varphi(x)\le\psi(x)]$ va $x=a$, $x=b$ chiziqlar bilan chegaralangan D yopiq sohaning yuzasini ifodalamaydi?

A) $\int_a^b\{\int_{\varphi(x)}^{\psi(x)}dy\}dx$. B) $-\int_a^b\{\int_{\psi(x)}^{\varphi(x)}dy\}dx$. C) $-\int_b^a\{\int_{\varphi(x)}^{\psi(x)}dy\}dx$. D) $\int_b^a\{\int_{\psi(x)}^{\varphi(x)}dy\}dx$.

E) keltirilgan barcha integrallar D yopiq sohaning yuzasini ifodalaydi .

6. $x=\varphi(y)$, $x=\psi(y)$ $[\varphi(y)\le\psi(y)]$ va $y=a$, $y=b$ chiziqlar bilan chegaralangan D yopiq sohaning S yuzasi hisoblash formulasi qayerda to'g'ri ifodalangan?

A) $S=\int_a^b\{\int_{\varphi(y)}^{\psi(y)}dx\}dy$. B) $S=\int_a^b\{\int_{\varphi(y)}^{\psi(y)}dy\}dx$. C) $S=\int_{\varphi(y)}^{\psi(y)}\{\int_a^b dx\}dy$.

D) $S=\int_{\varphi(y)}^{\psi(y)}\{\int_a^b dy\}dx$. E) to'g'ri javob keltirilmagan .

7. Quyidagi ikki karrali integrallardan qaysi biri $x=\varphi(y)$, $x=\psi(y)$ $[\varphi(y)\le\psi(y)]$ va $y=a$, $y=b$ chiziqlar bilan chegaralangan D yopiq sohaning S yuzasi hisoblash formulasi qayerda to'g'ri ifodalangan?

A) $\int_a^b\{\int_{\varphi(y)}^{\psi(y)}dx\}dy$. B) $-\int_a^b\{\int_{\varphi(y)}^{\psi(y)}dx\}dy$. C) $-\int_b^a\{\int_{\varphi(y)}^{\psi(y)}dx\}dy$. D) $\int_b^a\{\int_{\psi(y)}^{\varphi(y)}dx\}dy$.

E) keltirilgan barcha integrallar D yopiq sohaning yuzasini ifodalaydi .

8. $y=x^3$, $y=x$ va $x=0$, $x=1$ chiziqlar bilan chegaralangan D yopiq sohaning S yuzasini toping.

A) $S=1/2$. B) $S=1/3$. C) $S=1/4$. D) $S=1/5$. E) $S=1/6$.

9. $x=y^4$, $x=y^2$ va $y=0$, $y=1$ chiziqlar bilan chegaralangan D yopiq sohaning S yuzasini toping.

A) $S=1/3$. B) $S=4/15$. C) $S=1/5$. D) $S=2/15$.
E) $S=1/6$.

10. $z=f(x,y)$ funksiya bilan aniqlangan fazodagi σ sirtning XOY koordinata tekisligidagi proyeksiyasi D yopiq sohadan iborat bo'lsa, σ sirtning S yuzasi hisoblanadigan formula qayerda to'g'ri ifodalangan?

A) $S=\iint_D\sqrt{1+f^2(x,y)}dxdy$. B) $S=\iint_D\sqrt{1+[f'_x(x,y)]^2}dxdy$.

C) $S = \iint\limits_D \sqrt{1 + [f'_y(x,y)]^2}\, dxdy$. D) $S = \iint\limits_D \sqrt{1 + [f'_x(x,y)]^2 + [f'_y(x,y)]^2}\, dxdy$.

E) $S = \iint\limits_D \sqrt{1 + [f''_{xy}(x,y)]^2}\, dxdy$.

11. Sirt zichligi $\rho=\rho(x,y)$ funksiya bilan berilgan va yopiq D soha ko'rinishida bo'lgan moddiy yassi shaklning massasi m qaysi formula bilan aniqlnadi?

A) $m = \iint\limits_D \sqrt{\rho(x,y)}\, dxdy$. B) $m = \iint\limits_D \rho^2(x,y)\, dxdy$. C) $m = \iint\limits_D \rho(x,y)\, dxdy$. D) $m = \iint\limits_D [1/\rho(x,y)]\, dxdy$. E) to'g'ri javob keltirilmagan .

12. Markazi O(0,0) nuqtada joylashgan va tomoni $2a$ bo'lgan kvadratdan iborat yassi shaklning sirt zichligi $\rho(x,y)=(3xy)^2$ funksiya bilan berilgan bo'lsa, uning m massasi nimaga teng ?

A) $m=4a^2$. B) $m=4a^3$. C) $m=4a^4$. D) $m=4a^5$.

E) $m=4a^6$.

13: Sirt zichligi $\rho=\rho(x,y)$ funksiya bilan berilgan va yopiq D soha ko'rinishida bo'lgan moddiy yassi shaklning koordinata boshiga nisbatan inertsiya momenti I_0 formulasini ko'rsating .

A) $I_0 = \iint\limits_D \sqrt{x^2 + y^2}\, \rho(x,y)\, dxdy$. B) $I_0 = \iint\limits_D (x^2 + y^2)^2 \rho(x,y)\, dxdy$.

C) $I_0 = \iint\limits_D (x^2 + y^2)\rho(x,y)\, dxdy$. D) $I_0 = \iint\limits_D \frac{\rho(x,y)}{x^2 + y^2}\, dxdy$.

E) $I_0 = \iint\limits_D \frac{x^2 + y^2}{\rho(x,y)}\, dxdy$.

14. Sirt zichligi $\rho=\rho(x,y)$ funksiya bilan berilgan va yopiq D soha ko'rinishida bo'lgan moddiy yassi shaklning OX koordinata o'qiga nisbatan inersiya momenti I_X qaysi formula bilan topiladi?

