CS206 Data Structures

Graphs I

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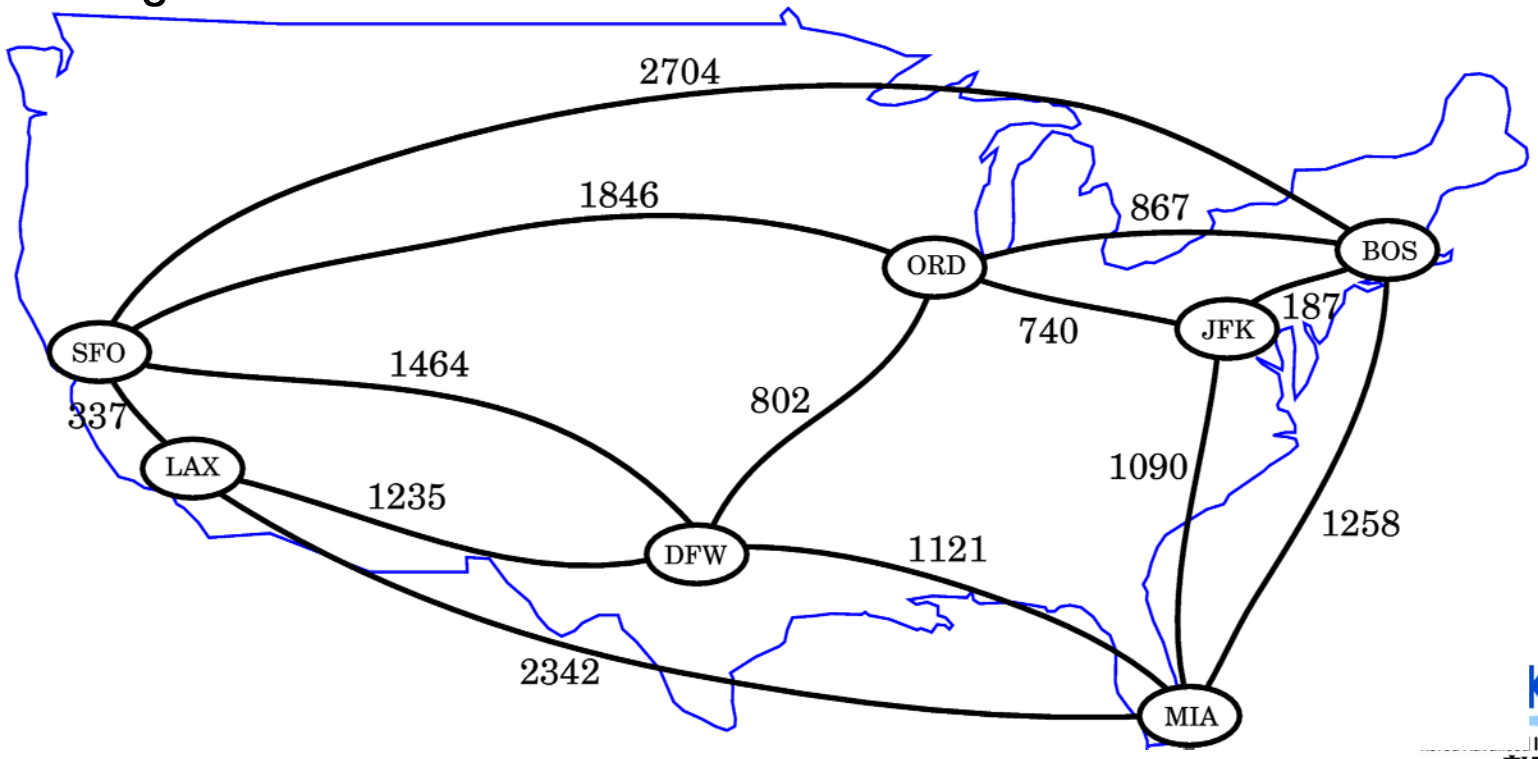
Class Objectives (Ch. 14)

- Get to know data representations of graphs and two different graph traversal methods

Graphs

Graphs

- A graph is a pair (V, E), where
  - V is a set of nodes, called vertices
  - E is a collection of pairs of vertices, called edges
  - Vertices and edges can store elements, i.e., weights
- Example:
  - A vertex represents an airport and stores the airport code
  - An edge represents a flight route between two airports and stores the mileage of the route



Edge Types

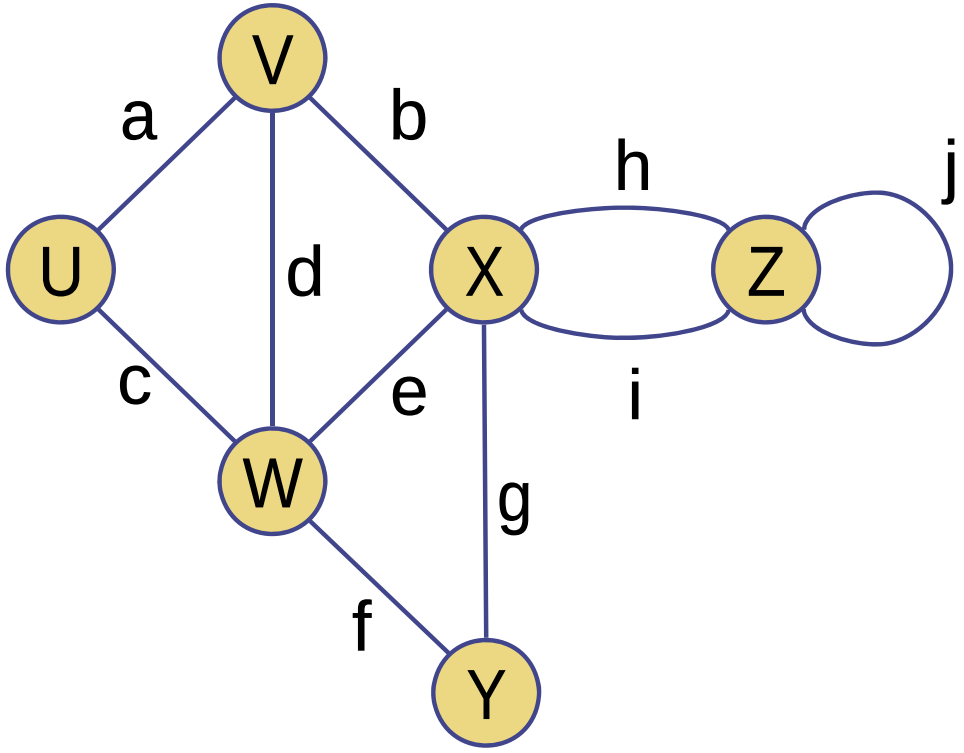
- Directed edge
  - ordered pair of vertices (u,v)
  - first vertex u is the origin
  - second vertex v is the destination
  - e.g., a flight
- Undirected edge
  - unordered pair of vertices (u,v)
  - e.g., a flight route
- Directed graph
  - all the edges are directed
  - e.g., route network
- Undirected graph
  - all the edges are undirected
  - e.g., flight network

Applications

- ☐ Electronic circuits
  - Printed circuit board
  - Integrated circuit
- ☐ Transportation networks
  - Highway network
  - Flight network
- ☐ Computer networks
  - Local area network
  - Internet
  - Web
- ☐ Databases
  - Entity-relationship diagram

Terminology

- End vertices (or endpoints) of an edge
  - U and V are the endpoints of a
- Edges incident on a vertex
  - a, d, and b are incident on V
- Adjacent vertices
  - U and V are adjacent
- Degree of a vertex
  - X has degree 5
- Parallel edges
  - h and i are parallel edges
- Self-loop
  - j is a self-loop



Terminology (cont.)

☐ Path

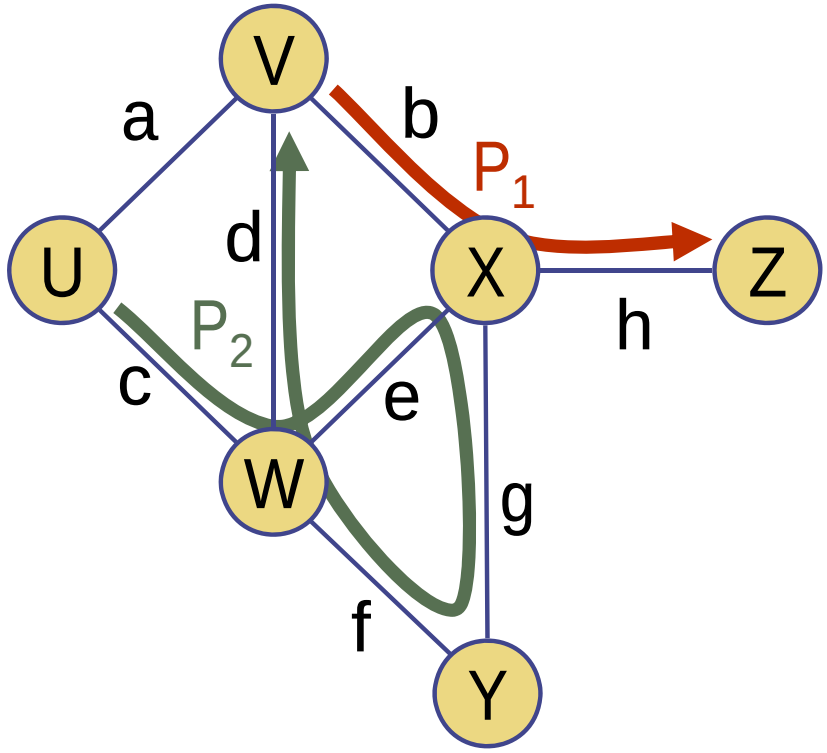
- sequence of alternating vertices and edges
- begins with a vertex
- ends with a vertex
- each edge is preceded and followed by its endpoints

