CS206 Data Structures

Sorting and Selection I

Sung-eui Yoon (윤성의)
Department of Computer Science
KAIST

http://sglab.kaist.ac.kr/~sungeui

Class Objectives (Ch. 13)

- Understand divide-and-conquer sorting methods: merge and quick sorts

Merge Sort

Divide-and-Conquer

- **Divide-and-conquer** is a general algorithm design paradigm:
  - **Divide**: divide the input data S in two disjoint subsets S_1 and S_2
  - **Recur**: solve the subproblems associated with S_1 and S_2
  - **Conquer**: combine the solutions for S_1 and S_2 into a solution for S
- The base case for the recursion are subproblems of size 0 or 1

- **Merge-sort** is a sorting algorithm based on the divide-and-conquer paradigm
- Like heap-sort
  - It uses a comparator
  - It has O(n log n) running time
- Unlike heap-sort
  - It does not use an auxiliary priority queue
  - It accesses data in a sequential manner (suitable to sort data on a disk)

Merge-Sort

- Merge-sort on an input sequence S with n elements consists of three steps:
  - Divide: partition S into two sequences S_1 and S_2 of about n/2 elements each
  - Recur: recursively sort S_1 and S_2
  - Conquer: merge S_1 and S_2 into a unique sorted sequence

Algorithm *mergeSort*(S, C)
  Input sequence S with n
    elements, comparator C
  Output sequence S sorted
    according to C
  if $S.size() > 1$
    $(S_1, S_2) \leftarrow partition(S, n/2)$
    $mergeSort(S_1, C)$
    $mergeSort(S_2, C)$
    $S \leftarrow merge(S_1, S_2)$

Merging Two Sorted Sequences

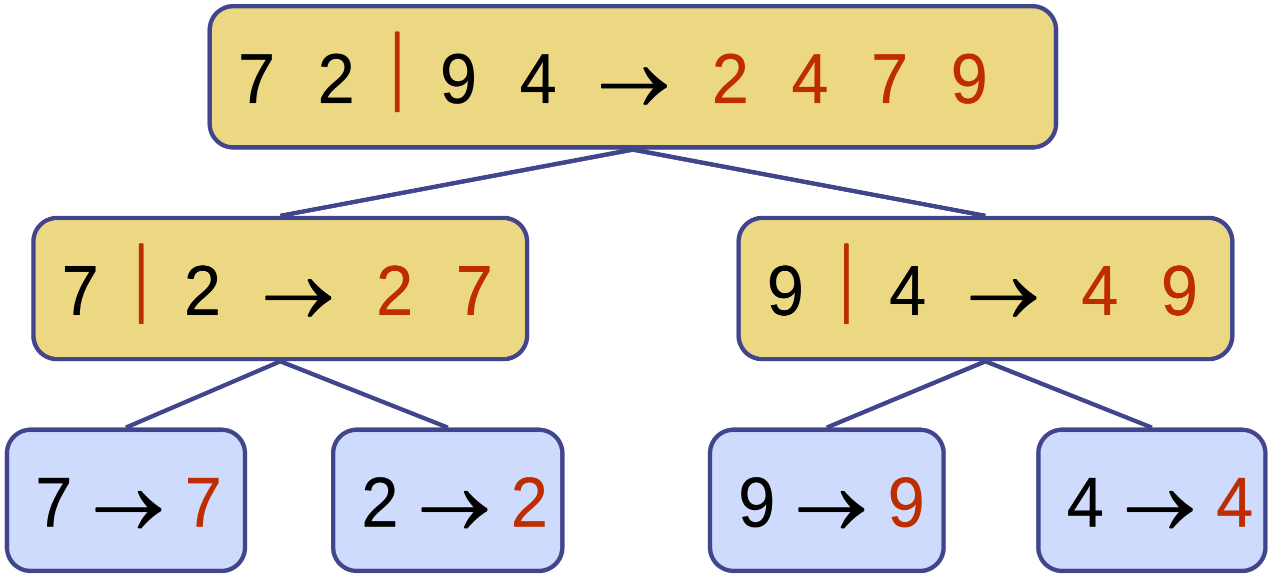
- The conquer step of merge-sort consists of merging two sorted sequences A and B into a sorted sequence S containing the union of the elements of A and B
- Merging two sorted sequences, each with n/2 elements and implemented by means of a doubly linked list, takes O(n) time

Algorithm *merge*(*A*, *B*)
  Input sequences *A* and *B* with
    n/2 elements each
  Output sorted sequence of $A \cup B$

  $S \leftarrow$ empty sequence
  while $\neg$*A.isEmpty*() $\wedge$ $\neg$*B.isEmpty*()
    if *A.first*().*element*() < *B.first*().*element*()
      S.insertLast(*A.remove*(*A.first*()))
    else
      S.insertLast(*B.remove*(*B.first*()))
  while $\neg$*A.isEmpty*()
    S.insertLast(*A.remove*(*A.first*()))
  while $\neg$*B.isEmpty*()
    S.insertLast(*B.remove*(*B.first*()))
  return *S*

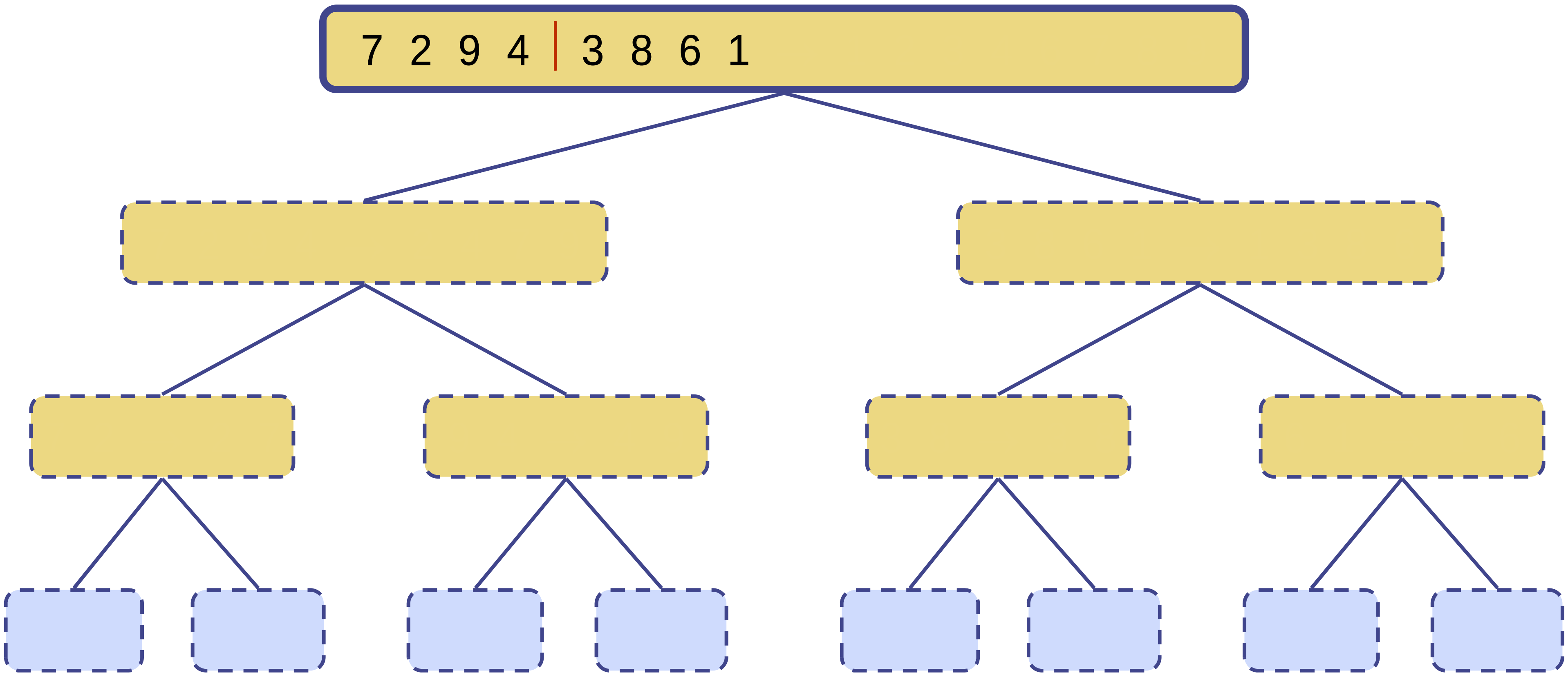
Merge-Sort Tree

- An execution of merge-sort is depicted by a binary tree
  - each node represents a recursive call of merge-sort and stores
    - unsorted sequence before the execution and its partition
    - sorted sequence at the end of the execution
  - the root is the initial call
  - the leaves are calls on subsequences of size 0 or 1



