Partial Differential Equations

An Accessible Route through Theory and Applications

András Vasy

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This book was based on lecture notes for the Math 220/CME 303 course at Stanford University and they benefitted a great deal from feedback from the students in these classes. These notes were also the basis of the notes for the Fourier transform component of the Math 172 course at Stanford University; again, comments from the students were beneficial for their development.

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Preface

This book is intended as an introduction to partial differential equations (PDE) for advanced undergraduate mathematics students or beginning graduate students in applied mathematics, the natural sciences and engineering. The assumption is that the students either have some background in basic real analysis, such as norms, metric spaces, ODE existence and uniqueness, or they are willing to learn the required material as the course goes on, with this material provided either in the text of the chapters or in the notes at the end of the chapters. The goal is to teach the students PDE in a mathematically complete manner, without using more advanced mathematics, but with an eye toward the larger PDE world that requires more background. For instance, distributions are introduced early because, although conceptually challenging, they are, nowadays, the basic language of PDE and they do not require a sophisticated setup (and they prevent one from worrying too much about differentiation!). Another example is that L^2-spaces are introduced as completions, their elements are shown to be distributions, and the L^2-theory of the Fourier series is developed based on this. This avoids the necessity of having the students learn measure theory and functional analysis, which are usually prerequisites of more advanced PDE texts, but which might be beyond the time constraints of students in these fields.

As for the aspects of PDE theory covered, the goal is to cover a wide range of PDE and emphasize phenomena that are general, beyond the cases which can be studied within the limitations of this book. While first order scalar PDE can be covered in great generality, beyond this the basic tools give more limited results, typically restricted to constant coefficient PDE. Nonetheless, when plausible, more general tools and results, such as energy estimates, are discussed even in the variable coefficient setting. At the end of the book these are used to show solvability of elliptic non-constant coefficient PDE via duality based arguments with the text also providing the basic Hilbert space tools required (Riesz representation).

In terms of mathematical outlook, this book is more advanced than Strauss's classic text [**6**]—but does not cover every topic Strauss covers—though it shares its general outlook on the field. It assumes much less background than Evans' [**1**] or Folland's [**2**] text; Folland's book covers many similar topics but with more assumption on the preparation of the students. For an even more advanced text see Taylor's book [**7**] (which has some overlaps with this book) which, however, in a sense has a similar outlook on the field: this would be a good potential continuation for students for a second PDE course. This text thus aims for a middle ground; it is hoped that this will bring at least aspects of modern PDE theory to those who cannot afford to go through a number of advanced mathematics courses to reach the latter.

Since PDE theory necessarily relies on basic real analysis as we recall, more advanced topics develop as we progress. Good references for further real analysis background are Simon's book [**4**] for multivariable calculus and basic real analysis topics, and Johnsonbaugh and Pfaffenberger [**3**] for the metric space material.

The chapters have many concrete PDE problems, but some of them also have some more abstract real analysis problems. The latter are not necessary for a good understanding of the main material, but give a more advanced overview.

The last two chapters of the text are more advanced than the rest of the book. They cover solvability by duality arguments and variational problems. While no additional background is required since the basic Hilbert space arguments are provided, the reader will probably find these chapters more difficult. However, these chapters do show that even sophisticated PDE theory is within reach after working through the previous chapters!

In practice, in a 10-week quarter at Stanford most of the (main chapter) material in Chapters 1–14 is covered in a very fast-paced manner. In a semester it should be possible to cover the whole book at a fast pace, or most of the book at a more moderate pace.

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This text on partial differential equations is intended for readers who want to understand the theoretical underpinnings of modern PDEs in settings that are important for the applications without using extensive analytic tools required by most advanced texts. The assumed mathematical background is at the level of multivariable calculus and basic metric space material, but the latter is recalled as relevant as the text progresses.

The key goal of this book is to be mathematically complete without overwhelming the reader, and to develop PDE theory in a manner that reflects how researchers would think about the material. A concrete example is that distribution theory and the concept of weak solutions are introduced early because while these ideas take some time for the students to get used to, they are fundamentally easy and, on the other hand, play a central role in the field. Then, Hilbert spaces that are quite important in the later development are introduced via completions which give essentially all the features one wants without the overhead of measure theory.

There is additional material provided for readers who would like to learn more than the core material, and there are numerous exercises to help solidify one's understanding. The text should be suitable for advanced undergraduates or for beginning graduate students including those in engineering or the sciences.

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