

Partial Differential Equations

An Accessible Route through
Theory and Applications

András Vasy

**Graduate Studies
in Mathematics**

Volume 169



American Mathematical Society

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Providence, Rhode Island

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This book was based on lecture notes for the Math 220/CME 303 course at Stanford University and they benefitted a great deal from feedback from the students in these classes. These notes were also the basis of the notes for the Fourier transform component of the Math 172 course at Stanford University; again, comments from the students were beneficial for their development.

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Contents

Preface	ix
Chapter 1. Introduction	1
§1. Preliminaries and notation	1
§2. Partial differential equations	6
Additional material: More on normed vector spaces and metric spaces	10
Problems	15
Chapter 2. Where do PDE come from?	19
§1. An example: Maxwell's equations	19
§2. Euler-Lagrange equations	21
Problems	25
Chapter 3. First order scalar semilinear equations	29
Additional material: More on ODE and the inverse function theorem	38
Problems	43
Chapter 4. First order scalar quasilinear equations	45
Problems	52
Chapter 5. Distributions and weak derivatives	55
Additional material: The space L^1	68
Problems	74
Chapter 6. Second order constant coefficient PDE: Types and d'Alembert's solution of the wave equation	81

§1. Classification of second order PDE	81
§2. Solving second order hyperbolic PDE on \mathbb{R}^2	85
Problems	90
Chapter 7. Properties of solutions of second order PDE: Propagation, energy estimates and the maximum principle	93
§1. Properties of solutions of the wave equation: Propagation phenomena	93
§2. Energy conservation for the wave equation	97
§3. The maximum principle for Laplace's equation and the heat equation	100
§4. Energy for Laplace's equation and the heat equation	103
Problems	108
Chapter 8. The Fourier transform: Basic properties, the inversion formula and the heat equation	113
§1. The definition and the basics	113
§2. The inversion formula	118
§3. The heat equation and convolutions	121
§4. Systems of PDE	123
§5. Integral transforms	126
Additional material: A heat kernel proof of the Fourier inversion formula	127
Problems	130
Chapter 9. The Fourier transform: Tempered distributions, the wave equation and Laplace's equation	133
§1. Tempered distributions	133
§2. The Fourier transform of tempered distributions	136
§3. The wave equation and the Fourier transform	138
§4. More on tempered distributions	140
Problems	141
Chapter 10. PDE and boundaries	147
§1. The wave equation on a half space	147
§2. The heat equation on a half space	150
§3. More complex geometries	153
§4. Boundaries and properties of solutions	154
§5. PDE on intervals and cubes	155

Problems	157
Chapter 11. Duhamel's principle	159
§1. The inhomogeneous heat equation	159
§2. The inhomogeneous wave equation	163
Problems	167
Chapter 12. Separation of variables	169
§1. The general method	169
§2. Interval geometries	171
§3. Circular geometries	173
Problems	176
Chapter 13. Inner product spaces, symmetric operators, orthogonality	179
§1. The basics of inner product spaces	179
§2. Symmetric operators	187
§3. Completeness of orthogonal sets and of the inner product space	191
Problems	196
Chapter 14. Convergence of the Fourier series and the Poisson formula on disks	201
§1. Notions of convergence	201
§2. Uniform convergence of the Fourier transform	203
§3. What does the Fourier series converge to?	206
§4. The Dirichlet problem on the disk	209
Additional material: The Dirichlet kernel	214
Problems	217
Chapter 15. Bessel functions	221
§1. The definition of Bessel functions	221
§2. The zeros of Bessel functions	226
§3. Higher dimensions	232
Problems	233
Chapter 16. The method of stationary phase	235
Problems	243
Chapter 17. Solvability via duality	245
§1. The general method	245
§2. An example: Laplace's equation	250

§3. Inner product spaces and solvability	252
Problems	260
Chapter 18. Variational problems	263
§1. The finite dimensional problem	263
§2. The infinite dimensional minimization	266
Problems	274
Bibliography	277
Index	279

Preface

This book is intended as an introduction to partial differential equations (PDE) for advanced undergraduate mathematics students or beginning graduate students in applied mathematics, the natural sciences and engineering. The assumption is that the students either have some background in basic real analysis, such as norms, metric spaces, ODE existence and uniqueness, or they are willing to learn the required material as the course goes on, with this material provided either in the text of the chapters or in the notes at the end of the chapters. The goal is to teach the students PDE in a mathematically complete manner, without using more advanced mathematics, but with an eye toward the larger PDE world that requires more background. For instance, distributions are introduced early because, although conceptually challenging, they are, nowadays, the basic language of PDE and they do not require a sophisticated setup (and they prevent one from worrying too much about differentiation!). Another example is that L^2 -spaces are introduced as completions, their elements are shown to be distributions, and the L^2 -theory of the Fourier series is developed based on this. This avoids the necessity of having the students learn measure theory and functional analysis, which are usually prerequisites of more advanced PDE texts, but which might be beyond the time constraints of students in these fields.