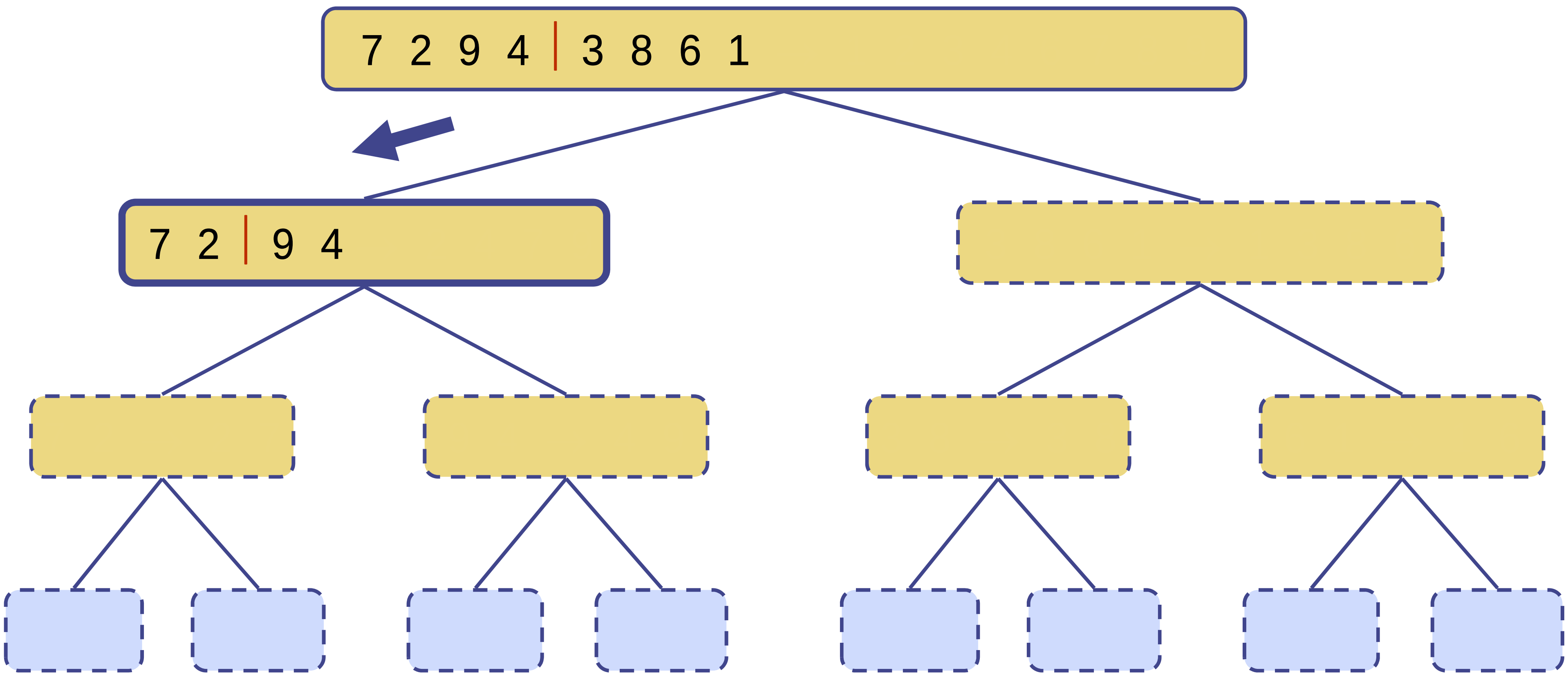
Execution Example

- Partition



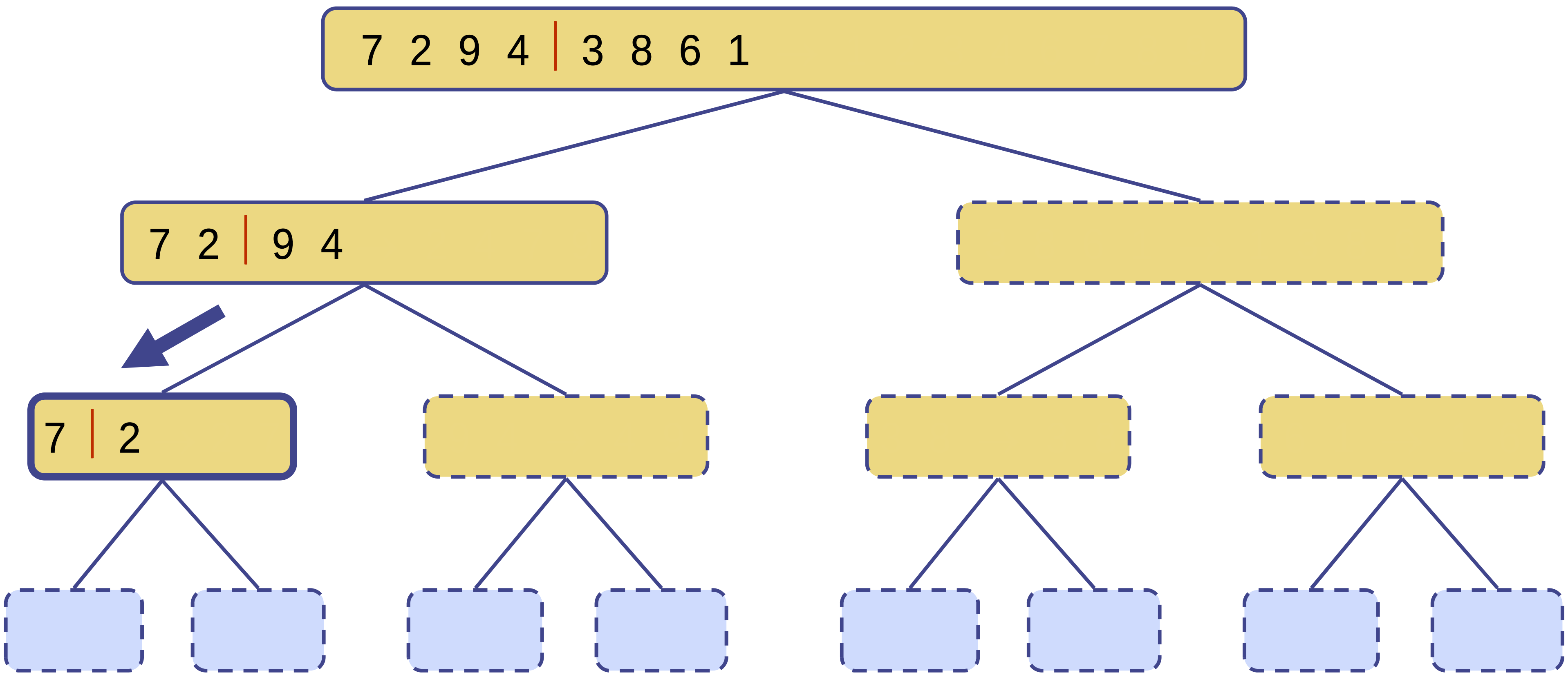
Execution Example (cont.)

- ☐ Recursive call, partition



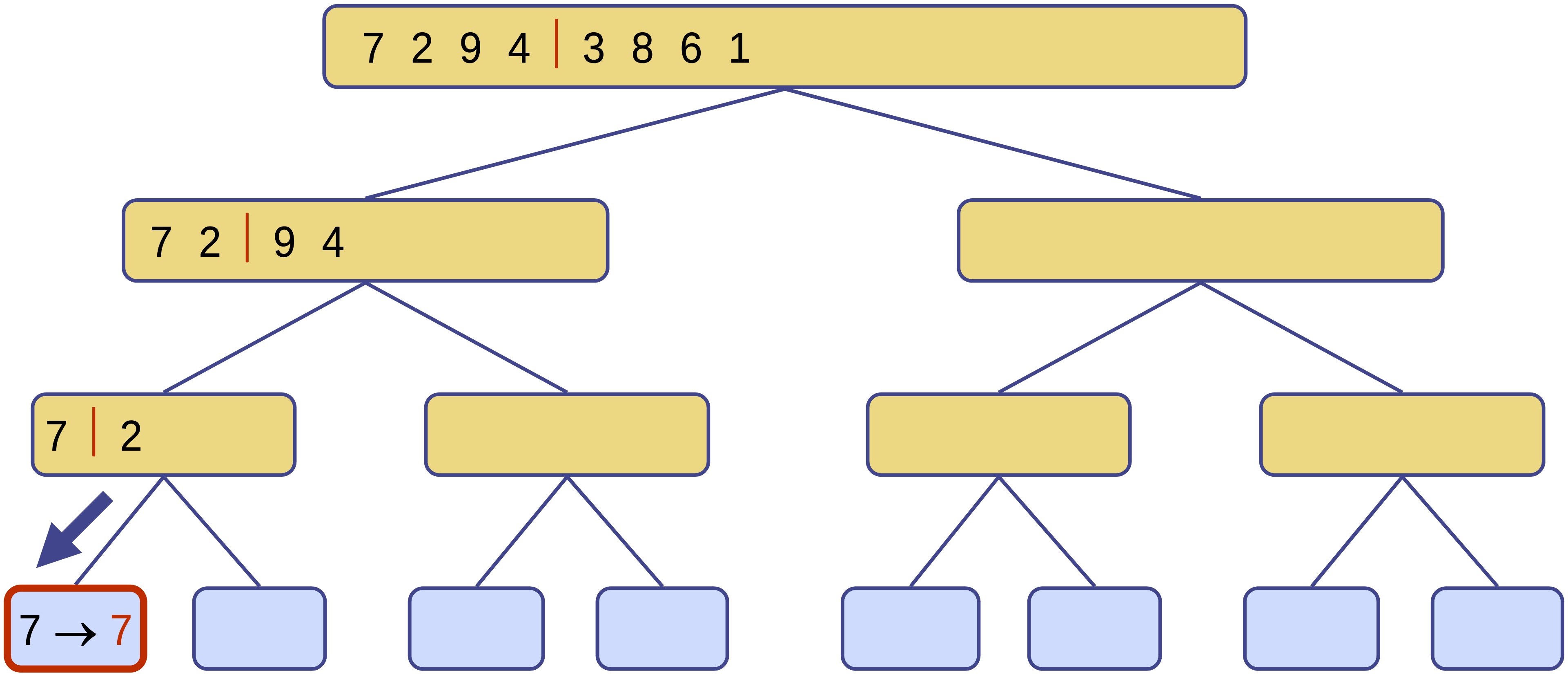
Execution Example (cont.)

- Recursive call, partition



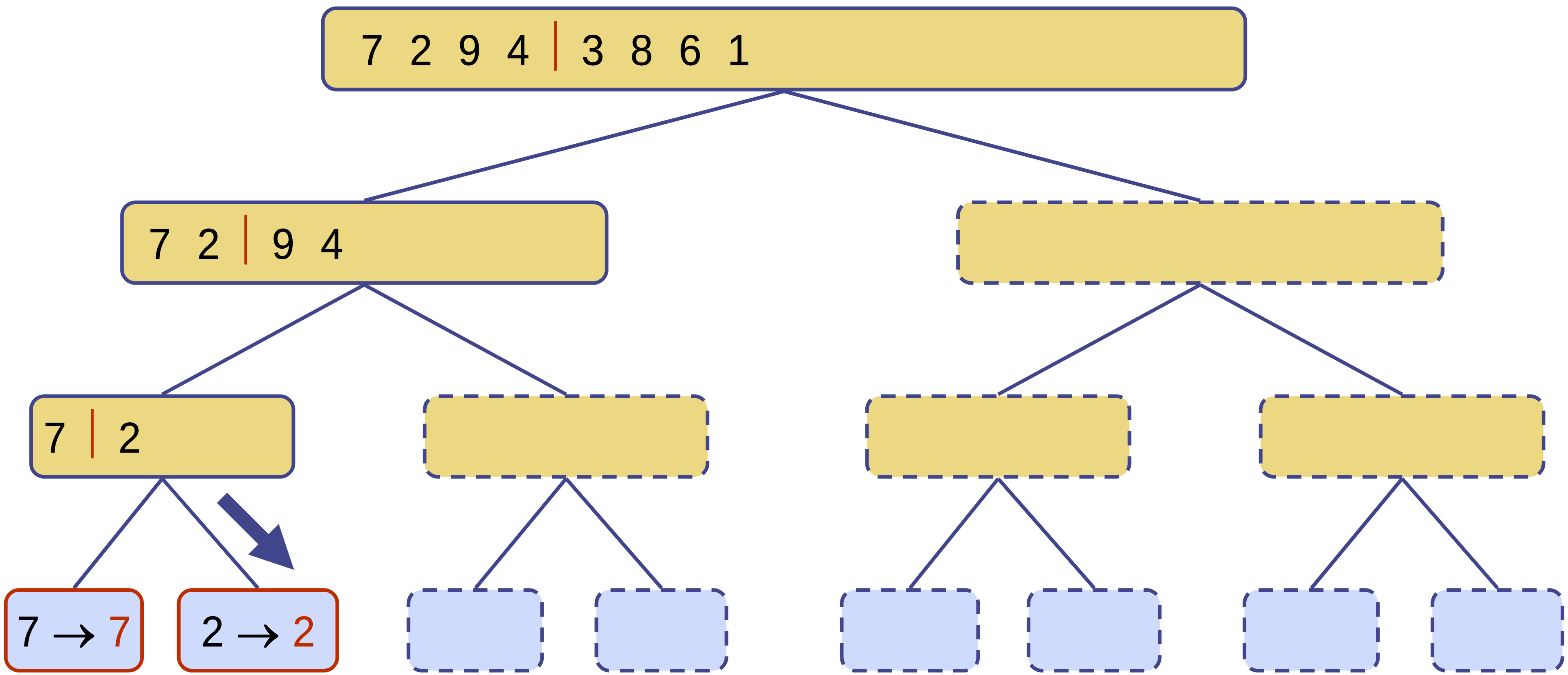
Execution Example (cont.)