☐ **Simple path**

- path such that all its vertices and edges are **distinct**

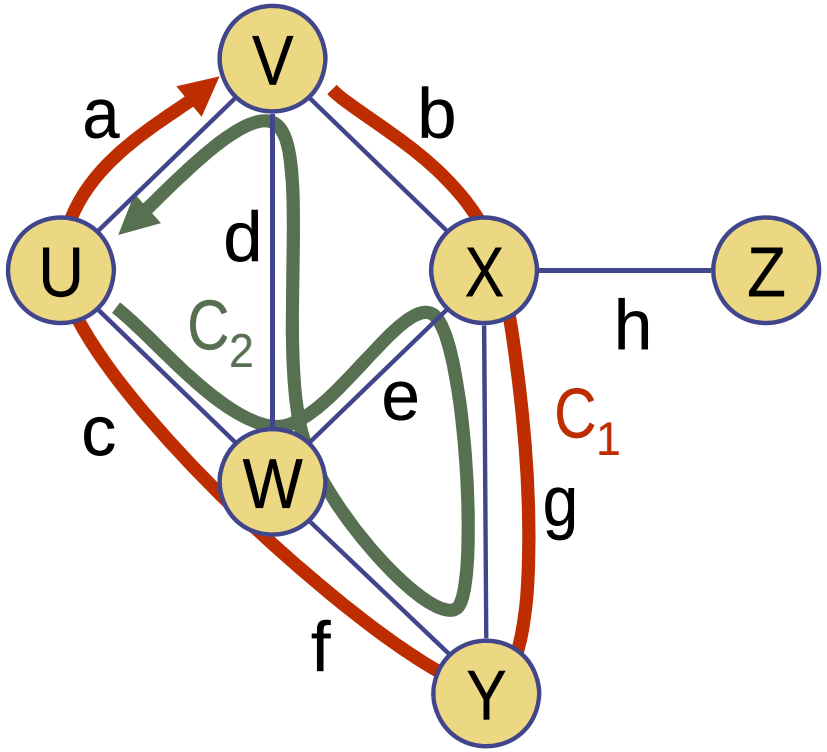
☐ Examples

- P_1=(V,b,X,h,Z) is a simple path
- P_2=(U,c,W,e,X,g,Y,f,W,d,V) is a path that is not simple



Terminology (cont.)

- Cycle
  - circular sequence of alternating vertices and edges
  - each edge is preceded and followed by its endpoints
- Simple cycle
  - cycle such that all its vertices and edges are distinct
- Examples
  - C_1=(V,b,X,g,Y,f,W,c,U,a,↵) is a simple cycle
  - C_2=(U,c,W,e,X,g,Y,f,W,d,V,a,↵) is a cycle that is not simple



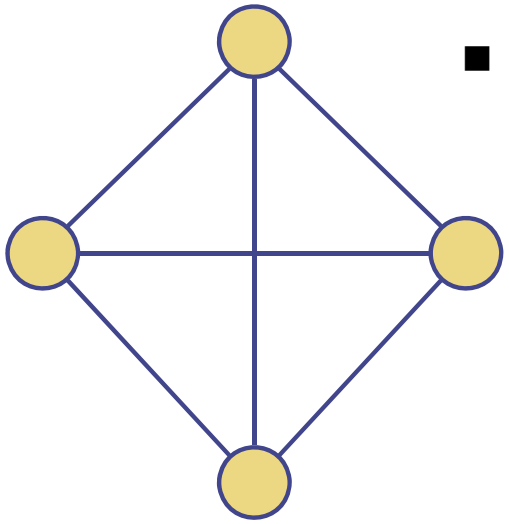
Properties

☐ Notation

- n : number of vertices
- m : number of edges
- deg(v) : degree of vertex v

Example

- $n = 4$
- $m = 6$
- $\deg(v) = 3$



☐ **Property 1**

- $\sum_v \deg(v) = 2m$
- Proof: each edge is counted twice

☐ **Property 2**

- In an undirected graph with no self-loops and no multiple edges

  $m \leq n\,(n - 1)/2$

- Proof: each vertex has degree at most (n - 1)

☐ **What is the bound for a directed graph?**

Main Methods of the Graph ADT

☐ Vertices and edges
- are positions
- store elements

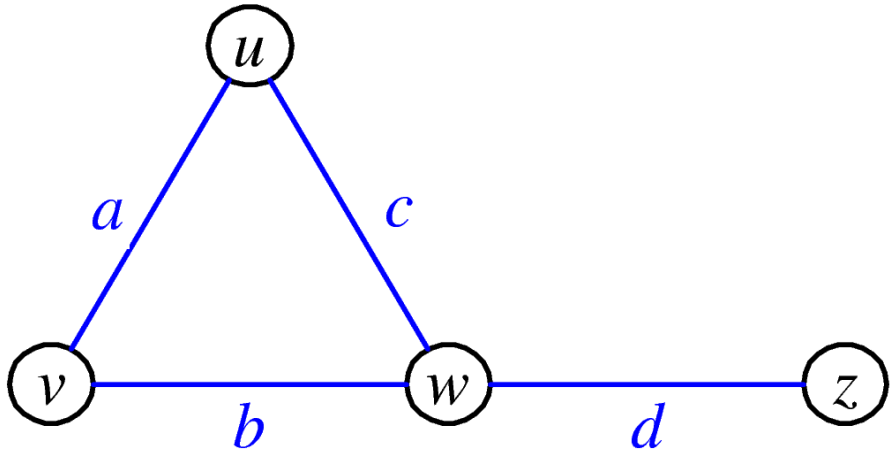
☐ Accessor methods
- endVertices(e): an array of the two endvertices of e
- opposite(v, e): the vertex opposite of v on e
- areAdjacent(v, w): true iff v and w are adjacent
- replace(v, x): replace element at vertex v with x
- replace(e, x): replace element at edge e with x

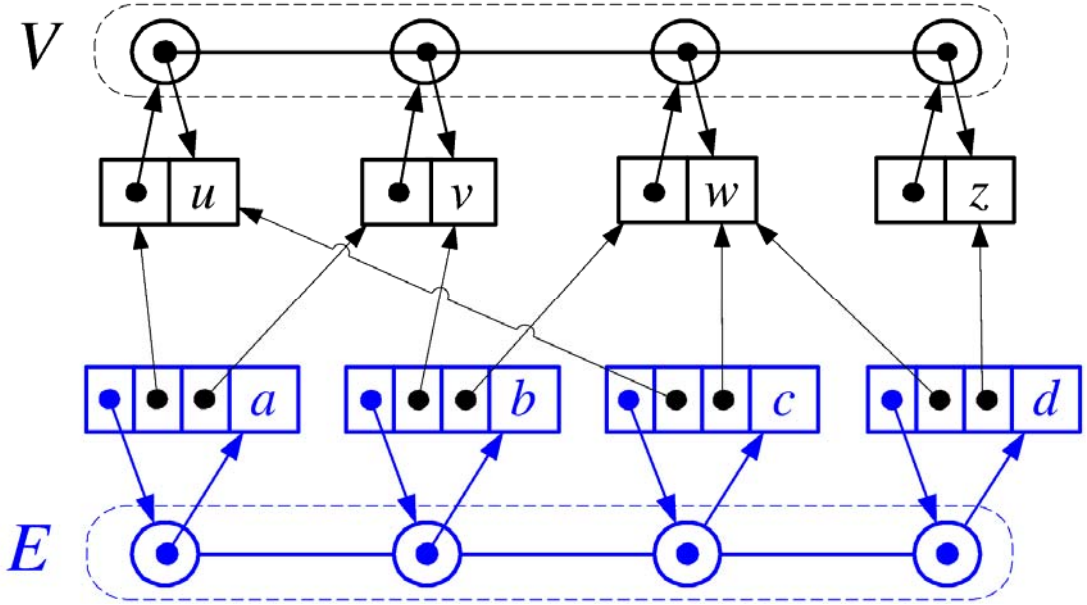
☐ Update methods
- insertVertex(o): insert a vertex storing element o
- insertEdge(v, w, o): insert an edge (v,w) storing element o
- removeVertex(v): remove vertex v (and its incident edges)
- removeEdge(e): remove edge e

☐ Iterator methods
- incidentEdges(v): edges incident to v
- vertices(): all vertices in the graph
- edges(): all edges in the graph