A) $I_X = \iint\limits_D x^2 \rho(x,y)\, dxdy$. B) $I_X = \iint\limits_D x\rho(x,y)\, dxdy$.

C) $I_X = \iint\limits_D x^2 \rho(x,y)\, dxdy$.

D) $I_X = \iint\limits_D y^2 \rho(x,y)\, dxdy$. E) $I_X = \iint\limits_D xy\rho(x,y)\, dxdy$.

15. Sirt zichligi $\rho=\rho(x,y)$ funksiya bilan berilgan va yopiq D soha ko'rinishida bo'lgan moddiy yassi shaklning OY koordinata o'qiga nisbatan inersiya momenti I_Y qaysi formula bilan topiladi?

A) $I_Y = \iint_D x^2 \rho(x,y)dxdy$. B) $I_Y = \iint_D x\rho(x,y)dxdy$.

C) $I_Y = \iint_D x^2 \rho(x,y)dxdy$.

D) $I_Y = \iint_D y^2 \rho(x,y)dxdy$. E) $I_Y = \iint_D xy\rho(x,y)dxdy$.

16. Sirt zichligi $\rho=\rho(x,y)$ funksiya bilan berilgan va yopiq D soha ko'rinishida bo'lgan moddiy yassi shaklning OX koordinata o'qiga nisbatan statik momenti M_X qaysi formula bilan hisoblanadi?

A) $M_X = \iint_D xy\rho(x,y)dxdy$. B) $M_X = \iint_D x\rho(x,y)dxdy$.

C) $M_X = \iint_D y\rho(x,y)dxdy$.

D) $M_X = \iint_D (x+y)\rho(x,y)dxdy$. E) $M_X = \iint_D (x^2+y^2)\rho(x,y)dxdy$.

17. Sirt zichligi $\rho=\rho(x,y)$ funksiya bilan berilgan va yopiq D soha ko'rinishida bo'lgan moddiy yassi shaklning OY koordinata o'qiga nisbatan statik momenti M_Y qaysi formula bilan hisoblanadi?

A) $M_Y = \iint_D xy\rho(x,y)dxdy$. B) $M_Y = \iint_D x\rho(x,y)dxdy$.

C) $M_Y = \iint_D y\rho(x,y)dxdy$.

D) $M_Y = \iint_D (x+y)\rho(x,y)dxdy$. E) $M_Y = \iint_D (x^2+y^2)\rho(x,y)dxdy$.

18. Sirt zichligi $\rho=\rho(x,y)$ funksiya bilan berilgan va yopiq D soha ko'rinishida bo'lgan moddiy yassi shaklning $M(x_0,y_0)$ og'irlik markazining x_0 abssissasining formulasini ko'rsating.

A) $x_0 = \dfrac{\iint_D xy\rho(x,y)dxdy}{\iint_D \rho(x,y)dxdy}$. B) $x_0 = \dfrac{\iint_D y\rho(x,y)dxdy}{\iint_D \rho(x,y)dxdy}$.

C) $x_0 = \dfrac{\iint_D x\rho(x,y)dxdy}{\iint_D \rho(x,y)dxdy}$.

D) $x_0 = \frac{\iint\limits_D (x+y)\rho(x,y)dxdy}{\iint\limits_D \rho(x,y)dxdy}$. E) $x_0 = \frac{\iint\limits_D (x^2+y^2)\rho(x,y)dxdy}{\iint\limits_D \rho(x,y)dxdy}$.

19. Sirt zichligi $\rho=\rho(x,y)$ funksiya bilan berilgan va yopiq D soha ko'rinishida bo'lgan moddiy yassi shaklning $M(x_0,y_0)$ og'irlik markazining y_0 ordinatasining formulasini ko'rsating.

A) $y_0 = \frac{\iint\limits_D xy\rho(x,y)dxdy}{\iint\limits_D \rho(x,y)dxdy}$. B) $y_0 = \frac{\iint\limits_D y\rho(x,y)dxdy}{\iint\limits_D \rho(x,y)dxdy}$.

C) $y_0 = \frac{\iint\limits_D x\rho(x,y)dxdy}{\iint\limits_D \rho(x,y)dxdy}$.

D) $y_0 = \frac{\iint\limits_D (x+y)\rho(x,y)dxdy}{\iint\limits_D \rho(x,y)dxdy}$. E) $y_0 = \frac{\iint\limits_D (x^2+y^2)\rho(x,y)dxdy}{\iint\limits_D \rho(x,y)dxdy}$.

20. Quyidagi masalalardan qaysi biri ikki karrali integral yordamida yechilmaydi?

A) σ sirt bilan chegaralangan to'g'ri silindrik jism hajmini hisoblash .
B) bir jinsli bo'lmagan sterjen massasini topish .
C) fazodagi sirt yuzasini aniqlash .
D) yassi figuraning og'irlik markazini topish .
E) barcha masalalar yechimi ikki karrali integral orqali ifodalanadi .

10. Uch karrali integral va uning xossalari

1. Uch o'zgaruvchili $w=f(x,y,z)$ funksiyadan fazodagi yopiq V soha bo'yicha integral yig'indini tuzishda quyidagilardan qaysi biri bajarilmaydi ?

A) V soha ixtiyoriy bir usulda ΔV_i (i=1,2, ..., n) kichik sohachalarga ajratiladi .

B) har bir ΔV_i (i=1,2, ..., n) sohachadan ixtiyoriy bir $M(x_i,y_i,z_i)$ nuqta tanlanadi va unda funksiya qiymati $f(x_i,y_i,z_i)$ hisoblanadi .

C) funksiyaning hisoblangan $f(x_i,y_i,z_i)$, i=1,2, ..., n, qiymatlari ΔV_i sohacha hajmi Δv_i ga ko'paytirilib, bu ko'paytmalar yig'indisi S_n topiladi .

D) S_n integral yig'indining absolut qiymati baholanadi .

E) ko'rsatilgan barcha amallar bajariladi .