- Recursive call, base case



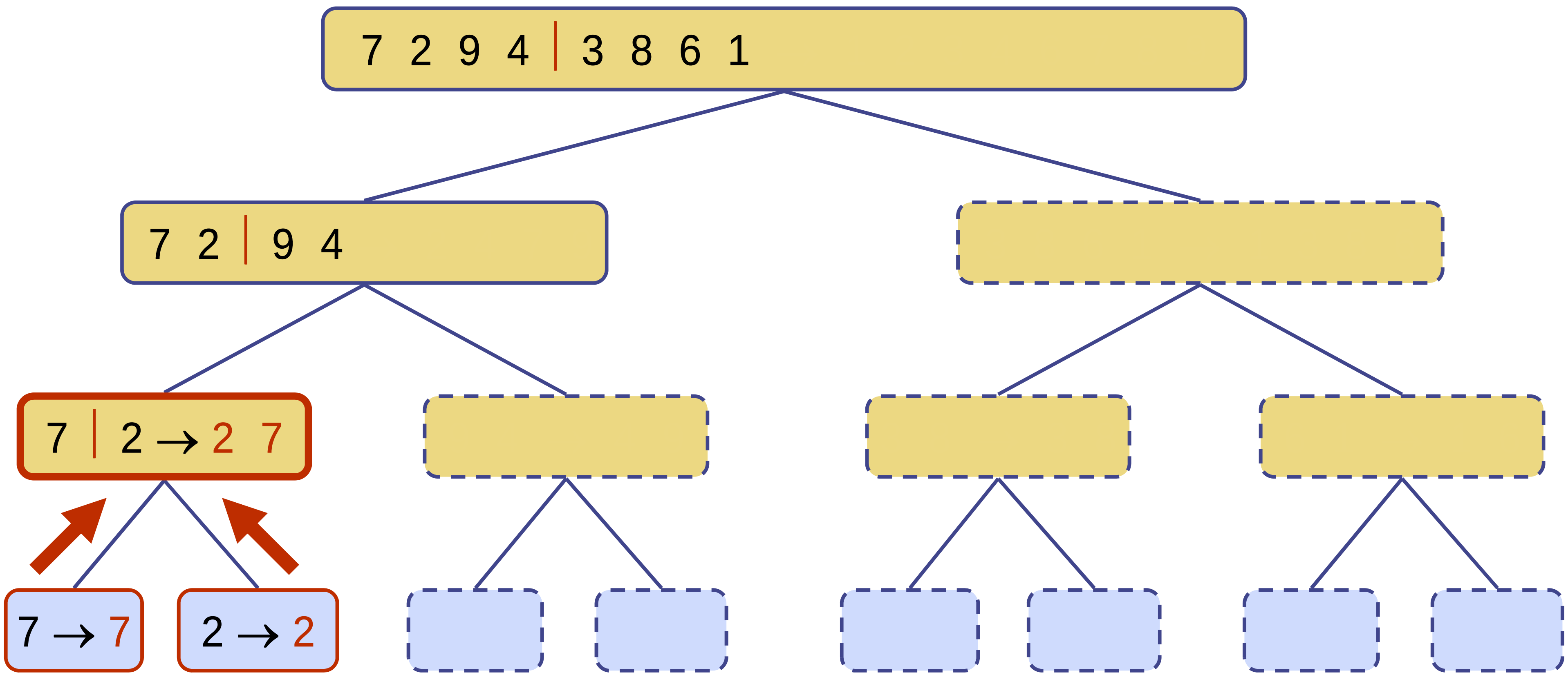
Execution Example (cont.)

- ☐ Recursive call, base case



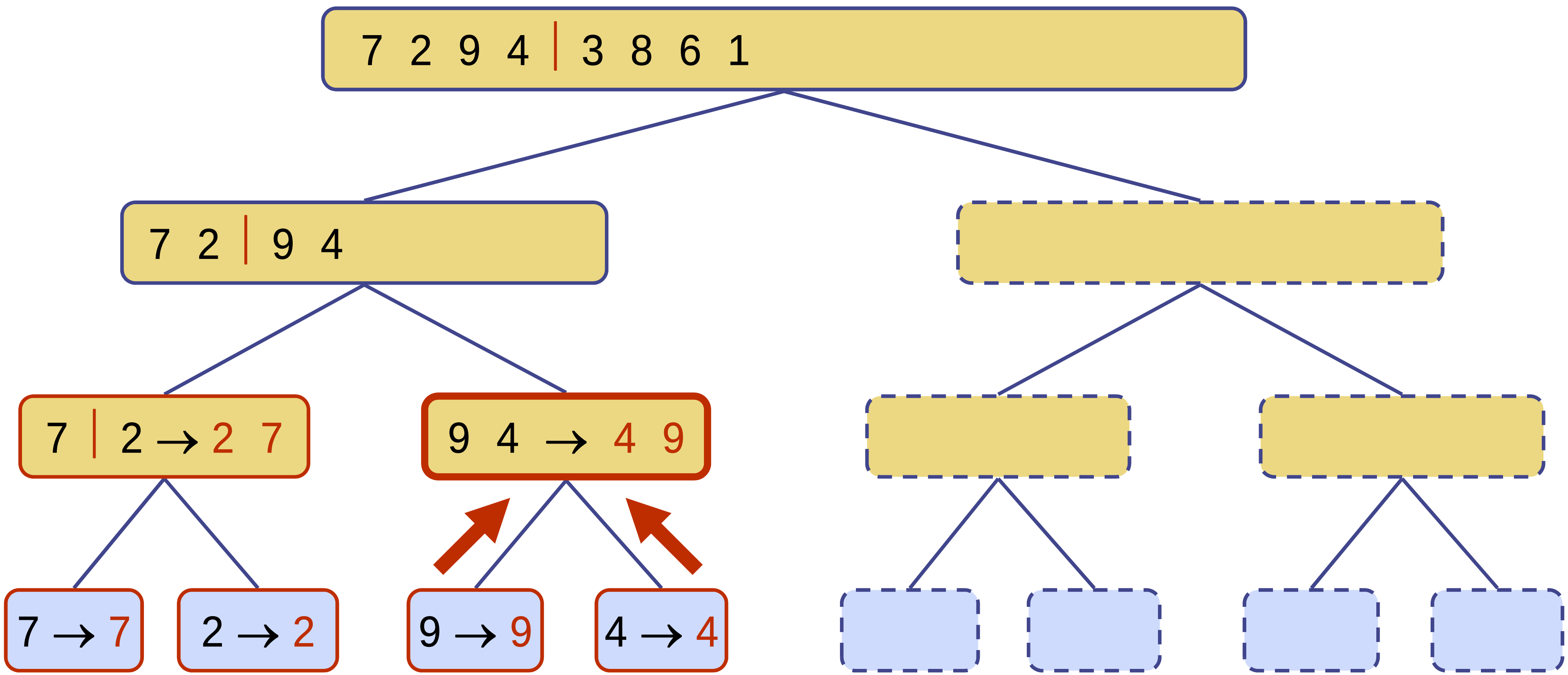
Execution Example (cont.)

- ☐ Merge



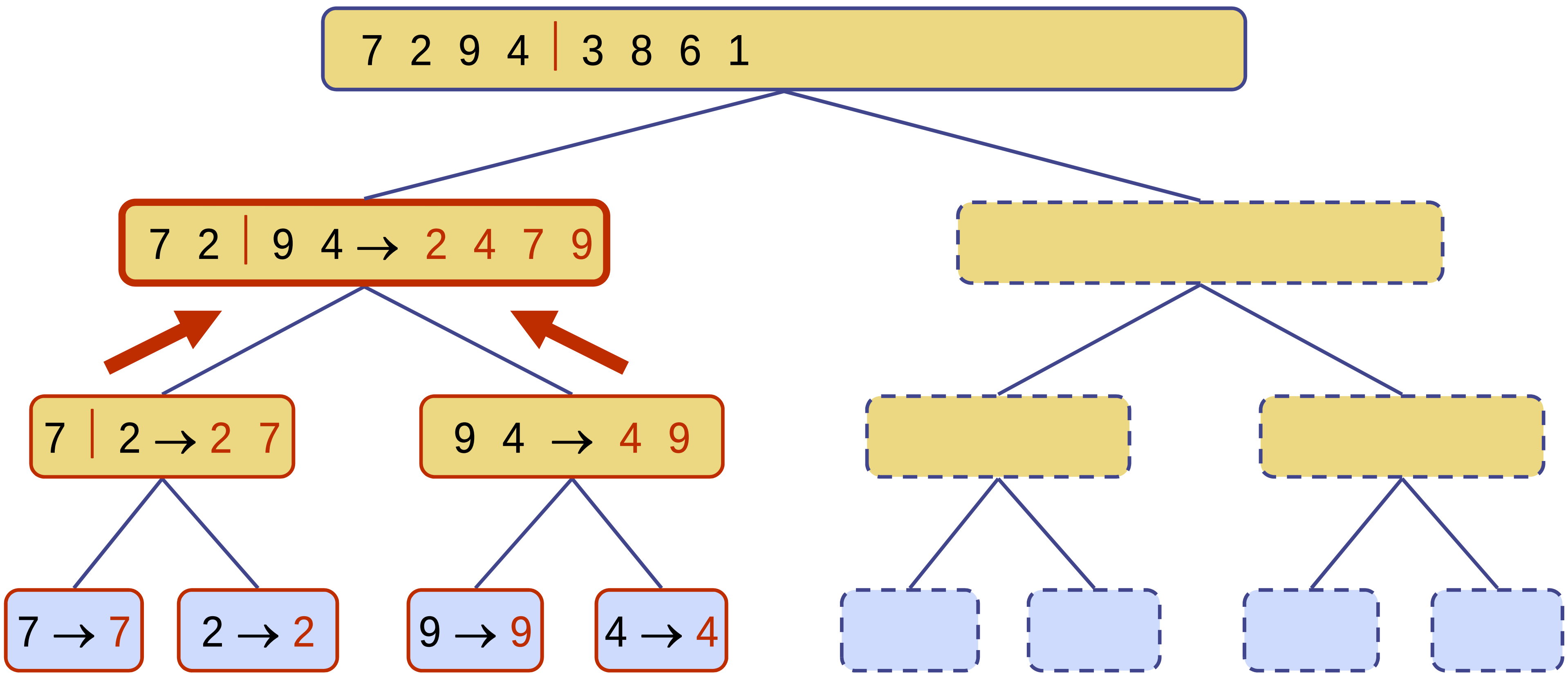
Execution Example (cont.)