Edge List Structure

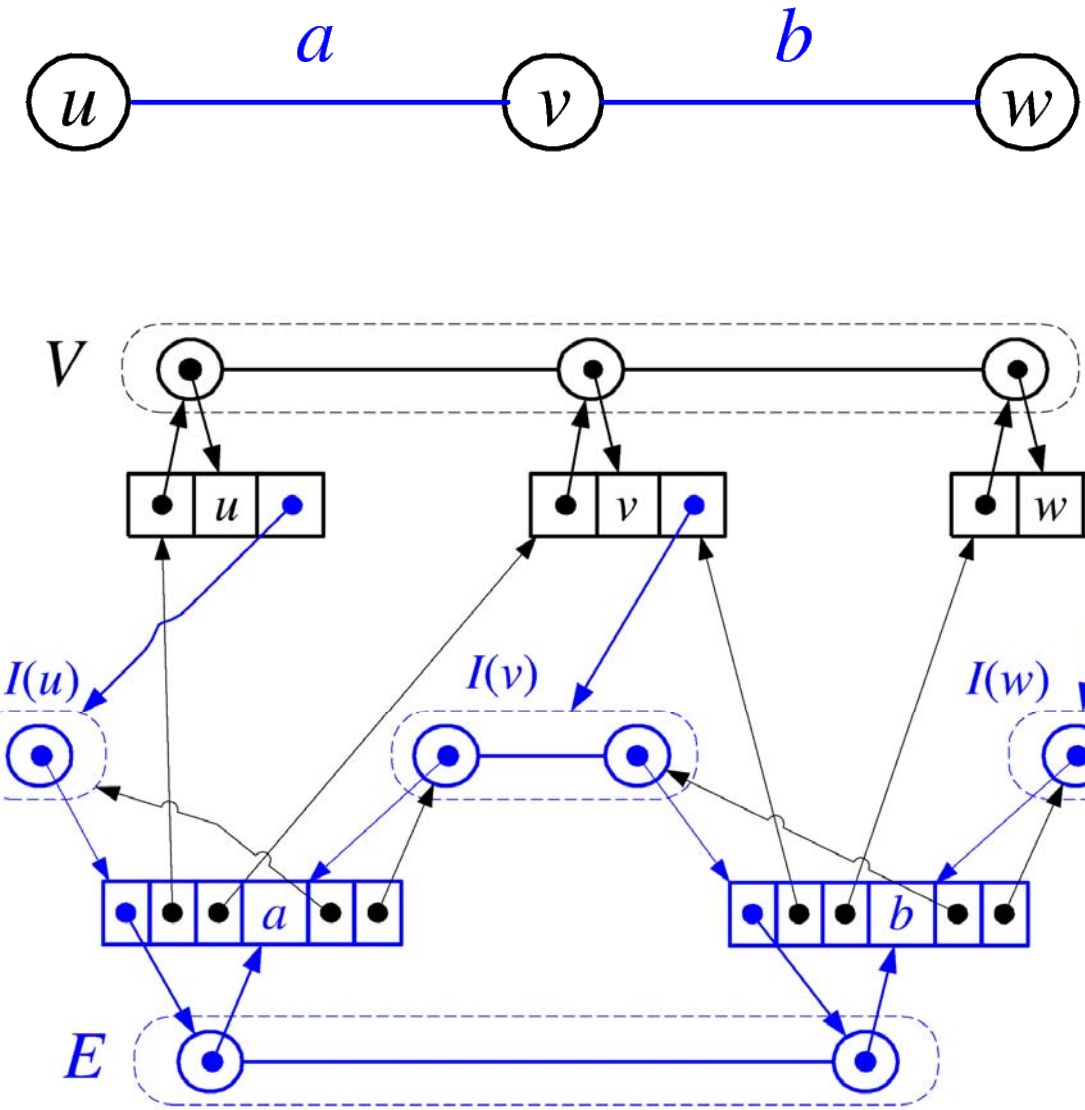
- Vertex sequence
  - sequence of vertex objects
- Edge sequence
  - sequence of edge objects
- Vertex object
  - element
  - reference to position in vertex sequence
- Edge object
  - element
  - origin vertex object
  - destination vertex object
  - reference to position in edge sequence





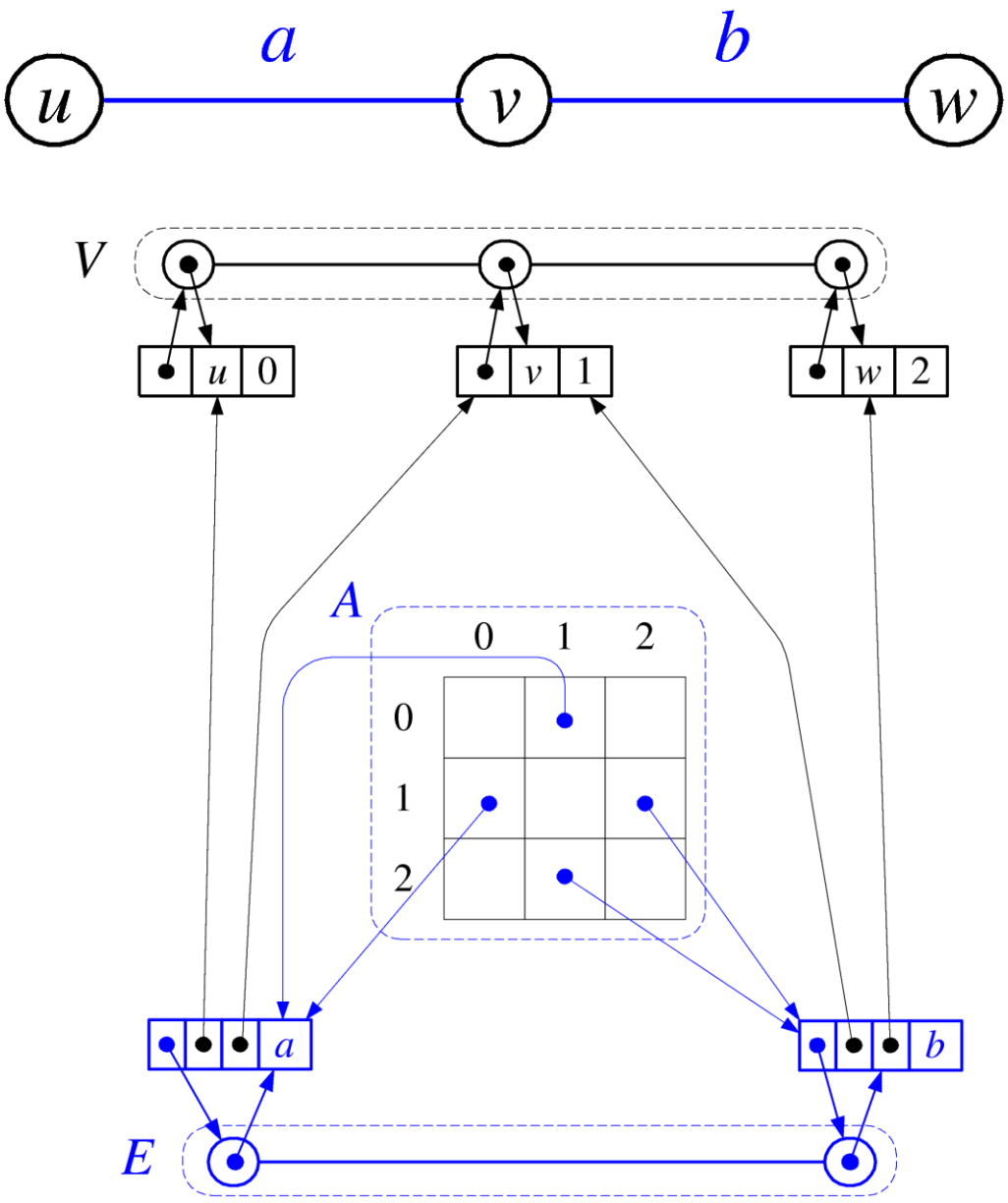
Adjacency List Structure

- Edge list structure
- Incidence sequence for each vertex
  - sequence of references to edge objects of incident edges
- Augmented edge objects
  - references to associated positions in incidence sequences of end vertices



Adjacency Matrix Structure

- Edge list structure
- Augmented vertex objects
  - Integer key (index) associated with vertex
- 2D-array adjacency array
  - Reference to edge object for adjacent vertices
  - Null for non nonadjacent vertices