2. $S_n = \sum_{i=1}^{n} f(x_i,y_i,z_i)\Delta v_i$ integral yig'indi orqali $I = \iiint_V f(x,y,z)dxdydz$ uch karrali integral qanday aniqlanadi ?

A) $I= maxV_n$. B) $I= minV_n$. C) $I = \lim_{n\to\infty} V_n$. D) $I= V_n$.

E) to'g'ri javob keltirilmagan .

3. ***Teoremaning shartini ko'rsating:*** Agar fazodagi V yopiq sohada $w=f(x,y,z)$ funksiya ··· bo'lsa , unda $I = \iiint_V f(x,y,z)dxdydz$ uch karrali integral mavjud bo'ladi .

A) monoton . B) uzluksiz . C) chegaralangan . D) davriy .

E) to'g'ri javob keltirilmagan .

4. Uch karrali integralning geometrik ma'nosi qayerda to'g'ri ko'rsatilgan ?

A) tekis shakl yuzasii . B) egri chiziq yoyi uzunligi . C) silindrik jism sirti . D) fazodagi jism hajmi . E) aylanma jism yon sirti .

5. Integrallash sohasi $V=\{(x, y, z): x^2+ y^2+z^2\leq 9\}$ yopiq shar bo'lgan uch karrali

$\iiint_V dxdydz$ integral qiymati nimaga teng?

A) 27π . B) 30π . C) 33π . D) 36π . E) 39π .

6. Integrallash sohasi $V=\{(x, y, z)\colon |x|\leq 2, |y|\leq 3, |z|\leq 1\}$ yopiq to‘g‘ri burchakli parallelepiped bo‘lgan uch karrali $\iiint\limits_V dxdydz$ integral qiymati nimaga teng?

A) 6π . B) 36 . C) 48 . D) 24 . E) 36π .

7. Uch karrali integralning mexanik ma’nosi qayerda to‘g‘ri ko‘rsatilgan ?

A) notekis harakatda bosib o‘tilgan masofa .

B) bir jinsli bo‘lmagan sterjen massasi . C) bir jinsli bo‘lmagan plastinka massasi .

D) o‘zgaruvchi kuch bajargan ish . E) bir jinsli bo‘lmagan jism massasi .

8. Uch karrali integral xossasi qayerda xato ko‘rsatilgan ?

A) $\iiint\limits_V f_1(x,y,z)\cdot f_2(x,y,z)dxdydz =$

$= \iiint\limits_V f_1(x,y,z)dxdydz \cdot \iiint\limits_V f_2(x,y,z)dxdydz$.

B) $\iiint\limits_V [f_1(x,y,z) + f_2(x,y,z)]dxdydz =$

$= \iiint\limits_V f_1(x,y,z)dxdydz + \iiint\limits_V f_2(x,y,z)dxdydz$.

S) $\iiint\limits_V [f_1(x,y,z) - f_2(x,y,z)]dxdydz =$

$= \iiint\limits_V f_1(x,y,z)dxdydz - \iiint\limits_V f_2(x,y,z)dxdydz$.

D) $\iiint\limits_V Cf(x,y,z)dxdydz = C\iiint\limits_V f(x,y,z)dxdydz$ (C – const.).

E) barcha xossalat to‘g‘ri ifodalangan .

9. Agar $\iiint\limits_V f(x,y,z)dxdydz = 6$ bo‘lsa, $\iiint\limits_V (-4)f(x,y,z)dxdydz$ integral qiymatini toping.

A) -6 . B) 12 . C) -24 . D) 0 .

E) bu integral qiymatini aniq ko‘rsatib bo‘lmaydi .

10. Agar $\iiint\limits_V f(x,y,z)dxdydz = 6$, $\iiint\limits_V g(x,y,z)dxdydz = -2$ bo‘lsa, $\iiint\limits_V f(x,y,z)g(x,y,z)dxdydz$ integral qiymati nimaga teng ?

A) -12 . B) 12 . C) -4 . D) 4 .

E) bu integral qiymatini aniq ko‘rsatib bo‘lmaydi .

11. Agar $\iiint_V f(x,y,z)dxdydz = 6$, $\iiint_V g(x,y,z)dxdydz = -2$ bo‘lsa,

$\iiint_V [f(x,y,z) + g(x,y,z)]dxdydz$ integral qiymati nimaga teng ?

A) −8 . B) 8 . C) −4 . D) 4 .
E) bu integral qiymatini aniq ko‘rsatib bo‘lmaydi .

12. Agar $\iiint_V f(x,y,z)dxdydz = 6$, $\iiint_V g(x,y,z)dxdydz = -2$ bo‘lsa,

$\iiint_V [f(x,y,z) - g(x,y,z)]dxdydz$ integral qiymati nimaga teng ?

A) −8 . B) 8 . C) −4 . D) 4 .
E) bu integral qiymatini aniq ko‘rsatib bo‘lmaydi .

13. Agar $\iiint_V f(x,y,z)dxdydz = 6$, $\iiint_V [f(x,y,z) + g(x,y,z)dxdydz = -2$ bo‘lsa,

$\iiint_V g(x,y,z)dxdydz$ integral qiymati nimaga teng ?

A) −8 . B) 8 . C) −4 . D) 4 .
E) bu integral qiymatini aniq ko‘rsatib bo‘lmaydi .

14. Agar $\iiint_V f(x,y,z)dxdydz = 6$, $\iiint_V [f(x,y,z) - g(x,y,z)dxdydz = -2$ bo‘lsa,

$\iiint_V g(x,y,z)dxdydz$ integral qiymati nimaga teng ?

A) −8 . B) 8 . C) −4 . D) 4 .
E) bu integral qiymatini aniq ko‘rsatib bo‘lmaydi .

15. Agar $\iiint_V [f(x,y,z) + g(x,y,z)]dxdydz = 6$ va

$\iiint_V [f(x,y,z) - g(x,y,z)dxdydz = -2$ bo‘lsa, $\iiint_V g(x,y,z)dxdydz$ integral qiymati nimaga teng ?