As for the aspects of PDE theory covered, the goal is to cover a wide range of PDE and emphasize phenomena that are general, beyond the cases which can be studied within the limitations of this book. While first order scalar PDE can be covered in great generality, beyond this the basic tools give more limited results, typically restricted to constant coefficient PDE. Nonetheless, when plausible, more general tools and results, such as energy estimates, are discussed even in the variable coefficient setting. At the end of

the book these are used to show solvability of elliptic non-constant coefficient PDE via duality based arguments with the text also providing the basic Hilbert space tools required (Riesz representation).

In terms of mathematical outlook, this book is more advanced than Strauss's classic text [6]—but does not cover every topic Strauss covers—though it shares its general outlook on the field. It assumes much less background than Evans' [1] or Folland's [2] text; Folland's book covers many similar topics but with more assumption on the preparation of the students. For an even more advanced text see Taylor's book [7] (which has some overlaps with this book) which, however, in a sense has a similar outlook on the field: this would be a good potential continuation for students for a second PDE course. This text thus aims for a middle ground; it is hoped that this will bring at least aspects of modern PDE theory to those who cannot afford to go through a number of advanced mathematics courses to reach the latter.

Since PDE theory necessarily relies on basic real analysis as we recall, more advanced topics develop as we progress. Good references for further real analysis background are Simon's book [4] for multivariable calculus and basic real analysis topics, and Johnsonbaugh and Pfaffenberger [3] for the metric space material.

The chapters have many concrete PDE problems, but some of them also have some more abstract real analysis problems. The latter are not necessary for a good understanding of the main material, but give a more advanced overview.

The last two chapters of the text are more advanced than the rest of the book. They cover solvability by duality arguments and variational problems. While no additional background is required since the basic Hilbert space arguments are provided, the reader will probably find these chapters more difficult. However, these chapters do show that even sophisticated PDE theory is within reach after working through the previous chapters!

In practice, in a 10-week quarter at Stanford most of the (main chapter) material in Chapters 1–14 is covered in a very fast-paced manner. In a semester it should be possible to cover the whole book at a fast pace, or most of the book at a more moderate pace.

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Index

- Abel summability, 214
- amplitude, 236
- asymptotic expansion, 229

- Bessel functions, 222
- Bessel's inequality, 192
- bounded linear map, 3
- bounded set, 14
- bump function, 57
- Burgers' equation, 49

- the space C^0 , 4
- the space C^1 , 5
- the space C^ℓ , 5
- Cantor's diagonal argument, 274
- Cauchy sequence, 38
- Cauchy sequence in normed space, 194
- Cauchy-Schwarz inequality, 184
- Cesàro summability, 214
- characteristic coordinates, 89
- characteristic curves, 30
- characteristic triangle, 94
- characteristics, 46
- closed sets, 13
- closure, 13
- compact set, 14
- compactly supported functions, 56
- complete normed space, 194
- complete metric space, 38
- complete orthogonal set, 192
- completion, 69
- constant coefficient PDE, 6
- continuous map in metric spaces, 12
- continuous map in normed spaces, 10
- contraction mapping, 38
- convergence, 3
- convergence of distributions, 61
- convolution, 123

- d'Alembert's solution, 87
- damped wave equation, 110
- degenerate PDE, 83
- delta distribution, 59
- dense subspace, 188
- density of test functions, 62
- differentiable function, 4
- directional derivative, 29
- Dirichlet boundary conditions, 8
- Dirichlet kernel, 215
- disk, 173
- distance function, 12
- distribution, 58
- distributional derivatives, 63
- domain of dependence, 93
- domain of influence, 95
- dual vector space Z^* of Z , 249
- Duhamel's principle, 160

- eigenvalue, 188
- eigenvalue equation, 171
- eigenvector, 188
- elliptic operator, 146
- elliptic regularity, 146
- elliptic second order PDE, 84
- energy conservation, 98
- energy estimate, 104

- entropy condition, 77
 equivalent norms, 3
 Euler-Lagrange equation, 23
 Euler-Lagrange functional, 21
 even extension, 149