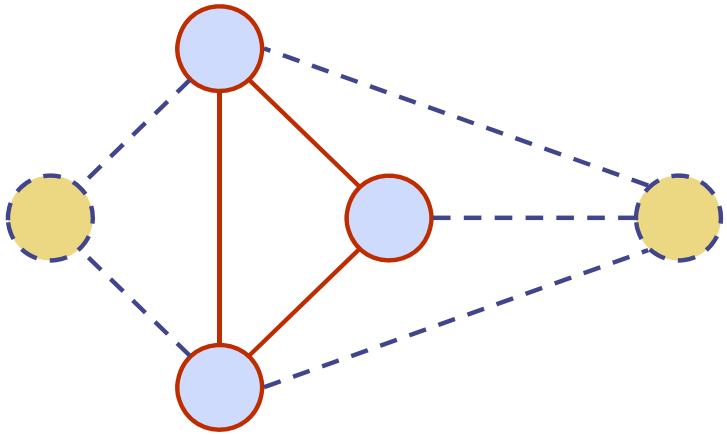
Asymptotic Performance

• n vertices, m edges • no parallel edges • no self-loops • Bounds are “big-Oh”	Edge List	Adjacency List	Adjacency Matrix
Space	$n + m$	$n + m$	n^2
incidentEdges(v)	m	deg(v)	n
areAdjacent (v, w)	m	min(deg(v), deg(w))	1
insertVertex(o)	1	1	n^2
insertEdge(v, w, o)	1	1	1
removeVertex(v)	m	deg(v)	n^2
removeEdge(e)	1	1	1

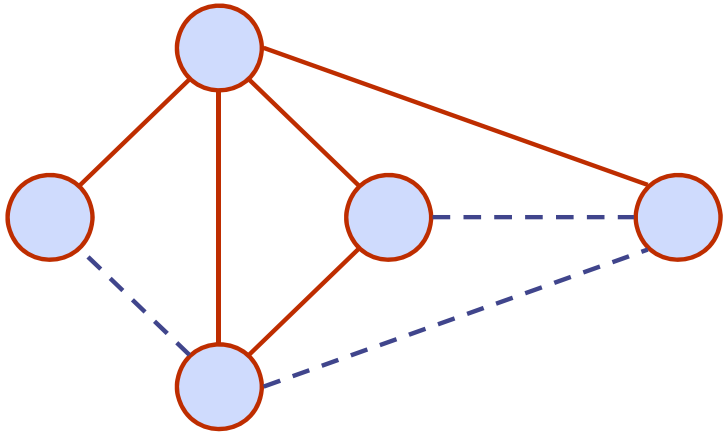
Graph Traversals
Depth-First Search

Subgraphs

- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A spanning subgraph of G is a subgraph that contains all the vertices of G



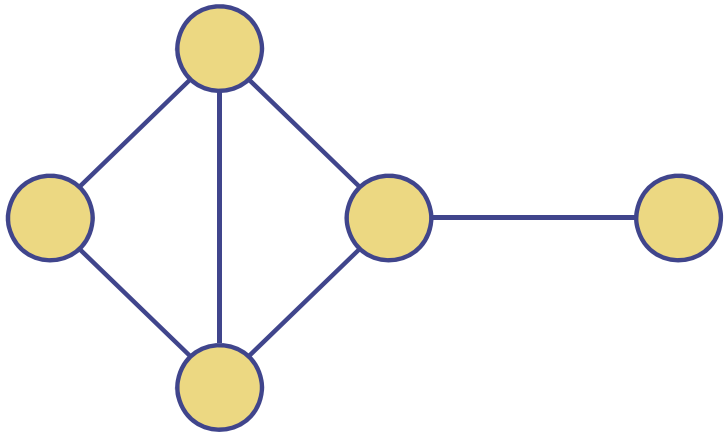
Subgraph



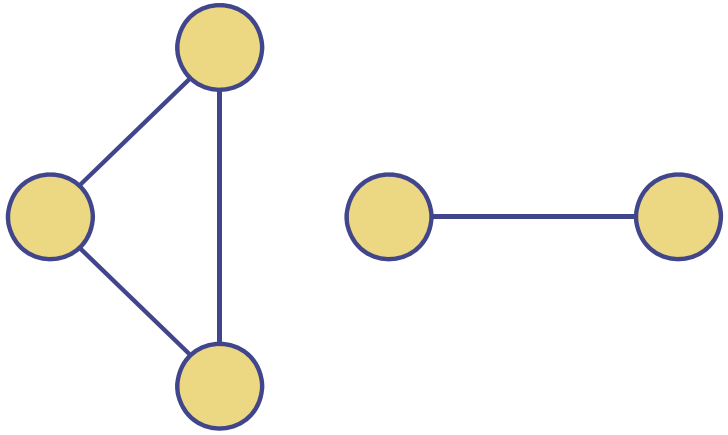
Spanning subgraph

Connectivity

- A graph is connected if there is a path between every pair of vertices
- A connected component of a graph G is a maximal connected subgraph of G



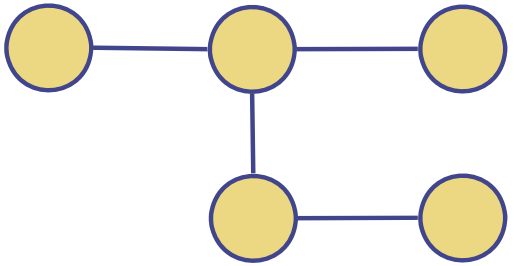
Connected graph



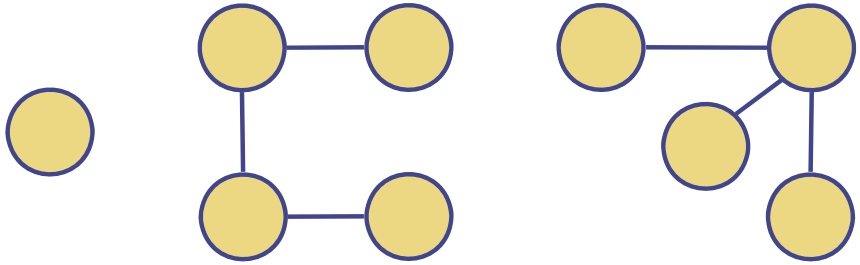
Non connected graph with two connected components

Trees and Forests

- A (free) tree is an undirected graph T such that
  - T is connected
  - T has no cycles
  - This definition of tree is different from the one of a rooted tree
- A forest is an undirected graph without cycles
- The connected components of a forest are trees



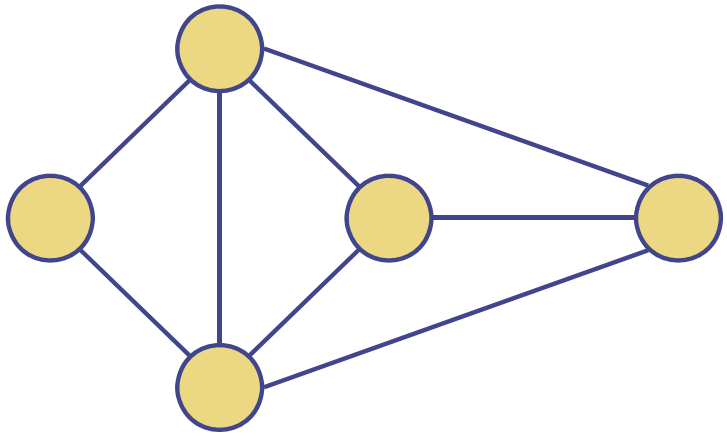
Tree



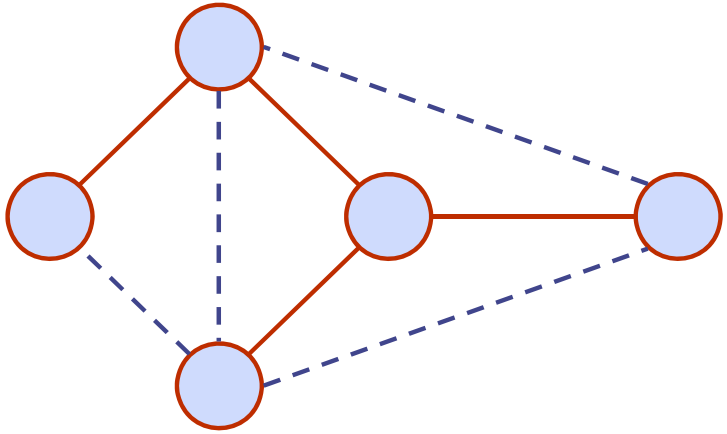
Forest

Spanning Trees and Forests

- A spanning tree of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A spanning forest of a graph is a spanning subgraph that is a forest



Graph



Spanning tree

Depth-First Search

- Depth-first search (DFS) is a general technique for traversing a graph
- A DFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- DFS on a graph with n vertices and m edges takes O(n + m) time
- DFS can be further extended to solve other graph problems
  - Find and report a path between two given vertices
  - Find a cycle in the graph