A) −8 . B) 8 . C) −4 . D) 4 .
E) bu integral qiymatini aniq ko‘rsatib bo‘lmaydi .

16. Agar $\iiint_V [f(x,y,z) + g(x,y,z)]dxdydz = 8$ va

$\iiint\limits_V [f(x,y,z) - g(x,y,z)dxdydz = -2$ bo‘lsa, $\iiint\limits_V f(x,y,z)dxdydz$ integral qiymati nimaga teng ?

A) −3 . B) 3 . C) −4 . D) 4 .

E) bu integral qiymatini aniq ko‘rsatib bo‘lmaydi .

17. Agar $\iiint\limits_V [f(x,y,z) + g(x,y,z)]dxdydz = 8$ va

$\iiint\limits_V [f(x,y,z) - g(x,y,z)dxdydz = -2$ bo‘lsa, $\iiint\limits_V [2f(x,y,z) + 3g(x,y,z)]dxdydz$ integral qiymati nimaga teng ?

A) 21 . B) 23 . C) 25 . D) 27 .

E) bu integral qiymatini aniq ko‘rsatib bo‘lmaydi .

18. Uch karrali integralning additivlik xossasi qayerda ifodalangan ?

A) $\iiint\limits_V [f(x,y,z) + g(x,y,z)]dxdydz =$

$= \iiint\limits_V f(x,y,z)dxdydz + \iiint\limits_V g(x,y,z)dxdydz$.

B) $\iiint\limits_V [f(x,y,z) \pm g(x,y,z)]dxdydz =$

$= \iiint\limits_V f(x,y,z)dxdydz \pm \iiint\limits_V g(x,y,z)dxdydz$.

C) $\iiint\limits_{V_1+V_2} f(x,y,z)dxdydz = \iiint\limits_{V_1} f(x,y,z)dxdydz + \iiint\limits_{V_2} f(x,y,z)dxdydz$.

D) $\iiint\limits_V f(x,y,z)dxdydz = f(x_0,y_0,z_0)\cdot v$.

E) Bu yerda integralning additivlik xossasi keltirilmagan .

19. Agar $\iiint\limits_{V_1} f(x,y,z)dxdydz = 5$, $\iiint\limits_{V_2} f(x,y,z)dxdydz = -3$ bo‘lsa,

$\iiint\limits_{V_1+V_2} f(x,y,z)dxdydz$ qiymati nimaga teng?

A) 8 . B) −8 . C) 2 . D) −2 .

E) bu integral qiymatini aniq ko‘rsatib bo‘lmaydi.

20. Agar integrallash sohasi V markazi koordinata boshida joylashgan va R=3 radiusili shardan iborat bo‘lib, unda $f(x,y,z) \le 2$ shartni

qanoatlantirsa, $I=\iiint\limits_V f(x,y,z)dxdydz$ integral uchun qaysi tengsizlik o‘rinli bo‘ladi?

A) $I\le 18\pi$. B) $I\le 36\pi$. C) $I\le 54\pi$. D) $I\le 72\pi$. E) $I\le 90\pi$.

11. Uch karrali integrallarni hisoblash.

1. Fazodagi V soha to'g'ri soha bo'lishi uchun quyidagi shartlardan qaysi biri talab etilmaydi ?

A) V soha S yopiq sirt bilan chegaralangan .

B) V sohaning ixtiyoriy ichki nuqtasidan OZ koordinata o'qiga parallel qilib o'tkazilgan har qanday to'g'ri chiziq bu sohaning S chegarasini faqat ikkita nuqtada kesib o'tadi .

C) V sohaning XOY koordinata tekisligidagi D proyeksiyasi to'g'ri sohadan iborat.

D) V sohaning chegaraviy nuqtalari bir bog'lamli to'plamni tashkil etadi .

E) keltirilgan barcha shartlar talab etiladi .

2. Fazodagi quyidagi sohalardan qaysi biri uch karrali to'g'ri soha bo'lmaydi ?

A) shar . B) piramida . C) parallelepiped . D) tor . E) aylanma ellipsoid .

3. Uch karrali integral qayerda to'g'ri ifodalangan?

A) $\int\limits_{\varphi_1(x)}^{\varphi_2(x)} \{\int\limits_a^b [\int\limits_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz\,]dx\,\}dy$.

B) $\int\limits_a^b \{\int\limits_{\psi_1(x,y)}^{\psi_2(x,y)} [\int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,y,z)dz\,]dy\,\}dx$.

C) $\int\limits_{\psi_1(x,y)}^{\psi_2(x,y)} \{\int\limits_{\varphi_1(x)}^{\varphi_2(x)} [\int\limits_a^b f(x,y,z)dz\,]dy\,\}dx$.

D) $\int\limits_a^b \{\int\limits_{\varphi_1(x)}^{\varphi_2(x)} [\int\limits_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz\,]dy\,\}dx$.

E) $\int\limits_{\varphi_1(x)}^{\varphi_2(x)} \{\int\limits_{\psi_1(x,y)}^{\psi_2(x,y)} [\int\limits_a^b f(x,y,z)dz\,]dy\,\}dx$.

4. $\int\limits_0^1 \{\int\limits_0^1 [\int\limits_0^1 (x+y+z)dz\,]dy\,\}dx$ uch karrali integral qiymatini toping.

A) 0 . B) 0.5 . C) 1.0 . D) 1.5 . E) −0.5 .

5. $\int\limits_0^1 \{\int\limits_0^x [\int\limits_0^{xy} (x+z)dz\,]dy\,\}dx$ uch karrali integral qiymatini toping.

A) 0 . B) 5/18 . C) 13/36 . D) 19/144 . E) 23/180 .

6. $\int_0^1\{\int_0^x[\int_0^{xy}(x+y+z)dz]dy\}dx$ uch karrali integral qiymatini toping.

A) 0 . B) 5/16 . C) 7/36 . D) −3/26 . E) −11/46 .