- Recursive call, …, base case, merge



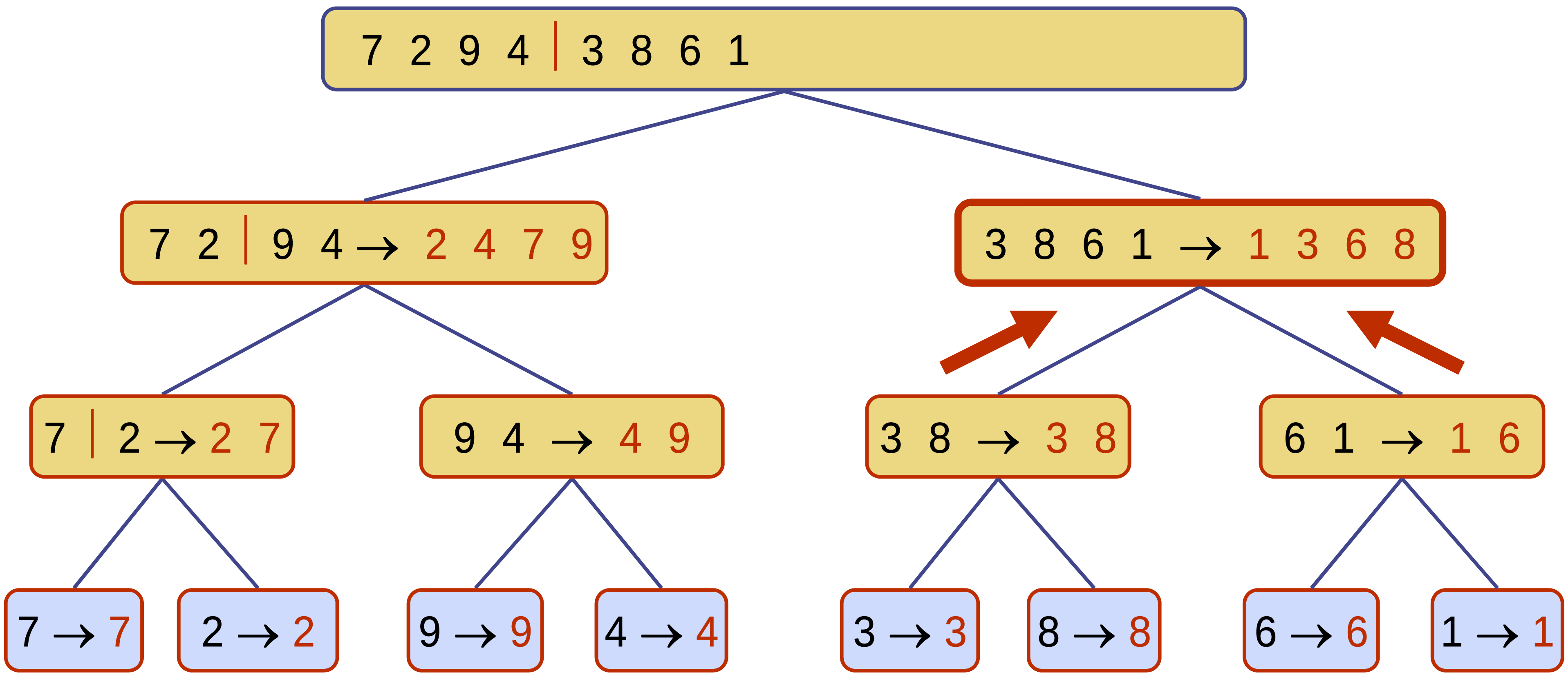
Execution Example (cont.)

☐ Merge



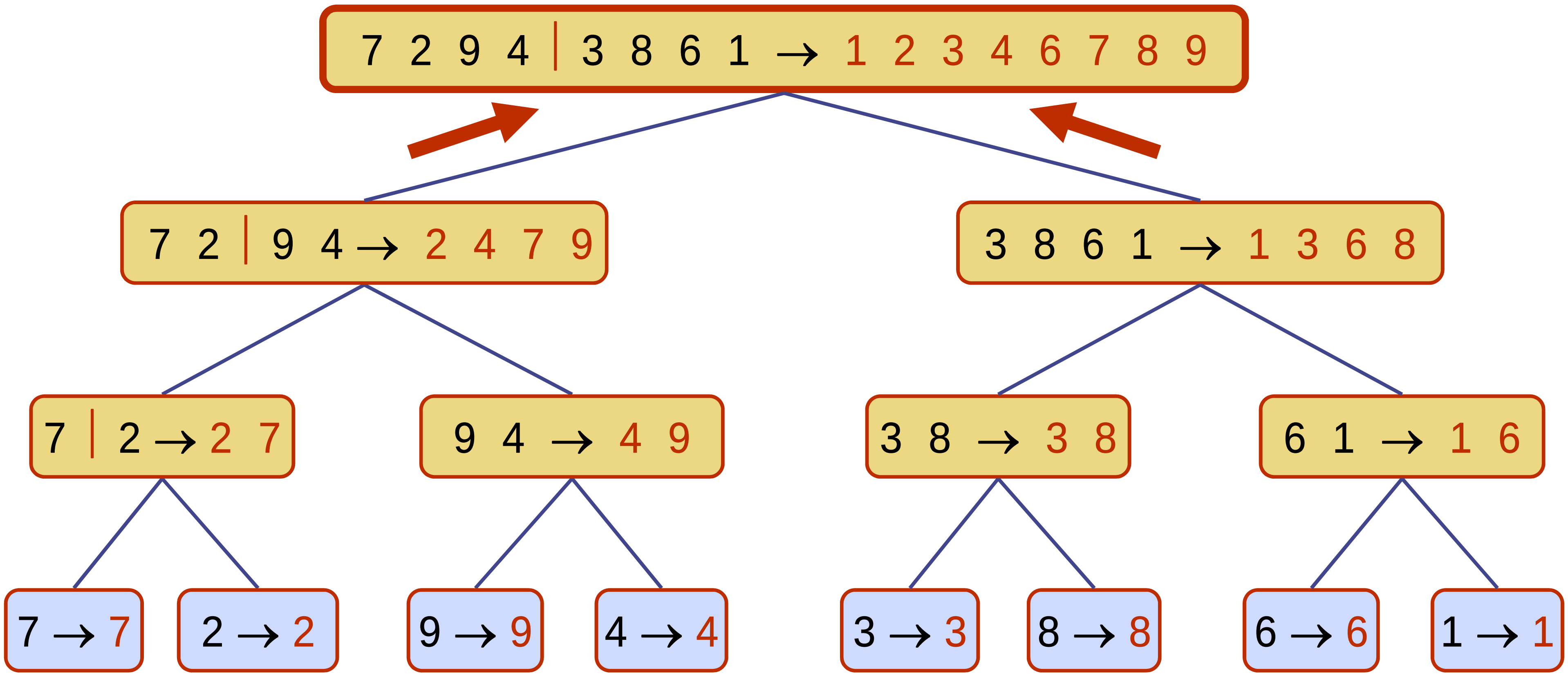
Execution Example (cont.)

- ☐ Recursive call, …, merge, merge



Execution Example (cont.)

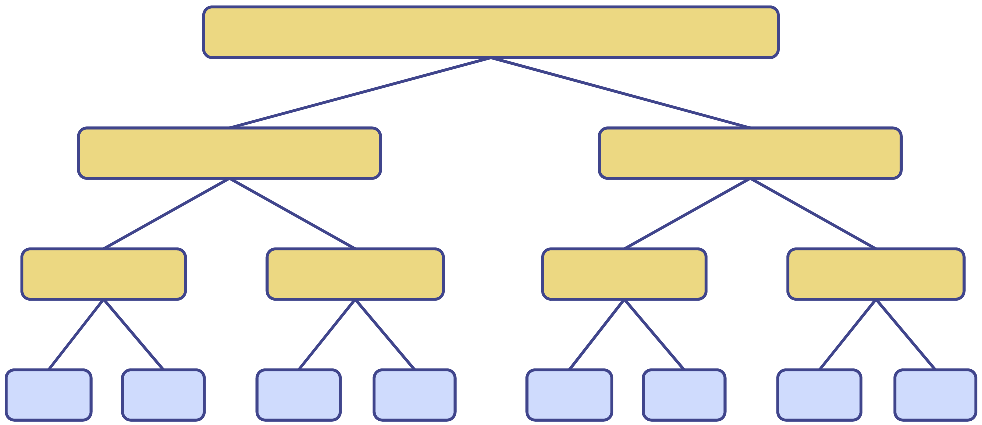
- Merge



Analysis of Merge-Sort

- The height h of the merge-sort tree is O(log n)
  - at each recursive call we divide in half the sequence,
- The overall amount or work done at the nodes of depth i is O(n)
  - we partition and merge 2^i sequences of size $n/2^i$
  - we make 2^{i+1} recursive calls
- Thus, the total running time of merge-sort is O(n log n)