 Fejér kernel, 220
 finite speed of propagation, 109
 fixed point, 38
 Fourier cosine basis, 190
 Fourier cosine series, 190
 Fourier inversion formula, 114
 Fourier sine basis, 189
 Fourier sine series, 190
 Fourier transform \mathcal{F} , 114
 Fourier transform of tempered distributions, 137
 full Fourier basis, 190
 full Fourier series, 190, 203
 full symbol of PDE, 117
 fully non-linear PDE, 7
 fundamental solution, 144

 generalized Fourier coefficients, 192
 generalized Fourier series, 187
 Gram-Schmidt orthogonalization, 198
 Green's function, 144

 standard H^1 -norm, 251
 the space $H^1(\Omega)$, 253
 the space $H_0^1(\Omega)$, 253
 heat equation, 6
 heat kernel, 123
 Helmholtz equation, 118
 Hermitian symmetry, 179
 Hilbert space, 248
 homogeneous boundary condition, 8
 homogeneous linear PDE, 6
 Huygens' principle, 95
 hyperbolic second order PDE, 84

 induced metric, 12
 inhomogeneous boundary condition, 8
 inhomogeneous linear PDE, 6
 initial conditions, 8
 inner product, 179
 integral curve, 29
 interior, 13
 invariant subspace, 264
 inverse Fourier transform \mathcal{F}^{-1} , 114

 the space $L^1(\mathbb{R}^n)$, 70
 the space $L^2(\Omega)$, 195

 the space ℓ^2 , 181
 Laplace's equation, 6
 Laplace-Beltrami operator, 26
 least squares, 193
 Lebesgue integral, 71
 left inverse, 246
 limit points, 13
 linear map, 2
 linear PDE, 6
 Lipschitz map, 41
 locally convex space, 248

 matrix transpose, 82
 maximum principle, 100
 Maxwell's equations, 19
 mean value property, 212
 method of images, 153
 metric, 12
 metric space, 12
 min-max for eigenvalues, 274
 multiindex notation, 5

 Neumann boundary conditions, 8
 non-characteristic initial value problem, 32
 norm, 2

 odd extension, 148
 odd periodic extension, 156
 open sets, 13
 orthocomplement, 259
 orthogonal projection to line, 184
 orthogonal projection to subspace, 258
 orthogonal set of vectors, 187
 overdetermined problem, 127

 parallelogram law, 196
 Parseval/Plancherel formula, 141
 partial derivative, 5
 partial Fourier transform, 121
 partition of unity, 74
 periodic boundary conditions, 219
 phase function, 236
 Poincaré inequality, 106
 pointwise convergence, 201
 Poisson formula, 212
 Poisson kernel, 213
 positive definite, 180
 propagation of singularities, 96
 Pythagoras' theorem, 183

 quasilinear PDE, 7

- Radon transform, 132
- Rankine-Hugoniot jump condition, 76
- rarefaction wave, 77
- Rayleigh quotient, 265
- rectangle, 177
- reflection of singularities, 155
- Rellich's lemma, 268
- Riesz' lemma, 255
- right inverse, 246
- Robin boundary conditions, 110

- Schrödinger equation, 131
- Schwartz functions, 116
- sector, 177
- semilinear PDE, 7
- separation of variables, 169
- sequential continuity, 12
- sesquilinear map, 182
- shock wave, 76
- singular support, 95
- Sobolev space, 253
- spectral methods, 171
- stationary phase, 237
- stationary points, 237
- subsequence, 14
- support, 56
- symmetric operator, 188

- Taylor's theorem, 15
- tempered distribution, 135
- test functions, 57
- torus, 197
- transpose operator, 65
- trial functions, 67
- triangle inequality, 3

- ultrahyperbolic second order PDE, 84
- uniform convergence, 201
- uniform with all derivatives
 - convergence, 201
- uniformly continuous map, 14

- wave equation, 6
- weak solution, 65
- weak-* topology, 61
- Weierstrass M-test, 202
- Weyl's law, 274

- X-ray transform, 126

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This text on partial differential equations is intended for readers who want to understand the theoretical underpinnings of modern PDEs in settings that are important for the applications without using extensive analytic tools required by most advanced texts. The assumed mathematical background is at the level of multivariable calculus and basic metric space material, but the latter is recalled as relevant as the text progresses.



Photo taken by Marguerite Wong, Yasy

The key goal of this book is to be mathematically complete without overwhelming the reader, and to develop PDE theory in a manner that reflects how researchers would think about the material. A concrete example is that distribution theory and the concept of weak solutions are introduced early because while these ideas take some time for the students to get used to, they are fundamentally easy and, on the other hand, play a central role in the field. Then, Hilbert spaces that are quite important in the later development are introduced via completions which give essentially all the features one wants without the overhead of measure theory.

There is additional material provided for readers who would like to learn more than the core material, and there are numerous exercises to help solidify one's understanding. The text should be suitable for advanced undergraduates or for beginning graduate students including those in engineering or the sciences.

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