7. Fazodagi V to'g'ri soha quyidan va yuqoridan $z=\psi_1(x,y)$ va $z=\psi_2(x,y)$ sirtlar bilan, uning XOY koordinata tekisligidagi proyeksiyasini ifodalovchi yassi D yopiq soha esa $y=\varphi_1(x)$, $y=\varphi_2(x)$ va $x=a$, $x=b$ chiziqlar bilan chegaralangan bo'lsin. Bu holda uch karrali $I=\iiint_V f(x,y,z)dxdydz$ integralni uch karrali integral orqali hisoblash formulasi qayerda to'g'ri ko'rsatilgan ?

A) $I=\int_{\varphi_1(x)}^{\varphi_2(x)}\{\int_a^b[\int_{\psi_1(x,y)}^{\psi_2(x,y)}f(x,y,z)dz]dx\}dy$.

B) $I=\int_a^b\{\int_{\psi_1(x,y)}^{\psi_2(x,y)}[\int_{\varphi_1(x)}^{\varphi_2(x)}f(x,y,z)dz]dy\}dx$.

C) $I=\int_{\psi_1(x,y)}^{\psi_2(x,y)}\{\int_{\varphi_1(x)}^{\varphi_2(x)}[\int_a^b f(x,y,z)dz]dy\}dx$.

D) $I=\int_a^b\{\int_{\varphi_1(x)}^{\varphi_2(x)}[\int_{\psi_1(x,y)}^{\psi_2(x,y)}f(x,y,z)dz]dy\}dx$.

E) $I=\int_{\varphi_1(x)}^{\varphi_2(x)}\{\int_{\psi_1(x,y)}^{\psi_2(x,y)}[\int_a^b f(x,y,z)dz]dy\}dx$.

8. Fazodagi V to'g'ri soha quyidan va yuqoridan $z=0$ va $z=xy$ sirtlar bilan, uning XOY koordinata tekisligidagi proyeksiyasini ifodalovchi yassi D yopiq soha esa $y=0$, $y=x$ va $x=0$, $x=1$ chiziqlar bilan chegaralangan bo'lsin. Uch karrali $I=\iiint_V(x+z)dxdydz$ integral qiymatini toping.

A) 0 . B) 5/36 . C) 23/180 . D) 21/216 . E) 1 .

9. Fazodagi V to'g'ri soha quyidan va yuqoridan $z=0$ va $z=xy$ sirtlar bilan, uning XOY koordinata tekisligidagi proyeksiyasini ifodalovchi yassi D yopiq soha esa $y=0$, $y=x$ va $x=0$, $x=1$ chiziqlar bilan chegaralangan bo'lsin. Uch karrali $I=\iiint_V(x+y+z)dxdydz$ integral qiymatini toping.

A) 7/36 . B) 5/46 . C) 3/56 . D) 1 . E) 0 .

10. Fazodagi V to'g'ri soha quyidan va yuqoridan $z=0$ va $z=xy$ sirtlar bilan, uning XOY koordinata tekisligidagi proyeksiyasini ifodalovchi

yassi D yopiq soha esa $y=0$, $y=x$ va $x=0, x=1$ chiziqlar bilan chegaralangan bo'lsin. Uch karrali $I = \iiint\limits_V xyzdxdydz$ integral qiymatini toping.

A) 1/16 . B) 1/32 . C) 1/48 . D) 1/ 64 . E) 1/80 .

11. Agar V soha $x=a_1$ va $x=b_1$, $y=a_2$ va $y=b_2$, $z=a_3$ va $z=b_3$ tekisliklar bilan chegaralangan parallelepipeddan iborat bo'lsa, unda uch karrali $I = \iiint\limits_V f(x,y,z)dxdydz$ integralni uch karrali integral orqali hisoblash formulasi qayerda noto'g'ri ko'rsatilgan ?

A) $I = \int\limits_{a_1}^{b_1}\{\int\limits_{a_2}^{b_2}[\int\limits_{a_3}^{b_3} f(x,y,z)dz]dy\}dx$. B) $I = \int\limits_{a_2}^{b_2}\{\int\limits_{a_3}^{b_3}[\int\limits_{a_1}^{b_1} f(x,y,z)dy]dz\}dx$.

C) $I = \int\limits_{a_3}^{b_3}\{\int\limits_{a_2}^{b_2}[\int\limits_{a_1}^{b_1} f(x,y,z)dx]dy\}dz$. D) $I = \int\limits_{a_2}^{b_2}\{\int\limits_{a_1}^{b_1}[\int\limits_{a_3}^{b_3} f(x,y,z)dz]dx\}dy$.

E) keltirilgan barcha formulalar to'g'ri ko'rsatilgan .

12. Integrallash sohasi V $x=0$ va $x=1$, $y=0$ va $y=1$, $z=0$ va $z=1$ tekisliklar bilan chegaralangan parallelepipeddan iborat bo'lgan uch karrali $I = \iiint\limits_V (x+y+z)dxdydz$ integralni hisoblang.

A) 0.5 . B) 1.0 . C) 1.5 . D) 2.0 . E) 2.5 .

13. Integrallash sohasi V $x=0$ va $x=1$, $y=0$ va $y=1$, $z=0$ va $z=1$ tekisliklar bilan chegaralangan parallelepipeddan iborat bo'lgan uch karrali $I = \iiint\limits_V xyzdxdydz$ integral qiymatini toping.

A) 1/2 . B) 3/2 . C) 1/4 . D) 3/4 . E) 1/8 .

14. Uch o'zgaruvchili $\varphi(u,v,t)$, $\psi(u,v,t)$ va $w(u,v,t)$ funksiyalarning jakobiani $I(u,v,t)$ qayerda to'g'ri ifodalangan?