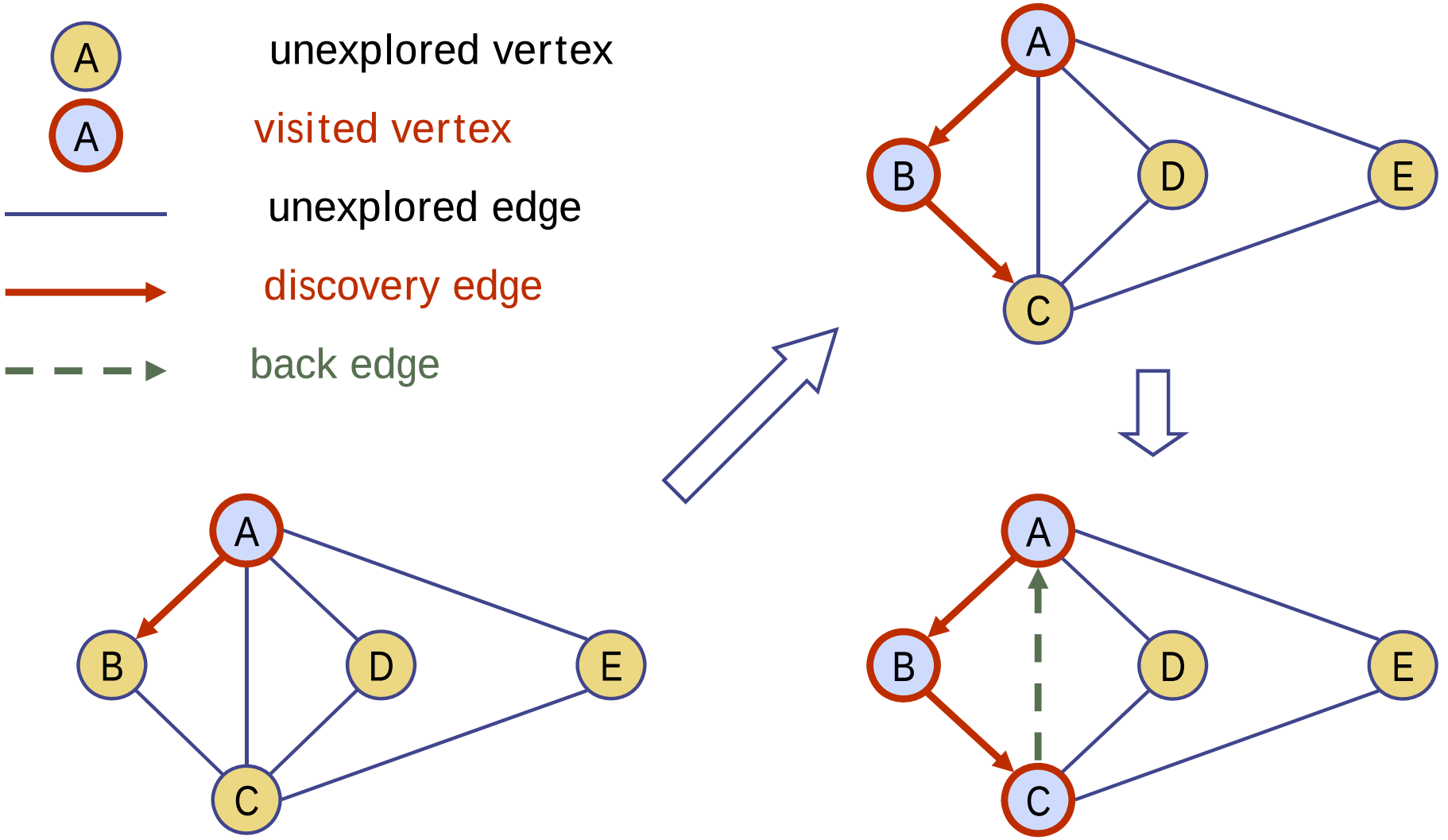
DFS Algorithm

- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

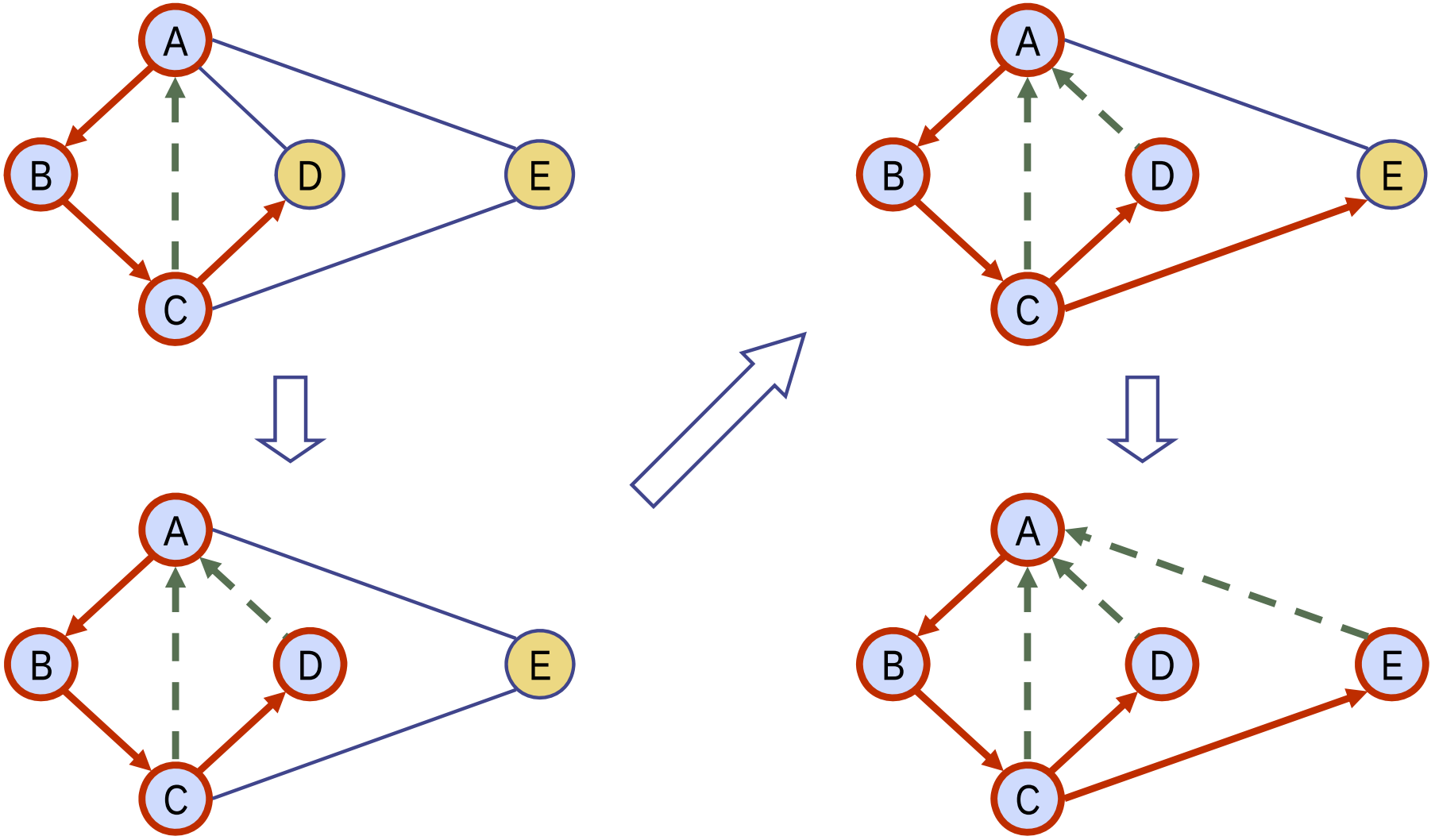
```
Algorithm DFS(G)
    Input graph G
    Output labeling of the edges of G
        as discovery edges and
        back edges
  for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
  for all e ∈ G.edges()
    setLabel(e, UNEXPLORED)
  for all v ∈ G.vertices()
    if getLabel(v) = UNEXPLORED
        DFS(G, v)
```

```
Algorithm DFS(G, v)
    Input graph G and a start vertex v of G
    Output labeling of the edges of G
        in the connected component of v
        as discovery edges and back edges
  setLabel(v, VISITED)
  for all e ∈ G.incidentEdges(v)
      if getLabel(e) = UNEXPLORED
          w ← opposite(v,e)
          if getLabel(w) = UNEXPLORED
              setLabel(e, DISCOVERY)
              DFS(G, w)
          else
              setLabel(e, BACK)
```