depth	#seqs	size
0	1	n
1	2	$n/2$
i	2^i	$n/2^i$
…	…	…



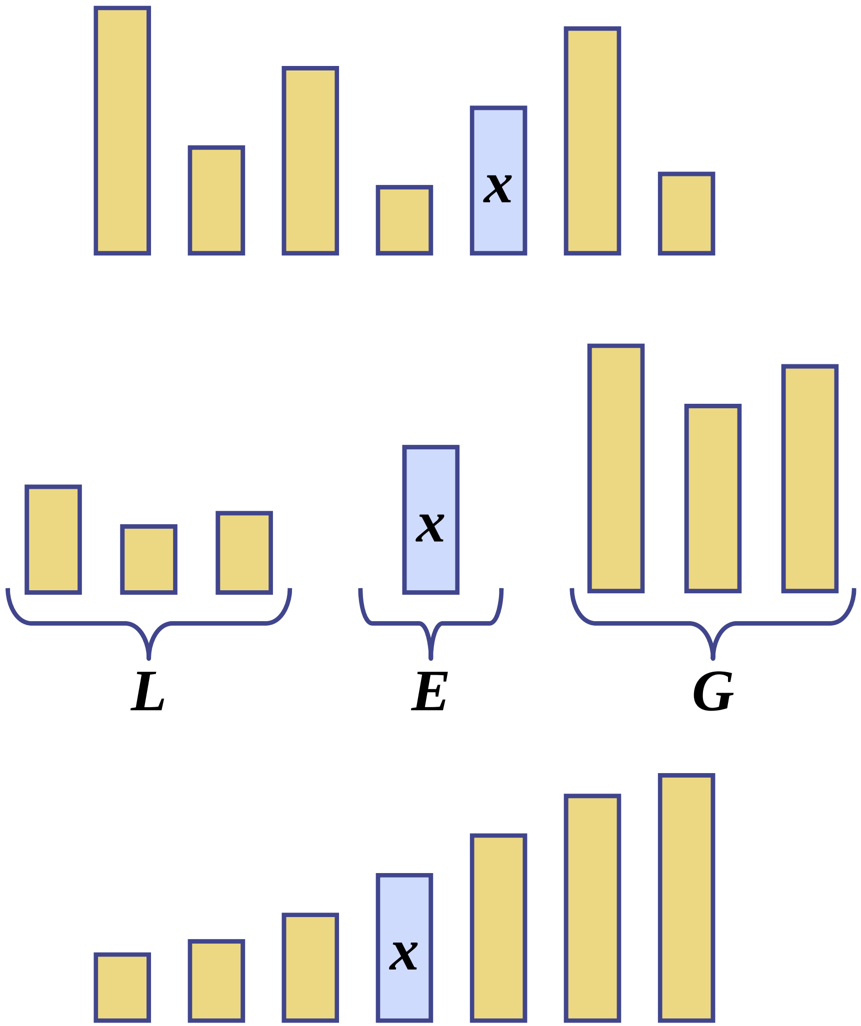
Summary of Sorting Algorithms

Algorithm	Time	Notes
selection-sort	$O(n^2)$	▪ slow ▪ in-place ▪ for small data sets (< 1K)
insertion-sort	$O(n^2)$	▪ slow ▪ in-place ▪ for small data sets (< 1K)
heap-sort	$O(n \log n)$	▪ fast ▪ in-place ▪ for large data sets (1K — 1M)
merge-sort	$O(n \log n)$	▪ fast ▪ sequential data access ▪ for huge data sets (> 1M)

Quick Sort

Quick-Sort

- Quick-sort is a randomized sorting algorithm based on the divide-and-conquer paradigm:
  - Divide: pick a random element x (called pivot) and partition S into
    - L elements less than x
    - E elements equal x
    - G elements greater than x
  - Recur: sort L and G
  - Conquer: join L, E and G



Partition

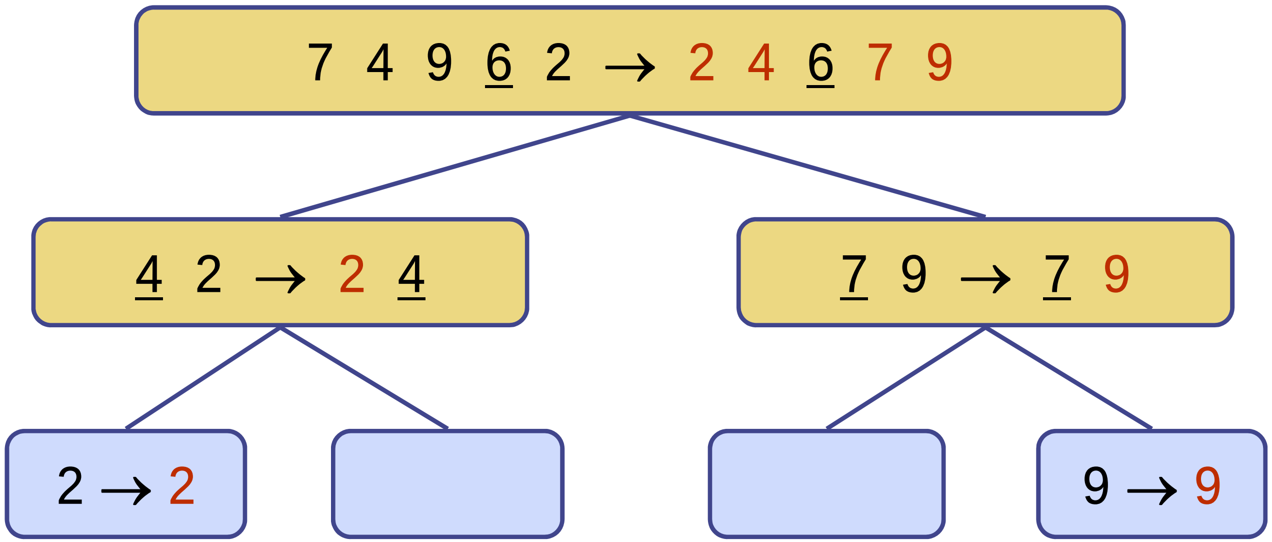
- We partition an input sequence as follows:
  - We remove, in turn, each element y from S and
  - We insert y into L, E or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes O(1) time
- Thus, the partition step of quick-sort takes O(n) time

Algorithm *partition*(S, p)
    Input sequence S, position p of pivot
    Output subsequences L, E, G of the elements of S less than, equal to, or greater than the pivot, resp.
    $L, E, G \leftarrow$ empty sequences
    $x \leftarrow S.remove(p)$
    while $\neg S.isEmpty()$
        $y \leftarrow S.remove(S.first())$
        if $y < x$
            $L.insertLast(y)$
        else if $y = x$
            $E.insertLast(y)$
        else $\{ y > x \}$
            $G.insertLast(y)$
    return L, E, G

Quick-Sort Tree

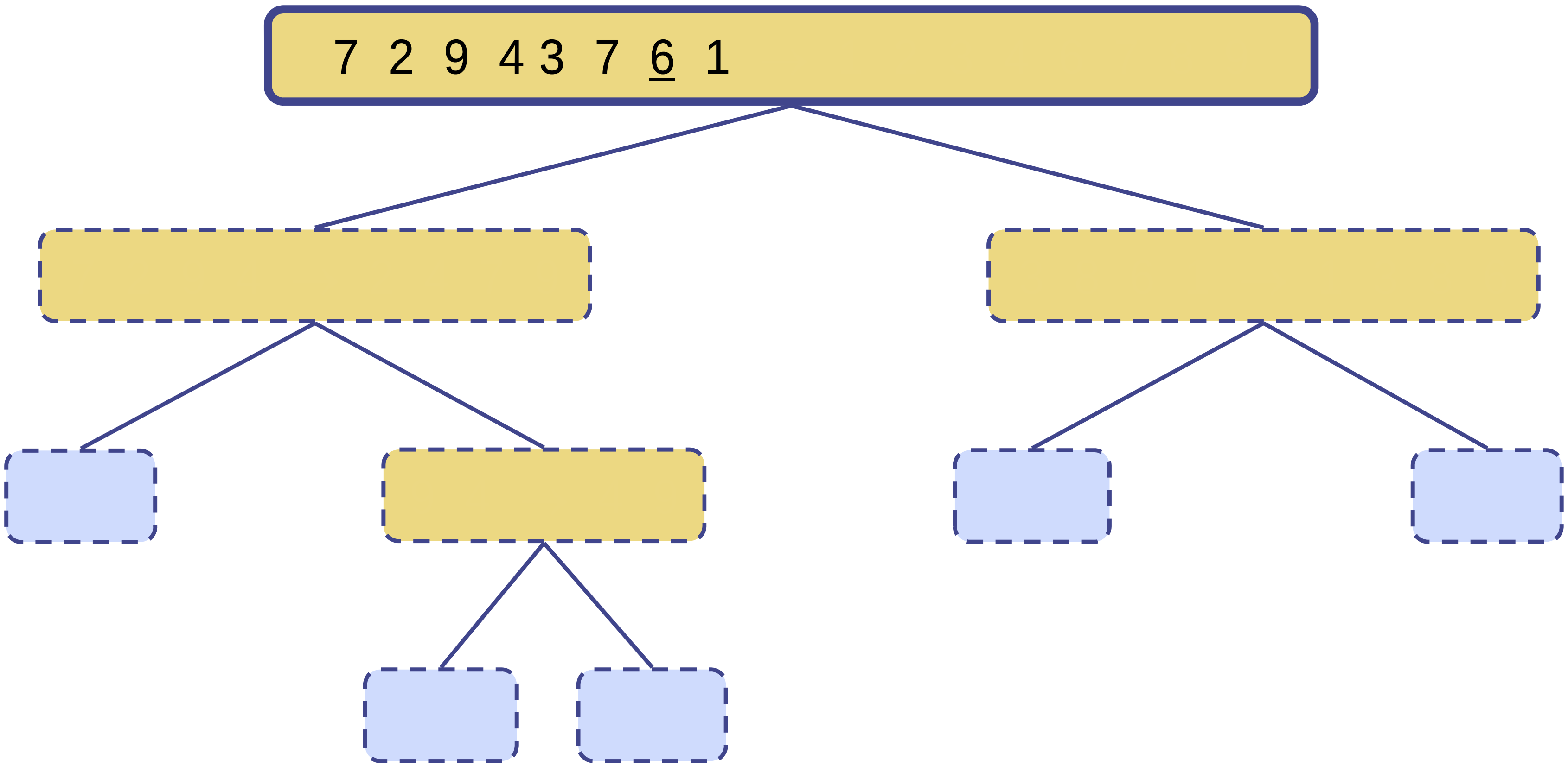
☐ An execution of quick-sort is depicted by a binary tree

- Each node represents a recursive call of quick-sort and stores
  - Unsorted sequence before the execution and its pivot
  - Sorted sequence at the end of the execution
- The root is the initial call
- The leaves are calls on subsequences of size 0 or 1