A) $I(u,v,t) = \begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} & \frac{\partial\varphi}{\partial t} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} & \frac{\partial\psi}{\partial t} \\ \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} & \frac{\partial w}{\partial t} \end{vmatrix}$. B) $I(u,v,t) = \begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\psi}{\partial u} & \frac{\partial w}{\partial u} \\ \frac{\partial\varphi}{\partial v} & \frac{\partial\psi}{\partial v} & \frac{\partial w}{\partial v} \\ \frac{\partial\varphi}{\partial t} & \frac{\partial\psi}{\partial t} & \frac{\partial w}{\partial t} \end{vmatrix}$.

C) $I(u,v,t)=\begin{vmatrix} \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} & \frac{\partial\psi}{\partial t} \\ \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} & \frac{\partial w}{\partial t} \\ \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} & \frac{\partial\varphi}{\partial t} \end{vmatrix}$. D) $I(u,v,t)=-\begin{vmatrix} \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} & \frac{\partial w}{\partial t} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} & \frac{\partial\psi}{\partial t} \\ \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} & \frac{\partial\varphi}{\partial t} \end{vmatrix}$.

E) keltirilgan barcha formulalar $I(u,v,t)$ yakobianni to‘g‘ri ifodalaydi .

15. $\varphi(u,v,t)=u-v+t$, $\psi(u,v,t)=u+v-t$ va $w(u,v,t)=-u+v+t$ funksiyalarning $I(u,v,t)$ jakobianini hisoblang.

a. $I(u,v,t)=uvt$. B) $I(u,v,t)=u+v+t$. C) $I(u,v,t)=1$.
D) $I(u,v,t)=0$. E) to‘g‘ri javob keltirilmagan .

16. x, y, z dekart va θ, ρ, z silindrik koordinatalar orasidagi bog‘lanish qayerda to‘g‘ri ifodalangan?

A) $x=\rho\sin\theta$, $y=\rho\cos\theta$, $z=\rho$. B) $x=\rho\cos\theta\sin\theta$, $y=\rho\cos\theta$, $z=\rho$.
C) $x=\rho\cos\theta\sin\theta$, $y=\rho\cos\theta$, $z=\rho\sin\theta$. D) $x=\rho\sin\theta$, $y=\rho\sin\theta\cos\theta$, $z=z$.

E) $x=\rho\cos\theta$, $y=\rho\sin\theta$, $z=z$.

17. $x=\rho\cos\theta$, $y=\rho\sin\theta$, $z=z$ silindrik koordinatalarning I yakobianini toping.

A) $I=\theta$. B) $I=\rho\theta$. C) $I=\rho$. D) $I=\rho/\theta$. E) $I=\rho\,\mathrm{tg}\theta$.

18. Dekart koordinatalaridagi uch karrali $I=\iiint\limits_V (x^2+y^2)z\,dx\,dy\,dz$ integral silindrik koordinatalarda qanday ifodalanadi?

A) $I=\iiint\limits_{V^*} \rho z\,d\rho\,d\theta\,dz$. B) $I=\iiint\limits_{V^*} \rho^2 z\,d\rho\,d\theta\,dz$.

C) $I=\iiint\limits_{V^*} \rho^3 z\,d\rho\,d\theta\,dz$.

D) $I=\iiint\limits_{V^*} \rho^2 \sin\theta\cos\theta\, z\,d\rho\,d\theta\,dz$. E) to‘g‘ri javob keltirilmagan .

19. $x=\rho\sin\varphi\cos\theta$, $y=\rho\sin\varphi\sin\theta$, $z=\rho\cos\varphi$ tengliklar orqali ρ, φ va θ sferik koordinatalarga o‘tish $I(\rho, \varphi, \theta)$ yakobianini hisoblang.

A) $I(\rho,\varphi,\theta)=\rho^2\sin\varphi$. B) $I(\rho,\varphi,\theta)=\rho^2\sin\theta$. C) $I(\rho,\varphi,\theta)=\rho^2\cos\varphi$.
D) $I(\rho,\varphi,\theta)=\rho^2\cos\theta$. E) $I(\rho,\varphi,\theta)=\rho^2\sin\varphi\sin\theta$.

20. Dekart koordinatalaridagi uch karrali $I = \iiint_V (x^2 + y^2 + z^2)dxdydz$ integral sferik koordinatalarda qanday ifodalanadi?

A) $I = \iiint_{V'} \rho \sin\varphi \, d\rho \, d\varphi \, d\theta$. B) $I = \iiint_{V'} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$.

C) $I = \iiint_{V'} \rho^3 \sin\varphi \, d\rho \, d\varphi \, d\theta$. D) $I = \iiint_{V'} \rho^4 \sin\varphi \, d\rho \, d\varphi \, d\theta$.

E) $I = \iiint_{V'} \rho^5 \sin\varphi \, d\rho \, d\varphi \, d\theta$.

12. Uch karrali integralning amaliy tatbiqlari

1. Fazoda to'g'ri sohani tashkil etuvchi T jismning V hajmi uch karrali integral orqali qaysi formula bilan hisoblanadi?

A) $V=\iiint\limits_T xyz\,dxdydz$. B) $V=\iiint\limits_T (x+y+z)dxdydz$. C) $V=\iiint\limits_T dxdydz$.

D) $V=\iiint\limits_T \sqrt{x^2+y^2+z^2}\,dxdydz$. E) $V=\iiint\limits_T (x^2+y^2+z^2)dxdydz$.

2. $z=x^2+y^2$, $z=a^2(x^2+y^2)$, $y=x$, $y=x^2$ sirtlar bilan chegaralangan jismning V hajmini toping.

A) $V=a^3$. B) $V=3(a^3-1)/35$. C) $V=a^2$. D) $V=3(a^2-1)/35$.
E) $8a\sqrt{2a}$.

3. $z=x+y$, $z=axy$, $y+x=1$, $y=0$, $x=0$ sirtlar bilan chegaralangan jismning V hajmini toping.

A) $V=(1-a)/8$. B) $V=(2-a)/12$. C) $V=(4-a)/16$.
D) $V=(6-a)/20$. E) $V=(8-a)/24$.