Example

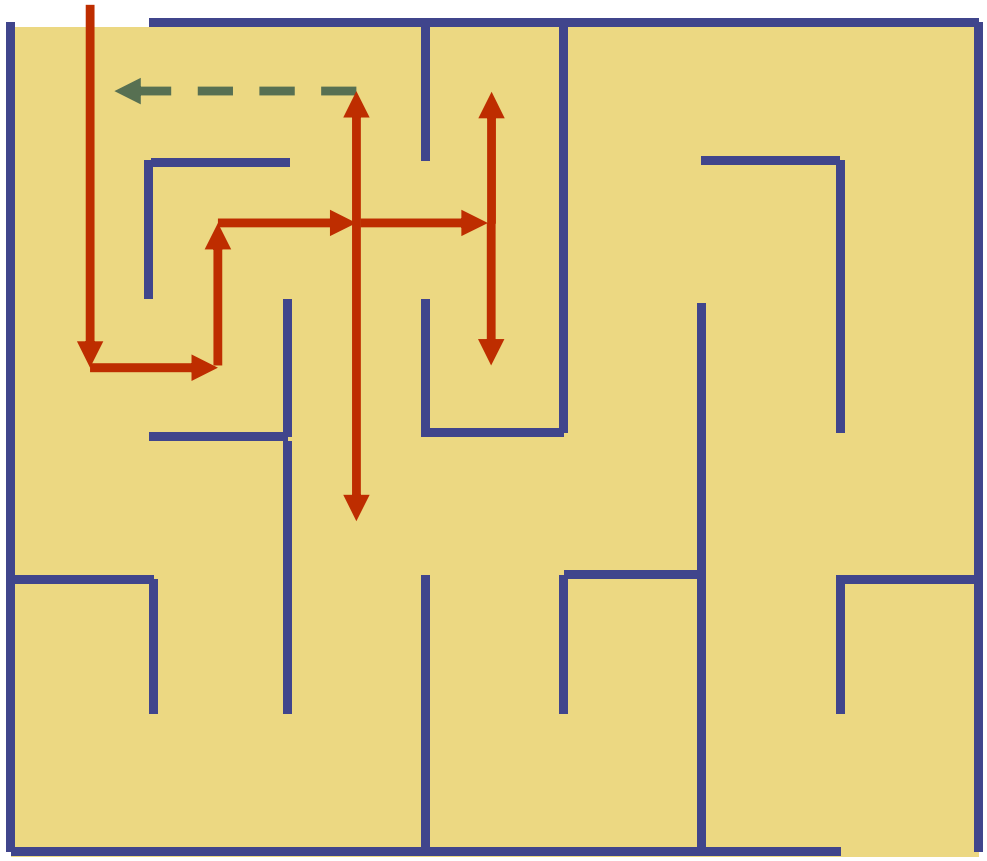


Example (cont.)



DFS and Maze Traversal

- The DFS algorithm is similar to a classic strategy for exploring a maze
  - We mark each intersection, corner and dead end (vertex) visited
  - We mark each corridor (edge) traversed
  - We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)



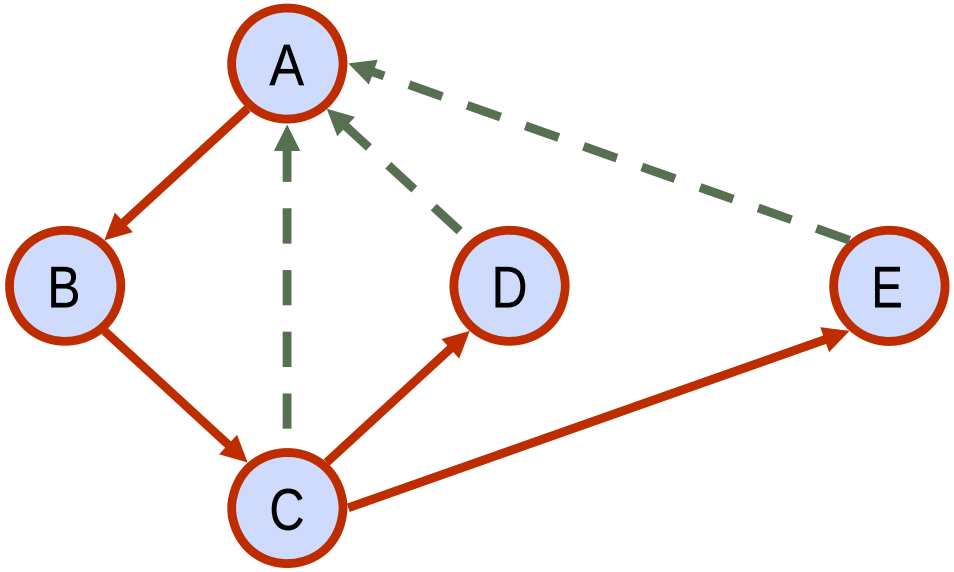
Properties of DFS

☐ Property 1

- DFS(G, v) visits all the vertices and edges in the connected component of v

☐ Property 2

- The discovery edges labeled by DFS(G, v) form a spanning tree of the connected component of v



Analysis of DFS

- Setting/getting a vertex/edge label takes O(1) time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or BACK
- Method incidentEdges is called once for each vertex
- DFS runs in O(n + m) time provided the graph is represented by the adjacency list structure
  - Recall that $\sum_v \deg(v) = 2m$

Path Finding

- We can specialize the DFS algorithm to find a path between two given vertices u and z using the template method pattern
- We call DFS(G, u) with u as the start vertex
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as destination vertex z is encountered, we return the path as the contents of the stack

```
Algorithm pathDFS(G, v, z)
  setLabel(v, VISITED)
  S.push(v)
  if v = z
    return S.elements()
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        S.push(e)
        pathDFS(G, w, z)
        S.pop(e)
      else
        setLabel(e, BACK)
  S.pop(v)
```

Cycle Finding

- We can specialize the DFS algorithm to find a simple cycle using the template method pattern
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as a back edge (v, w) is encountered, we return the cycle as the portion of the stack from the top to vertex w

```
Algorithm cycleDFS(G, v, z)
    setLabel(v, VISITED)
    S.push(v)
    for all e ∈ G.incidentEdges(v)
        if getLabel(e) = UNEXPLORED
            w ← opposite(v,e)
            S.push(e)
            if getLabel(w) = UNEXPLORED
                setLabel(e, DISCOVERY)
                pathDFS(G, w, z)
                S.pop(e)
            else
                // Check whether there is a
                // cycle. If so, return it.
                // Otherwise, keep continue.
                …..
                S.pop(e)
    S.pop(v)
```

Graph Traversals
Breadth-First Search

Breadth-First Search

- Breadth-first search (BFS) is a general technique for traversing a graph
- BFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- BFS on a graph with n vertices and m edges takes O(n + m) time
- BFS can be further extended to solve other graph problems
  - Find and report a path with the minimum number of edges between two given vertices
  - Find a simple cycle, if there is one