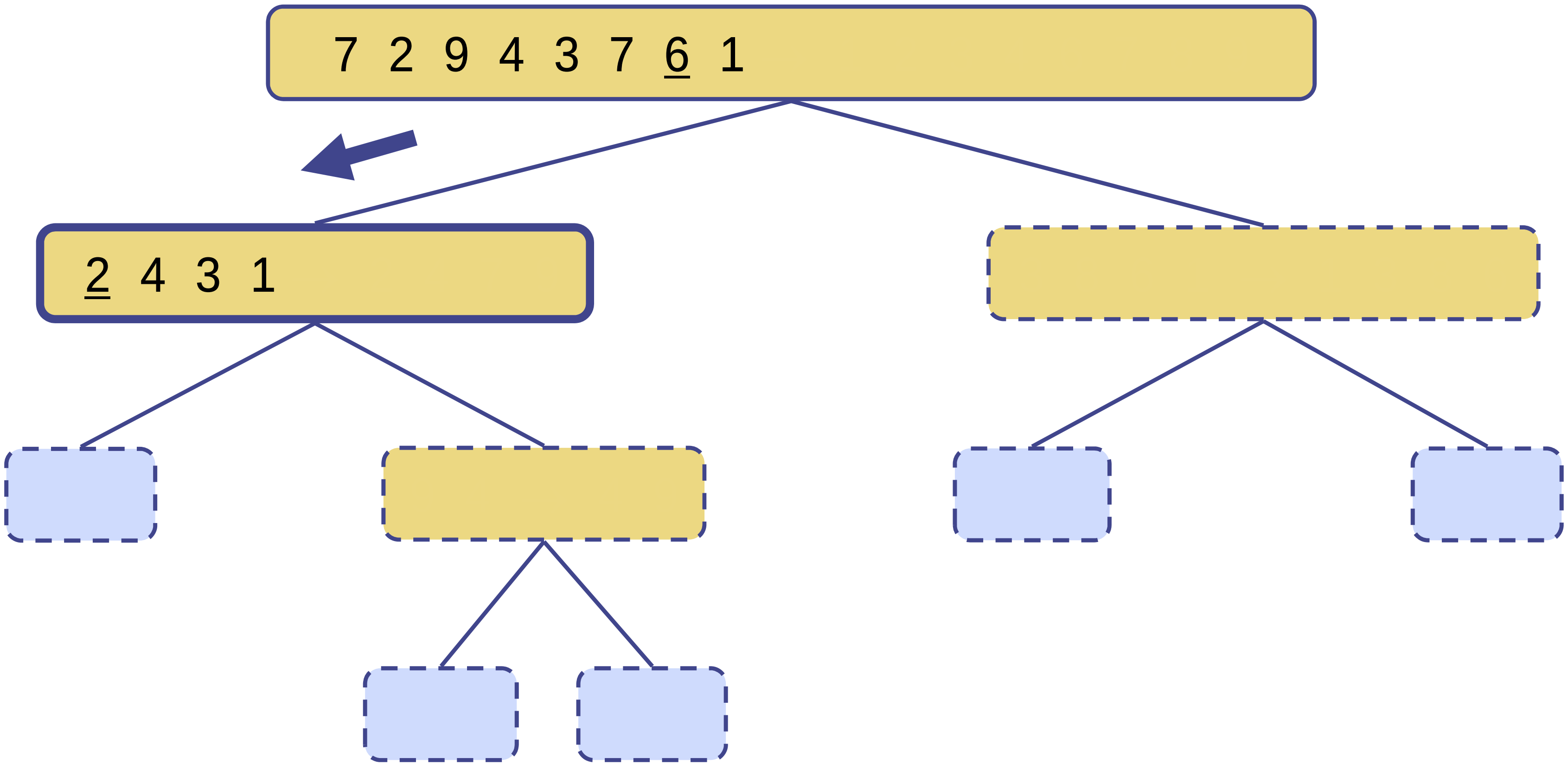
Execution Example

☐ Pivot selection



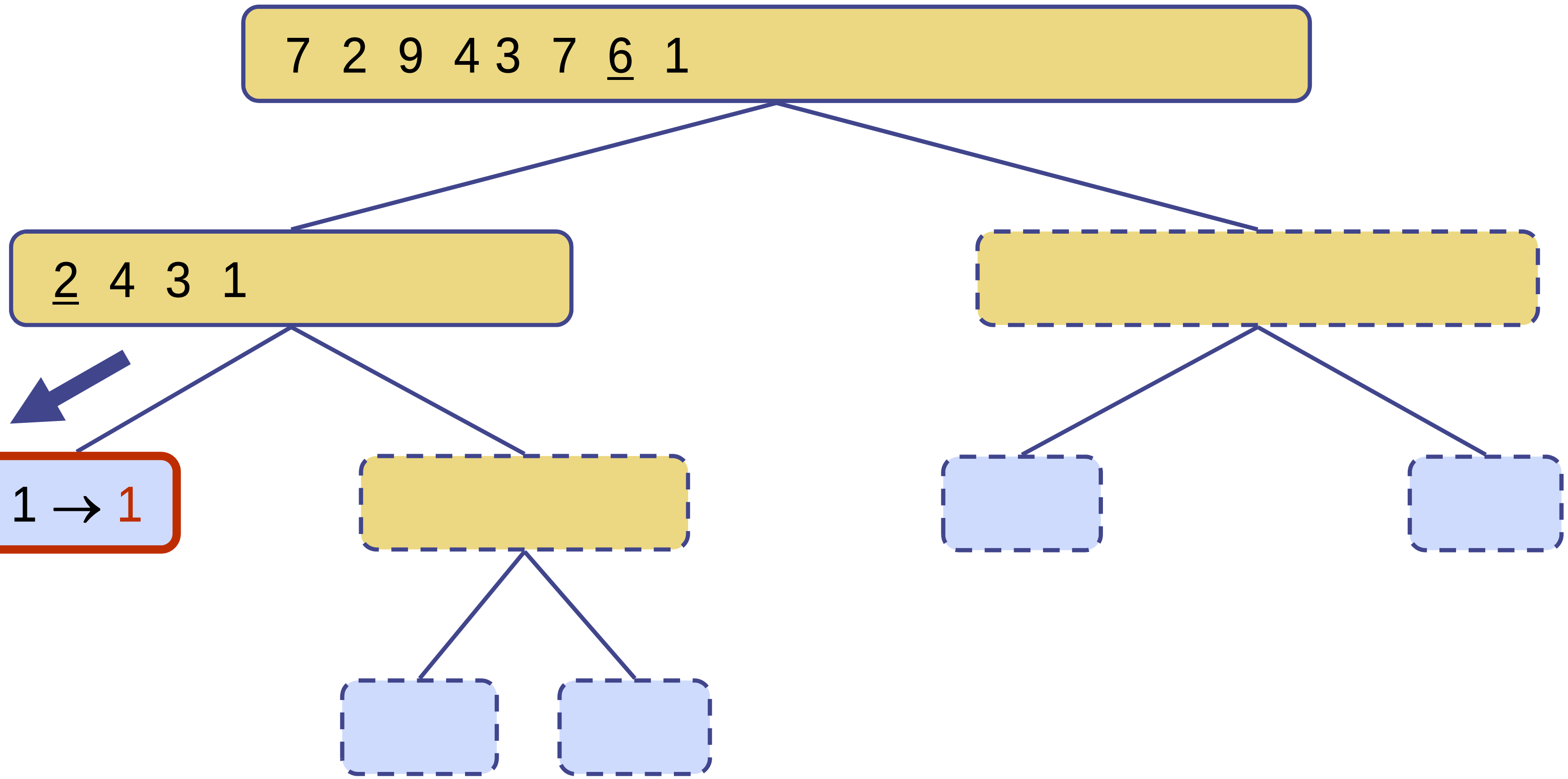
Execution Example (cont.)

☐ Partition, recursive call, pivot selection



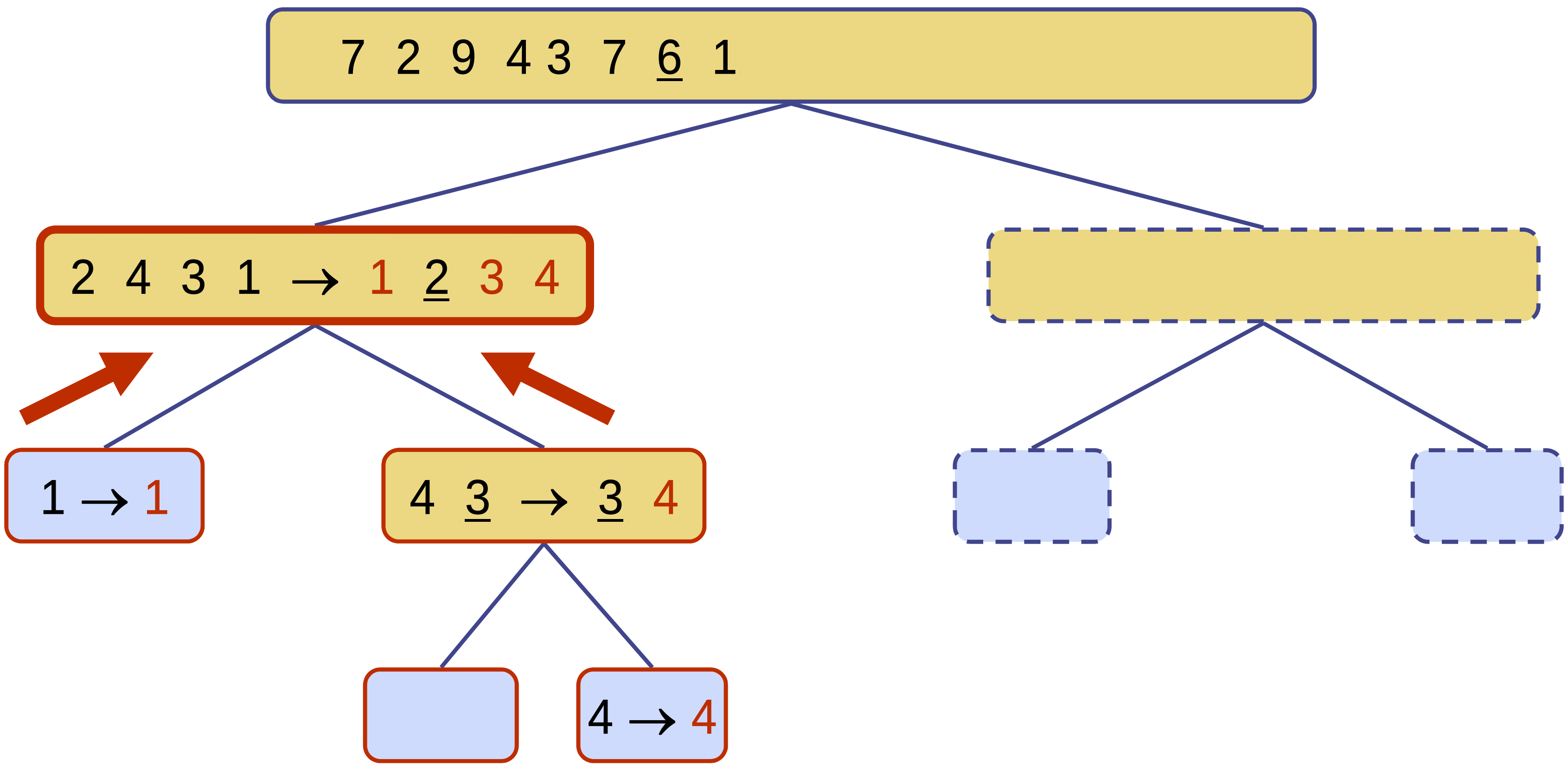
Execution Example (cont.)