4. Fazoda to'g'ri sohani tashkil etuvchi va har bir $M(x,y,z)$ nuqtasidagi zichligi $\rho=\rho(x,y,z)$ funksiya bilan aniqlanadigan bir jinsli bo'lmagan T jismning m massasi qaysi formula bilan topiladi ?

A) $m=\iiint\limits_T xyz\,\rho(x,y,z)dxdydz$. B) $m=\iiint\limits_T (x+y+z)\rho(x,y,z)dxdydz$.

C) $m=\iiint\limits_T (x^2+y^2+z^2)\rho(x,y,z)dxdydz$.

D) $m=\iiint\limits_T \rho(x,y,z)dxdydz$.

E) $m=\iiint\limits_T \sqrt{x^2+y^2+z^2}\,\rho(x,y,z)dxdydz$.

5. Har bir $M(x,y,z)$ nuqtasidagi zichligi $\rho(x,y,z)=a(x+y+z)$ funksiya bilan aniqlanadigan $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$ kubning m massasini toping.

A) $a/2$. B) $a/3$. C) $3a/2$. D) $2a/3$. E) $3a^2$.

6. Har bir $M(x,y,z)$ nuqtasidagi zichligi $\rho(x,y,z)=x+y+z$ funksiya bilan aniqlanadigan $0\le x\le a$, $0\le y\le a$, $0\le z\le a$ kubning m massasini toping.

A) $3a/2$. B) $3a^2/2$. C) $3a^3/2$. D) $3a^4/2$. E) $3a^5/2$.

7. Har bir $M(x,y,z)$ nuqtasidagi zichligi $\rho(x,y,z)=x+y+z$ funksiya bilan aniqlanadigan $0\le x\le a$, $0\le y\le b$, $0\le z\le c$ parallelepipedning m massasini toping.

A) $abc/2$. B) $(a+b+c)/2$. C) $abc(a+b+c)/2$.
D) $abc/2(a+b+c)$. E) $(a+b+c)/2abc$.

8. Zichligi $\rho=\rho(x,y,z)$ funksiya bilan aniqlanadigan bir jinsli bo‘lmagan T jismning XOY koordinata tekisligiga nisbatan M_{xy} statik momenti qaysi uch karrali integral orqali ifodalanadi?

A) $\iiint_T y\rho(x,y,z)dxdydz$. B) $\iiint_T x\rho(x,y,z)dxdydz$.

C) $\iiint_T z\rho(x,y,z)dxdydz$.

D) $\iiint_T xy\rho(x,y,z)dxdydz$. E) to‘g‘ri javob keltirilmagan .

9. Zichligi $\rho=\rho(x,y,z)$ funksiya bilan aniqlanadigan bir jinsli bo‘lmagan T jismning XOZ koordinata tekisligiga nisbatan M_{xz} statik momenti qaysi uch karrali integral orqali ifodalanadi?

A) $\iiint_T y\rho(x,y,z)dxdydz$. B) $\iiint_T x\rho(x,y,z)dxdydz$.

C) $\iiint_T z\rho(x,y,z)dxdydz$.

D) $\iiint_T xz\rho(x,y,z)dxdydz$. E) to‘g‘ri javob keltirilmagan .

10. Zichligi $\rho=\rho(x,y,z)$ funksiya bilan aniqlanadigan bir jinsli bo‘lmagan T jismning YOZ koordinata tekisligiga nisbatan M_{yz} statik momenti qaysi uch karrali integral orqali ifodalanadi?

A) $\iiint_T y\rho(x,y,z)dxdydz$. B) $\iiint_T x\rho(x,y,z)dxdydz$.

C) $\iiint_T z\rho(x,y,z)dxdydz$.

D) $\iiint_T yz\rho(x,y,z)dxdydz$. E) to‘g‘ri javob keltirilmagan .

11. Har bir $M(x,y,z)$ nuqtasidagi zichligi $\rho(x,y,z)=x+y+z$ funksiya bilan aniqlanadigan $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$ kubning YOZ koordinata tekisligiga nisbatan S_{yz} statik momenti nimaga teng ?

A) $S_{yz}=2/3$. B) $S_{yz}=3/4$. C) $S_{yz}=4/5$. D) $S_{yz}=5/6$. E) $S_{yz}=6/7$.

12. Har bir $M(x,y,z)$ nuqtasidagi zichligi $\rho(x,y,z)=x+y+z$ funksiya bilan aniqlanadigan $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$ kubning XOZ koordinata tekisligiga nisbatan S_{xz} statik momenti nimaga teng ?

A) $S_{xz}=13/12$. B) $S_{xz}=11/9$. C) $S_{xz}=9/7$. D) $S_{xz}=7/5$.

E) $S_{xz}=5/3$.

13. Bir jinsli bo'lmagan , m massali T jism og'irlik markazining x_0 abssissasini hisoblash formulasi qayerda to'g'ri ko'rsatilgan?

A) $x_0 = \iiint\limits_T z\rho(x,y,z)dxdydz \; / m$. B) $x_0 = \iiint\limits_T y\rho(x,y,z)dxdydz \; / m$.

C) $x_0 = \iiint\limits_T x\rho(x,y,z)dxdydz \; / m$. D) $x_0 = \iiint\limits_T zy\rho(x,y,z)dxdydz \; / m$.

E) $x_0 = \iiint\limits_T (z+y)\rho(x,y,z)dxdydz \; / m$.

14. Bir jinsli bo'lmagan , m massali T jism og'irlik markazining y_0 ordinatasini hisoblash formulasi qayerda to'g'ri ko'rsatilgan?

A) $y_0 = \iiint\limits_T z\rho(x,y,z)dxdydz \; / m$. B) $y_0 = \iiint\limits_T y\rho(x,y,z)dxdydz \; / m$.

C) $y_0 = \iiint\limits_T x\rho(x,y,z)dxdydz \; / m$. D) $y_0 = \iiint\limits_T xz\rho(x,y,z)dxdydz \; / m$.