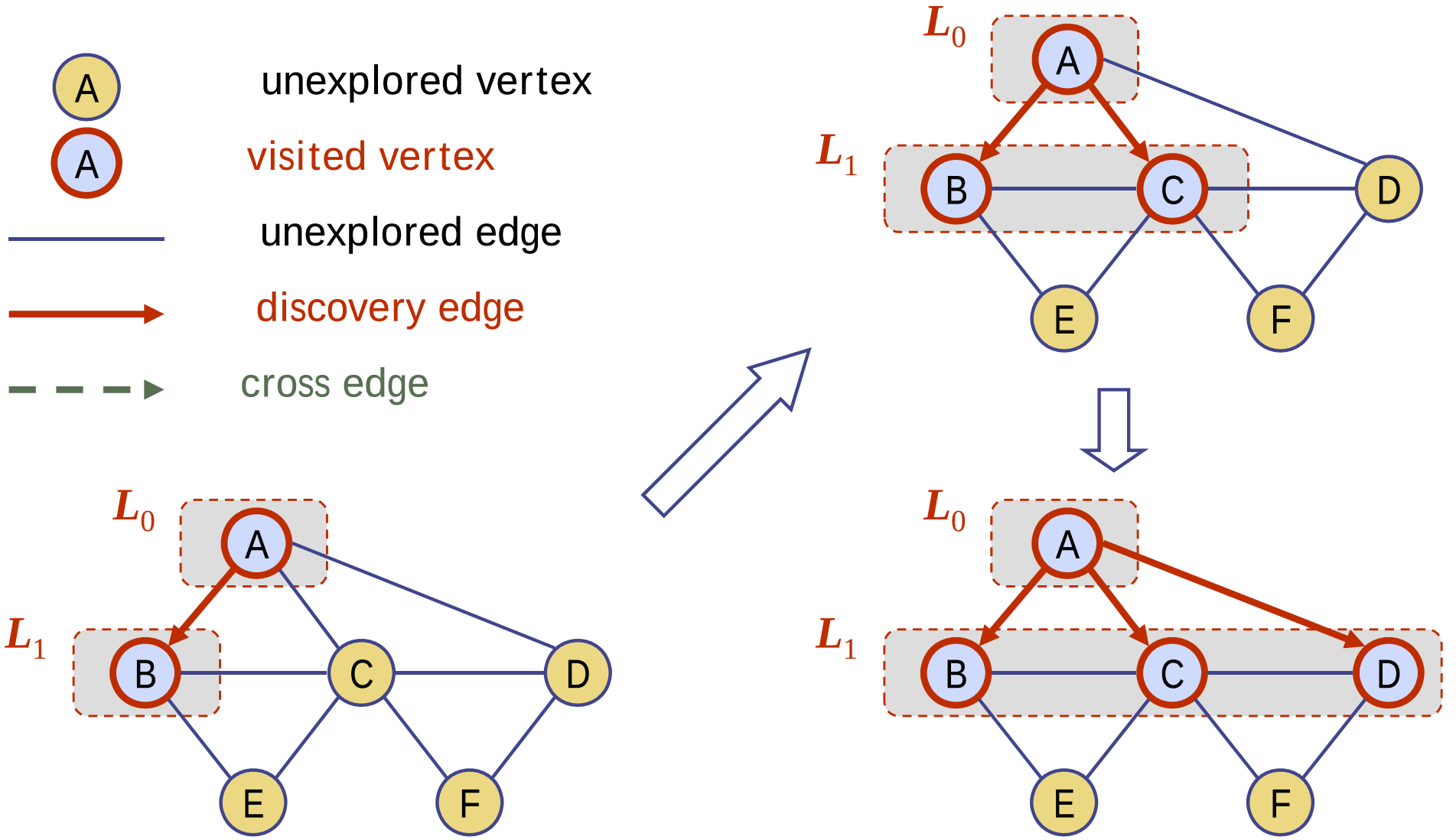
BFS Algorithm

- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

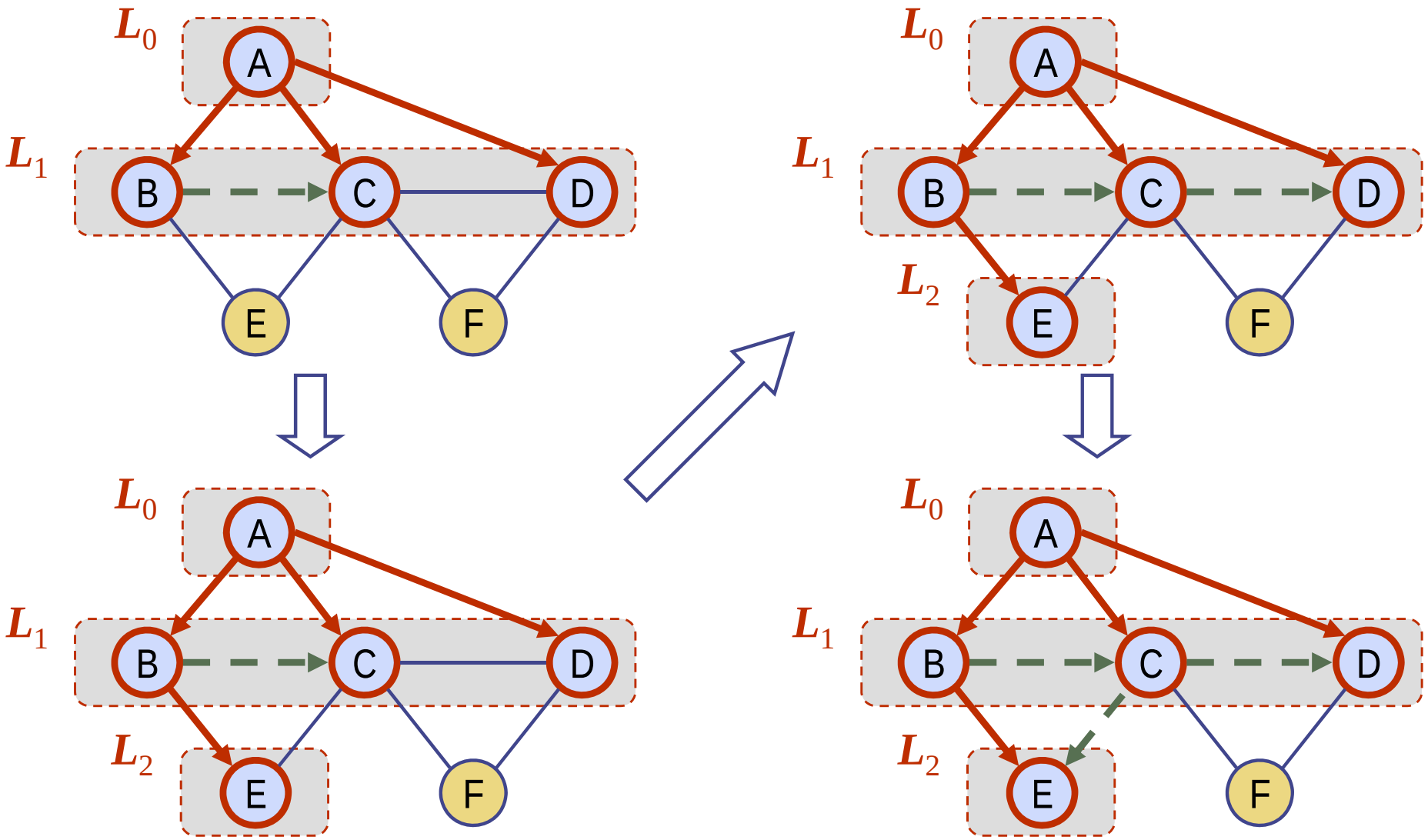
```
Algorithm BFS(G)
    Input graph G
    Output labeling of the edges
        and partition of the
        vertices of G
  for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
  for all e ∈ G.edges()
    setLabel(e, UNEXPLORED)
  for all v ∈ G.vertices()
    if getLabel(v) = UNEXPLORED
      BFS(G, v)
```

```
Algorithm BFS(G, s)
  L0 ← new empty sequence
  L0.insertLast(s)
  setLabel(s, VISITED)
  i ← 0
  while ¬Li.isEmpty()
    Li+1 ← new empty sequence
    for all v ∈ Li.elements()
      for all e ∈ G.incidentEdges(v)
        if getLabel(e) = UNEXPLORED
          w ← opposite(v,e)
          if getLabel(w) = UNEXPLORED
            setLabel(e, DISCOVERY)
            setLabel(w, VISITED)
            Li+1.insertLast(w)
          else
            setLabel(e, CROSS)
    i ← i +1
```

Example



Example (cont.)



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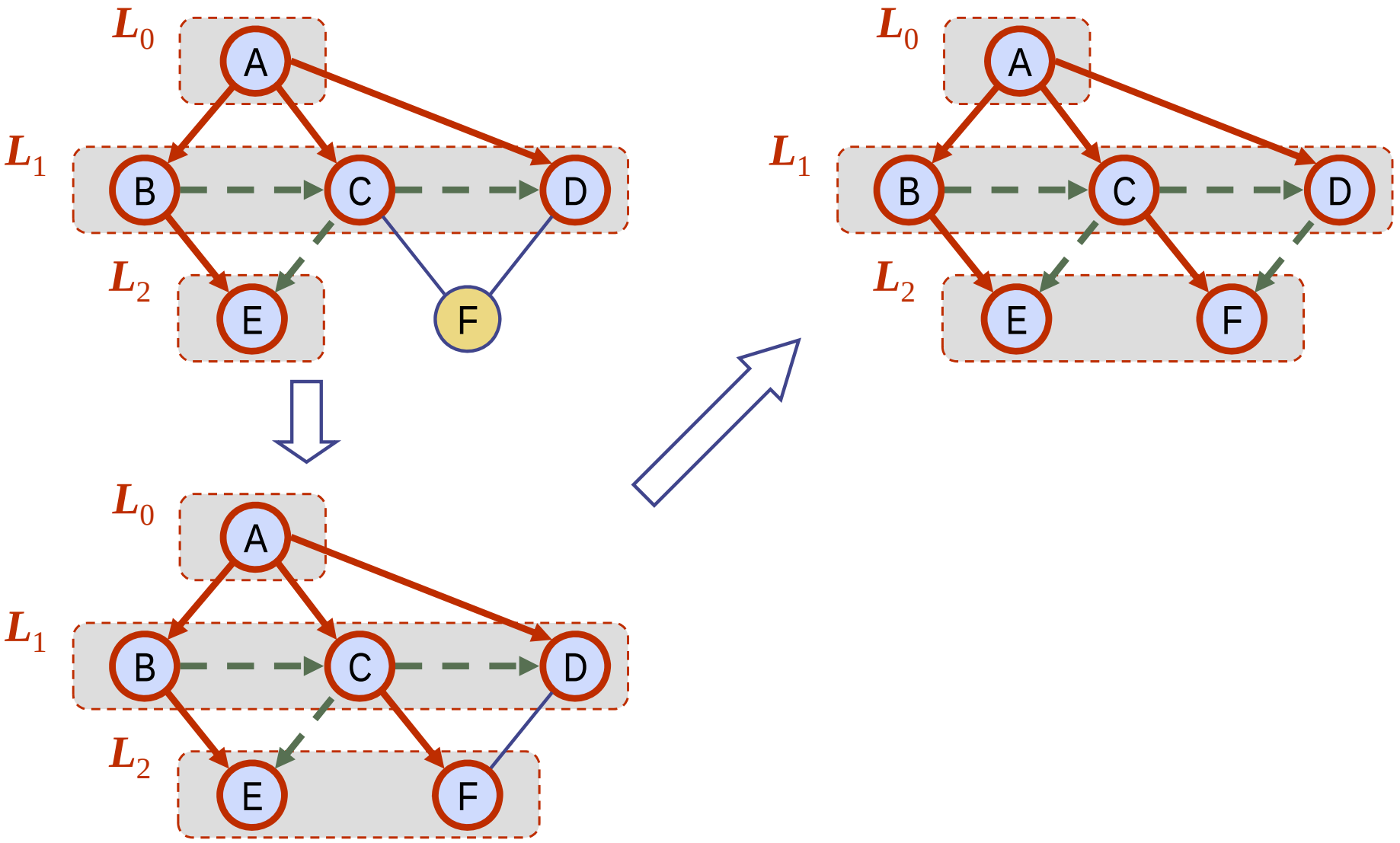
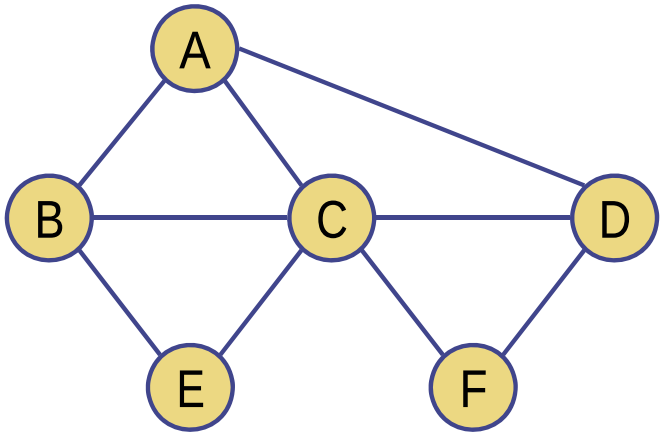
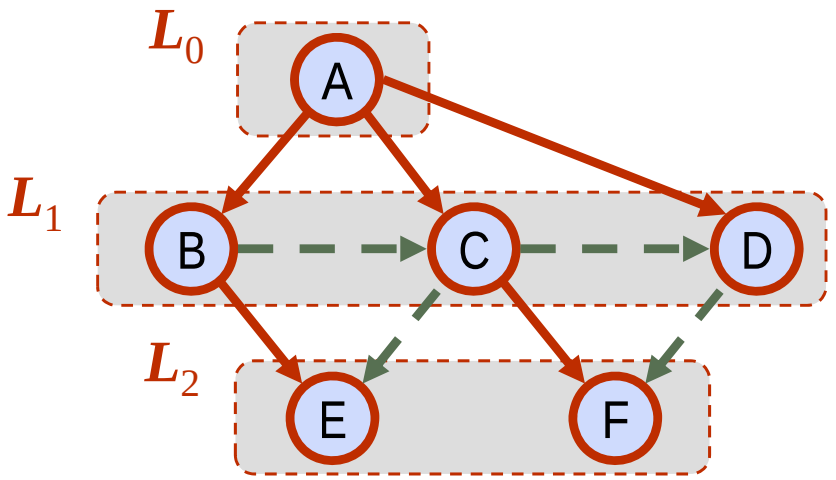