☐ Partition, recursive call, base case



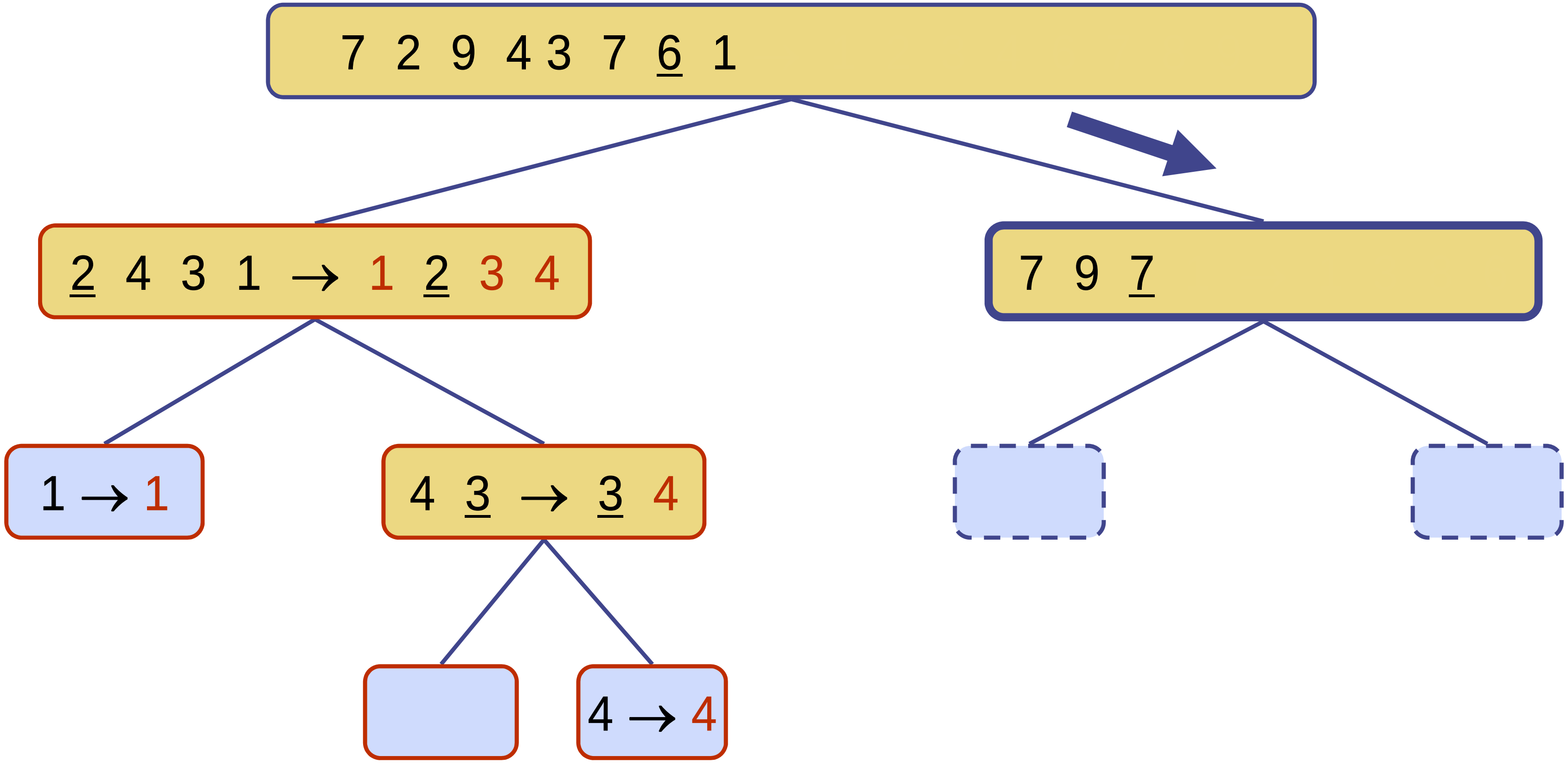
Execution Example (cont.)

- Recursive call, …, base case, join



Execution Example (cont.)

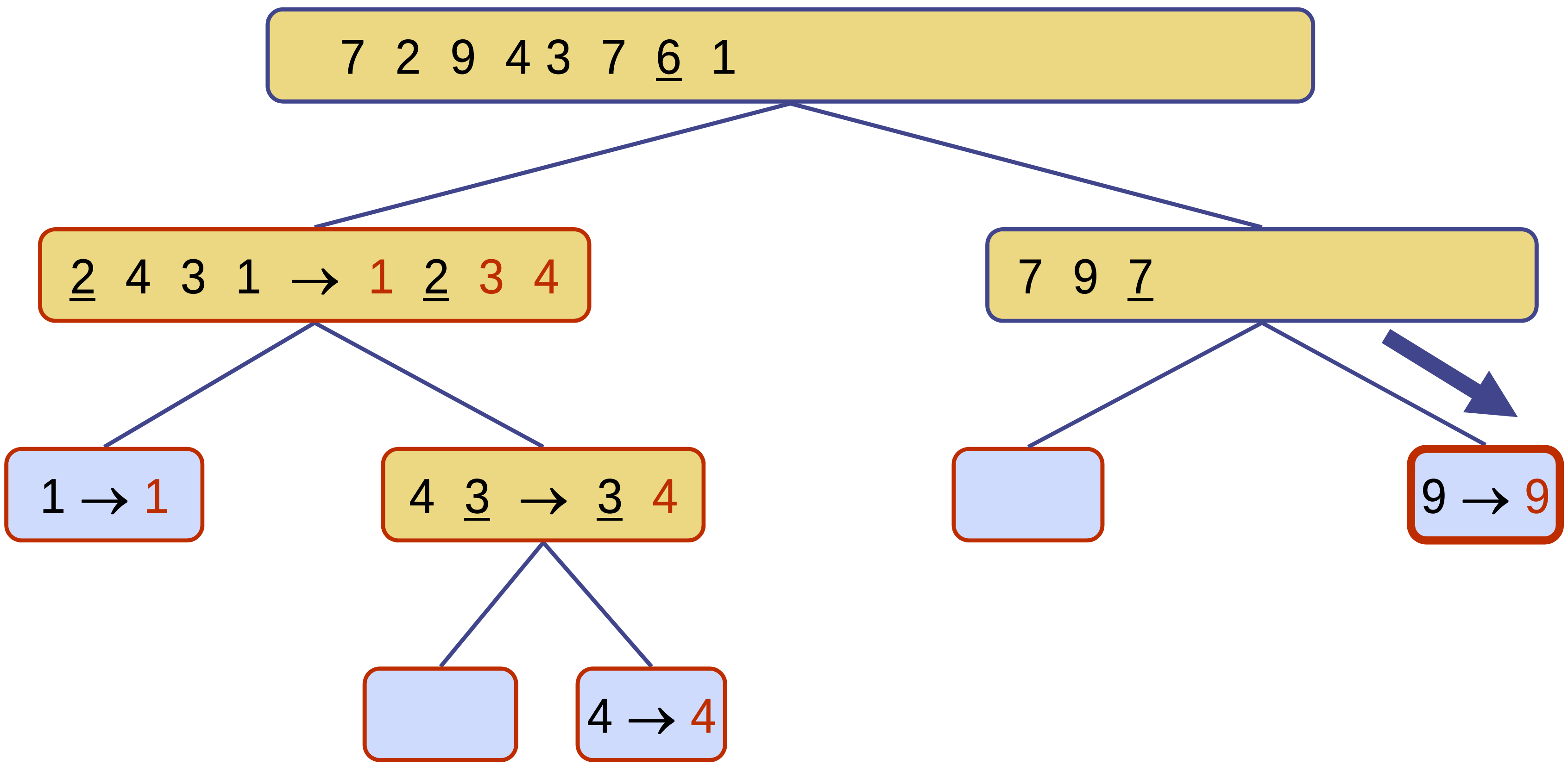
☐ Recursive call, pivot selection



KAIST
Korea Advanced Institute of Science and Technology
한국과학기술원

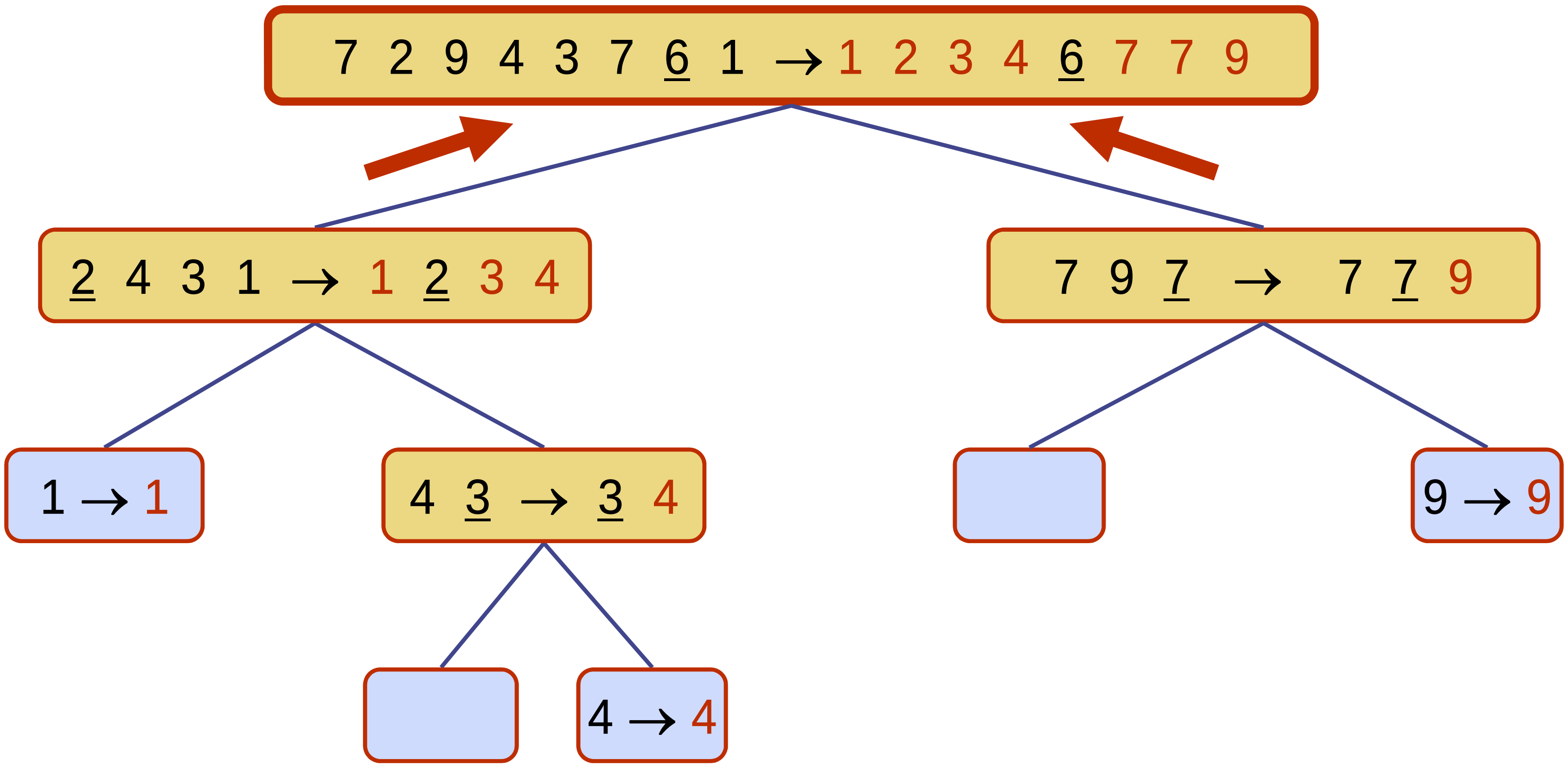
Execution Example (cont.)