E) $y_0 = \iiint\limits_T (x+z)\rho(x,y,z)dxdydz \; / m$.

15. Bir jinsli bo'lmagan , m massali T jism og'irlik markazining z_0 aplikatasini hisoblash formulasi qayerda to'g'ri ko'rsatilgan?

A) $z_0 = \iiint\limits_T z\rho(x,y,z)dxdydz \; / m$. B) $z_0 = \iiint\limits_T y\rho(x,y,z)dxdydz \; / m$.

C) $z_0 = \iiint\limits_T x\rho(x,y,z)dxdydz \; / m$. D) $z_0 = \iiint\limits_T xy\rho(x,y,z)dxdydz \; / m$.

E) $z_0 = \iiint\limits_T (x+y)\rho(x,y,z)dxdydz \; / m$.

16. Har bir $M(x,y,z)$ nuqtasidagi zichligi $\rho(x,y,z)=\rho$ o'zgarmas bo'lgan va $x+y+z=8$, $x=0$, $x=2$, $y=0$, $z=0$, $y=4$ sirtlar bilan chegaralangan bir jinsli kesik parallelepiped og'irlik markazining x_0 abssissasini toping.

A) $x_0=8/3$. B) $x_0=26/15$. C) $x_0=14/15$. D) $x_0=14/3$. E) $x_0=12/5$.

17. Har bir $M(x,y,z)$ nuqtasidagi zichligi $\rho(x,y,z)=\rho$ o'zgarmas bo'lgan va $x+y+z=8$, $x=0$, $x=2$, $y=0$, $z=0$, $y=4$ sirtlar bilan chegaralangan bir jinsli kesik parallelepiped og'irlik markazining y_0 ordinatasini toping.

A) $y_0=8/3$. B) $y_0=26/15$. C) $y_0=14/15$. D) $y_0=14/3$. E) $y_0=12/5$.

18. Har bir $M(x,y,z)$ nuqtasidagi zichligi $\rho(x,y,z)=\rho$ o'zgarmas bo'lgan va $x+y+z=8$, $x=0$, $x=2$, $y=0$, $z=0$, $y=4$ sirtlar bilan chegaralangan bir jinsli kesik parallelepiped og'irlik markazining z_0 aplikatasini toping.

A) $z_0=8/3$. B) $z_0=26/15$. C) $z_0=14/15$. D) $z_0=14/3$. E) $z_0=12/5$.

19. Bir jinsli bo'lmagan T jismning XOY koordinata tekisligiga nisbatan inersiya momenti I_{xy} qaysi formula bilan hisoblanadi?

A) $I_{xy}=\iiint\limits_T x^2\rho(x,y,z)dxdydz$. B) $I_{xy}=\iiint\limits_T z^2\rho(x,y,z)dxdydz$.

C) $I_{xy}=\iiint\limits_T y^2\rho(x,y,z)dxdydz$. D) $I_{xy}=\iiint\limits_T xy\rho(x,y,z)dxdydz$.

E) to'g'ri javob keltirilmagan .

20. Bir jinsli bo'lmagan T jismning XOZ koordinata tekisligiga nisbatan inersiya momenti I_{xz} qaysi formula bilan hisoblanadi?

A) $I_{xz}=\iiint\limits_T x^2\rho(x,y,z)dxdydz$. B) $I_{xz}=\iiint\limits_T z^2\rho(x,y,z)dxdydz$.

C) $I_{xz}=\iiint\limits_T y^2\rho(x,y,z)dxdydz$. D) $I_{xz}=\iiint\limits_T xz\rho(x,y,z)dxdydz$.

E) to'g'ri javob keltirilmagan .

TESTLAR KALITI

1. Funksiyani differensiallash.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
D	C	E	C	D	D	E	B	E	B	D	C	D	E	C	A
17	18	19	20												
C	D	B	E												

2. Boshlang'ich funksiya va aniqmas integral.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
C	D	C	C	D	C	E	E	B	B	E	E	C	E	C	E
17	18	19	20												
C	D	E	D												

3. Aniqmas integralni hisoblash .

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A	C	D	C	C	E	D	D	B	A	A	C	C	A	C	B
17	18	19	20												
B	B	A	D												

4. Ratsional funksiyalar va ularni integrallash

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
C	E	C	C	D	C	D	E	A	B	A	C	B	D	B	B
17	18	19	20												
D	A	C	C												

5. Ayrim irratsional ifodali integrallarni hisoblash

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
C	A	C	C	B	E	C	B	D	C	D	C	D	C	B	B
17	18	19	20												
C	D	B	C												

6. Ayrim trigonometrik ifodali integrallarni hisoblash

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
E	E	A	C	B	D	B	B	A	A	D	C	C	C	D	C
17	18	19	20												
E	C	A	C												

7. Ikki karrali integral va uning xossalari

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A	C	B	C	E	C	C	A	C	E	D	A	C	A	C	D
17	18	19	20												
D	C	C	D												

8. Aniq integralni hisoblash usullari

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
D	C	D	D	E	B	E	A	C	B	A	E	C	D	C	B
17	18	19	20												
E	D	A	D												

9. Ikki karrali integralning amaliy tatbiqlari

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
B	D	E	B	E	A	E	C	D	D	C	E	C	D	A	C
17	18	19	20												
B	C	B	B												
10. Uch karrali integral va uning xossalari															
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
D	C	B	D	D	C	E	A	C	E	D	B	A	B	D	B
17	18	19	20												
A	C	C	D												
11. Uch karrali integrallarni hisoblash.															
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	1 6
D	D	D	D	E	C	D	C	A	D	B	C	E	A	E	E
17	18	19	20												
C	C	A	D												
12. Uch karrali integralning amaliy tatbiqlari															
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	1 6
D	D	E	D	C	D	C	C	A	B	D	A	C	B	A	C
17	18	19	20												
B	A	B	C												

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