Properties

- Notation
  - G_s: connected component of s
- Property 1
  - BFS(G, s) visits all the vertices and edges of G_s
- Property 2
  - The discovery edges labeled by BFS(G, s) form a spanning tree T_s of G_s
- Property 3

  For each vertex v in L_i
  - The path of T_s from s to v has i edges
  - Every path from s to v in G_s has at least i edges





Analysis

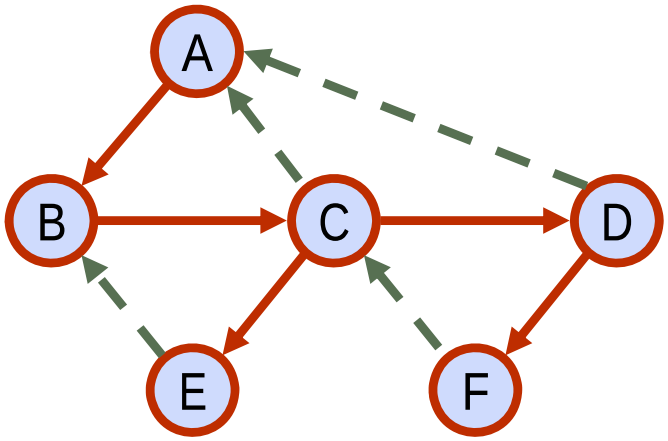
- Setting/getting a vertex/edge label takes O(1) time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or CROSS
- Each vertex is inserted once into a sequence L_i
- Method incidentEdges is called once for each vertex
- BFS runs in O(n + m) time provided the graph is represented by the adjacency list structure
  - Recall that $\sum_v \deg(v) = 2m$

Applications

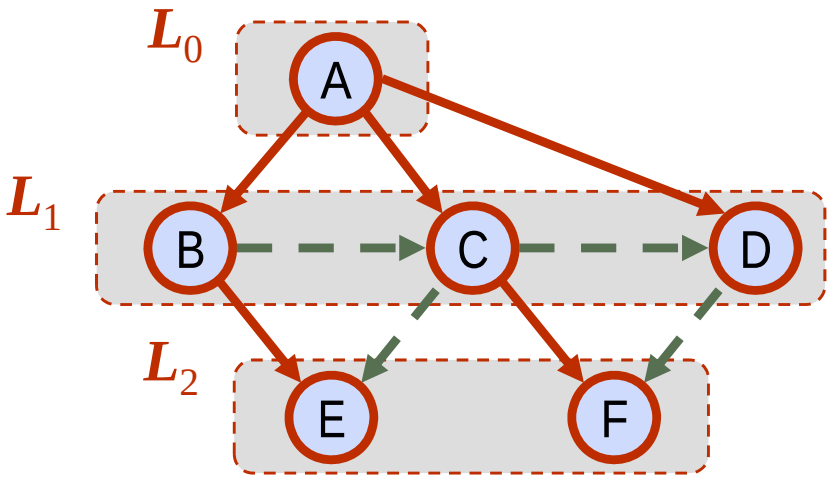
- Using the template method pattern, we can specialize the BFS traversal of a graph G to solve the following problems in O(n + m) time
  - Compute the connected components of G
  - Compute a spanning forest of G
  - Find a simple cycle in G, or report that G is a forest
  - Given two vertices of G, find a path in G between them with the minimum number of edges, or report that no such path exists

DFS vs. BFS

Applications	DFS	BFS
Spanning forest, connected components, paths, cycles	√	√
Shortest paths		√
Biconnected components	√	



DFS



BFS

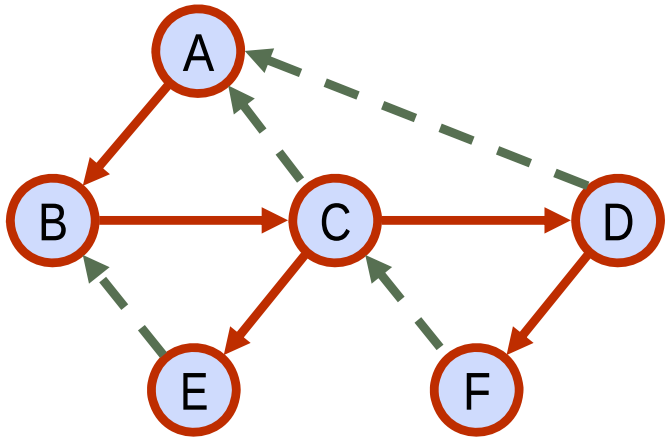
DFS vs. BFS (cont.)

☐ Back edge (v,w)

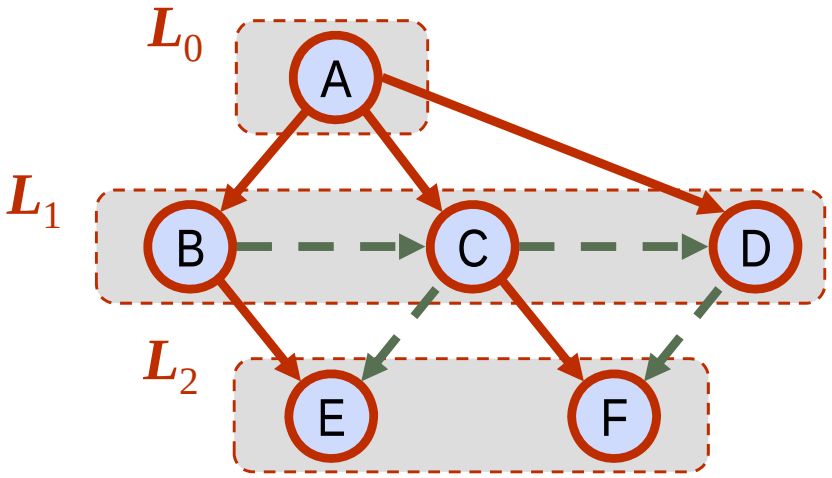
- w is an ancestor of v in the tree of discovery edges

☐ Cross edge (v,w)

- w is in the same level as v or in the next level in the tree of discovery edges



DFS



BFS

Class Objectives were: (Ch. 14)

- Get to know data representations of graphs and two different graph traversal methods

Next Time

☐ Shortest paths

☐ Questions:

- Come up with one question on what we have discussed in the class and submit at the end of the class
- 1 for typical questions and 2 for questions with thoughts or that surprised me
- Write questions at least 4 times; you can type at KLMS

☐ HW:

- Go over the next lecture slides before the class
- Just 10 min ~ 20 min should be okay