- Partition, …, recursive call, base case



Execution Example (cont.)

- Join, join



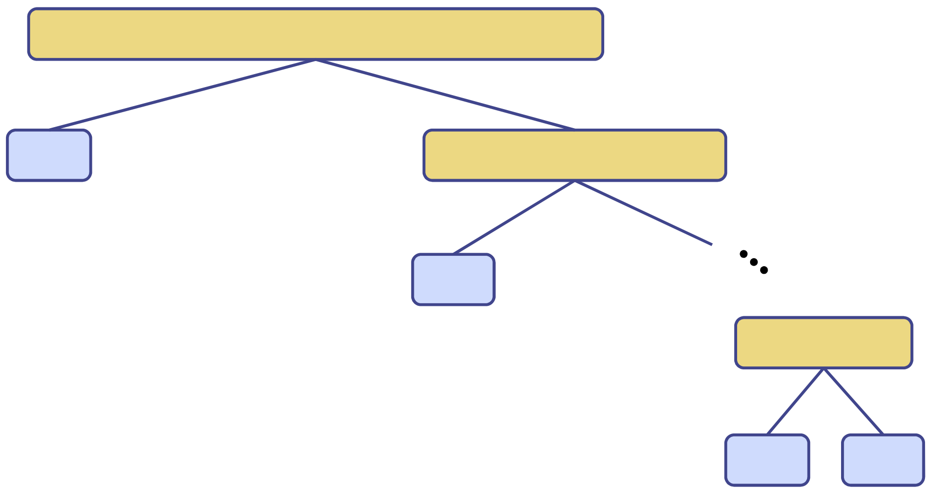
Worst-case Running Time

- The worst case for quick-sort occurs when the pivot is the unique minimum or maximum element
- One of L and G has size n - 1 and the other has size 0
- The running time is proportional to the sum

$$n + (n - 1) + \ldots + 2 + 1$$

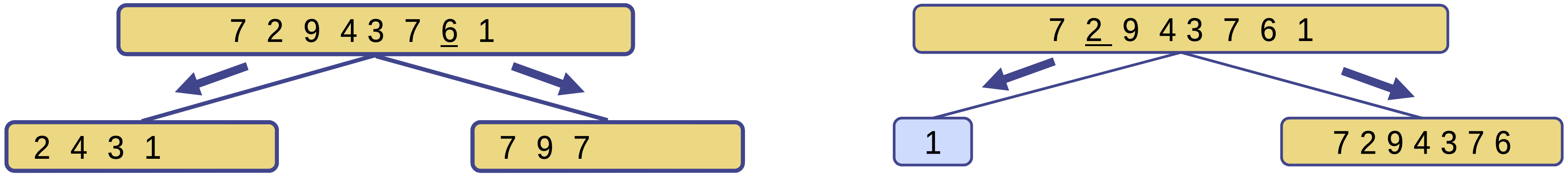
- Thus, the worst-case running time of quick-sort is $O(n^2)$

depth	time
0	n
1	$n-1$
$\cdots$	$\cdots$
$n-1$	1



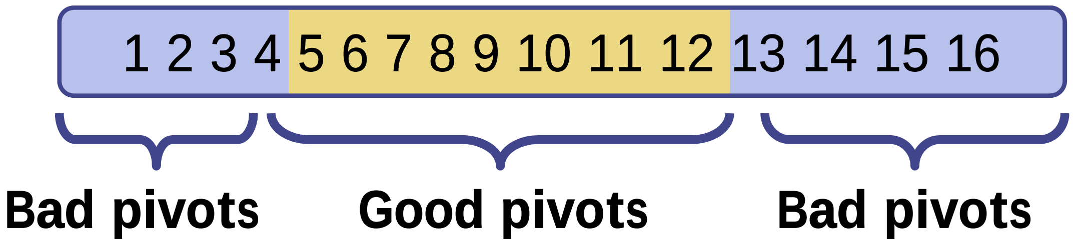
Expected Running Time

- Consider a recursive call of quick-sort on sequence of size s
  - **Good call**: the sizes of L and G are each less than 3s/4
  - **Bad call**: one of L and G has size greater than 3s/4



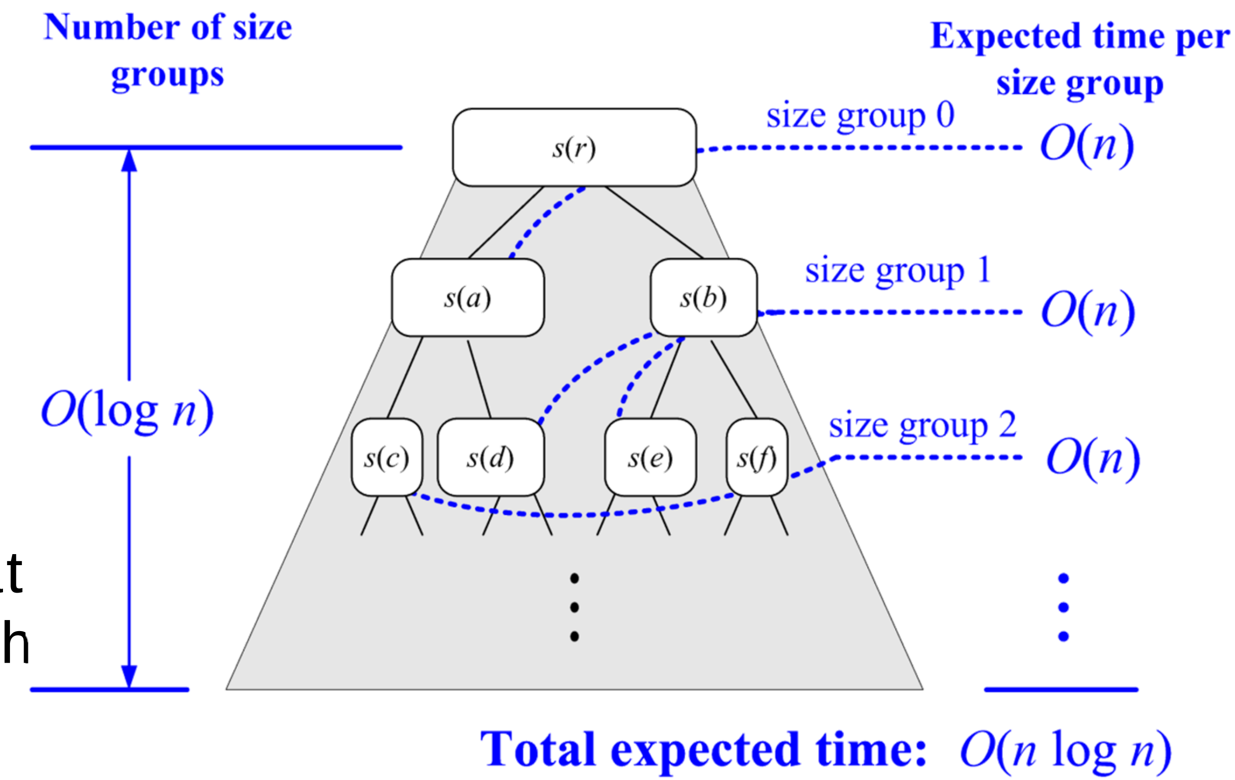
Good call **Bad call**

- A call is good with probability 1/2
  - 1/2 of the possible pivots cause good calls:



Expected Running Time (continued)

- Probabilistic Fact: The expected number of coin tosses required in order to get k heads is 2k
- For a node of depth i, we expect
  - i/2 ancestors are good calls
  - The size of the input sequence for the current call is at most $(3/4)^{i/2}n$
- Therefore, we have
  - For a node of depth $2\log_{4/3}n$, the expected input size is one
  - The expected height of the quick-sort tree is O(log n)
- The amount or work done at the nodes of the same depth is O(n)
- Thus, the expected running time of quick-sort is O(n log n)



In-Place Quick-Sort

- ☐ Quick-sort can be implemented to run in-place
- ☐ In the partition step, we use replace operations to rearrange the elements of the input sequence such that
  - the elements less than the pivot have rank less than h
  - the elements equal to the pivot have rank between h and k
  - the elements greater than the pivot have rank greater than k
- ☐ The recursive calls consider
  - elements with rank less than h
  - elements with rank greater than k

Algorithm *inPlaceQuickSort*(*S*, *l*, *r*)
    Input sequence *S*, ranks *l* and *r*
    Output sequence *S* with the elements of rank between *l* and *r* rearranged in increasing order
    if $l \geq r$
        return
    $i \leftarrow$ a random integer between *l* and *r*
    $x \leftarrow S.elemAtRank(i)$
    $(h, k) \leftarrow inPlacePartition(x)$
    $inPlaceQuickSort(S, l, h-1)$
    $inPlaceQuickSort(S, k+1, r)$

In-Place Partitioning

- Perform the partition using two indices to split S into L and E U G (a similar method can split E U G into E and G).

j k

3 2 5 1 0 7 3 5 9 2 7 9 8 9 7 **<u>6</u>** 9 (pivot = 6)

- Repeat until j and k cross:
  - Scan j to the right until finding an element $\geq$ x.
  - Scan k to the left until finding an element < x.
  - Swap elements at indices j and k

j k

3 2 5 1 0 $\boxed{7}$ 3 5 9 $\boxed{2}$ 7 9 8 9 7 **<u>6</u>** 9

Summary of Sorting Algorithms

Algorithm	Time	Notes
selection-sort	$O(n^2)$	◆ in-place ◆ slow (good for small inputs)
insertion-sort	$O(n^2)$	◆ in-place ◆ slow (good for small inputs)
quick-sort	$O(n \log n)$ expected	◆ in-place, randomized ◆ fastest (good for large inputs)
heap-sort	$O(n \log n)$	◆ in-place ◆ fast (good for large inputs)
merge-sort	$O(n \log n)$	◆ sequential data access ◆ fast (good for huge inputs)

Class Objectives were:

- ☐ Understand divide-and-conquer sorting methods: merge and quick sorts

PA 6

- Implement the quick sort

Next Time

- ☐ Radix sort and selection

- ☐ Questions:
  - Come up with one question on what we have discussed in the class and submit at the end of the class
  - 1 for typical questions and 2 for questions with thoughts or that surprised me
  - Write questions at least 4 times; you can type at KLMS

- ☐ HW:
  - Go over the next lecture slides before the class
  - Just 10 min ~ 20 min should be okay