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Abstract

Purpose – The traditional data envelopment analysis (DEA) model as a non-parametric technique can measure the relative efficiencies of a decision-making units (DMUs) set with exact values of inputs and outputs, but it cannot handle the imprecise data. The purpose of this paper is to establish a super efficiency interval data envelopment analysis (IDEA) model, an IDEA model based on cross-evaluation and a cross evaluation-based measure of super efficiency IDEA model. And the authors apply the proposed approach to data on the 29 public secondary schools in Greece, and further demonstrate the feasibility of the proposed approach.

Design/methodology/approach – In this paper, based on the IDEA model, the authors propose an improved version of establishing a super efficiency IDEA model, an IDEA model based on cross-evaluation, and then present a cross evaluation-based measure of super efficiency IDEA model by combining the super efficiency method with cross-evaluation. The proposed model cannot only discriminate the performance of efficient DMUs from inefficient ones, but also can distinguish between the efficient DMUs. By using the proposed approach, the overall performance of all DMUs with interval data can be fully ranked.

Findings – A numerical example is presented to illustrate the application of the proposed methodology. The result shows that the proposed approach is an effective and practical method to measure the efficiency of the DMUs with imprecise data.

Practical implications – The proposed model can avoid the fact that the original DEA model can only distinguish the performance of efficient DMUs from inefficient ones, but cannot discriminate between the efficient DMUs.

Originality/value – This paper introduces the effective method to obtain the complete rank of all DMUs with interval data.

Keywords Decision making, Operational research, Optimization techniques, Cross-efficiency, Interval DEA, Super efficiency

Paper type Research paper

1. Introduction

Data envelopment analysis (DEA), originally proposed by Charnes *et al.* (1978), is a linear and non-parametric programming technique for measuring the relative efficiency of decision-making units (DMUs) with multiple inputs and multiple outputs. When DEA models are used to calculate the efficiency of DMUs, a number of them may have an equal efficiency score of one. In order to distinguish the efficient DMUs, the modified CCR model called super efficiency model was developed by Andersen and Petersen (1993), where a DMU under evaluation is excluded from the reference set. For efficient DMUs, super efficiency scores are not less than one under the assumption of input-orientation.

In addition, in real-life evaluation problems, there exists some situation of imprecise data due to the existence of uncertainty and incompleteness. In such case, the interval data envelopment analysis (IDEA) model (Cooper *et al.*, 1999, 2001a, b) was presented

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to evaluate the efficiency of a set of DMUs with imprecise data such as bounded data. Lee *et al.* (2002) extended the idea of IDEA to the additive model to deal with imprecise data in evaluations of performance. Despotis and Smirlis (2002) developed an alternative approach for dealing with imprecise data in DEA. Wang and Yang (2007) investigated the efficiencies of DMUs within the range of an interval, whose upper bound is set to one and the lower bound is determined through introducing a virtual anti-ideal DMU, whose performance is definitely inferior to any DMUs. Entani *et al.* (2002) proposed an interval efficiency for crisp data by adding an evaluation from the pessimistic viewpoint to the conventional DEA which is an evaluation from the optimistic viewpoint. Smirlis *et al.* (2006) introduced an approach based on IDEA that allows the evaluation of the units with missing values along with the other units with available crisp data. Wang *et al.* (2005) developed a new pair of IDEA models for dealing with imprecise data such as interval data, ordinal preference information, fuzzy data and their mixture. Foroughi and Aouni (2012) proposed an approach for determining efficiency intervals and setting up a full ranking of DMUs based on the intervals. Gouveia *et al.* (2013) studied the problem of finding the range of efficiency for each DMU considering uncertain data. Azizi (2013) pointed out the drawbacks in the IDEA models presented by Smirlis *et al.* (2006) and to present new IDEA models so that the assessment of the interval efficiencies of the DMUs can be done according to a fixed and unified production frontier. Esmaeili (2012) developed a new approach based upon the Enhanced Russell Measure for dealing with interval data in DEA.

In the traditional DEA model, a DMU is allowed to choose the most favorable weights arbitrarily to achieve its best possible relative efficiency, which make the DMU under evaluation heavily weighs the inputs and outputs of a few favorable DMUs and ignores those of the others. Thus, the cross-efficiency method as an extension of the classic DEA was proposed by Sexton *et al.* (1986) and Doyle and Green (1994), which is a typical peer evaluation technique different from the self-evaluation one. Due to its powerful discrimination ability, cross-efficiency model has been widely used in various fields (Oral *et al.*, 1991; Chen, 2002; Lu and Lo, 2007; Wu *et al.*, 2009; Yu *et al.*, 2010; Yang *et al.*, 2012, 2013). Washio *et al.* (2013) proposed an algorithm to calculate cross-efficiency scores which used the equations forming the efficient frontier in DEA. Wang and Chin (2010a) proposed a neutral DEA model for cross-efficiency evaluation. Furthermore, Wang and Chin (2010b) developed some new alternative models for DEA cross-efficiency evaluation to provide more methodological options for the decision maker (DM) to choose from. Jahanshahloo *et al.* (2011) investigated the symmetric weight assignment technique that does not affect feasibility and rewards DMUs that make a symmetric selection of weights. Wang *et al.* (2012) developed some alternative DEA models to minimize the virtual disparity in the cross-efficiency evaluation. Ruiz and Sirvent (2012) discussed the issue of total weight flexibility of DEA in the context of the cross-efficiency evaluation. Wu *et al.* (2013) introduced a cross-efficiency method into the DEA model to calculate the interval of cross-efficiency values, based on which a new TOPSIS method is proposed to rank the DMUs.

The above mentioned approaches have already been proven effective and feasible for measuring the DMUs. However, most evaluation value of inputs and outputs are represented by imprecise information due to the imprecise knowledge or subjective cognition. For such cases, different from the methods discussed above, the main purpose of this paper is to propose an interval super efficiency model, an IDEA model based on cross-evaluation and the cross evaluation-based measure of super efficiency in IDEA. In a word, the contributions of this paper are presented as follows. First, the super efficiency idea is introduced for distinguishing the efficient DMUs with interval data. Second, the IDEA model based on cross-evaluation is established. Third, based on the interval super efficiency model and the interval cross-efficiency evaluation, the cross evaluation-based measure of super efficiency in DEA with interval data is proposed to obtain the more reasonable result. The proposed models can exactly rank the overall performance of all DMUs with imprecise data not only from the perspective of self-evaluation but also from the perspective of the peer evaluation.

The rest of the paper is organized as follows. Section 2 presents the basic notations of IDEA models for measuring the efficiencies of DMUs. Section 3 develops the super efficiency model in DEA with interval data based on cross-evaluation. Section 4 applies the proposed approach to data on the 29 public secondary schools in Greece, and we further demonstrate the feasibility of our proposed approach. Conclusions are presented in Section 5.

2. IDEA models

In the original DEA model, assume that there are n DMUs to be evaluated. Each DMU uses m different inputs to produce s different outputs. For the jth ($j=1, \ldots, n$) unit DMU_j, the level of outputs and inputs are denoted by y_{rj} ($r=1, \ldots, s$) and x_{ij} ($i=1, \ldots, m$), respectively.

However, in some cases, the input and output data x_{ij} and y_{rj} are not assumed to be exactly obtained because of uncertainty. In IDEA model, it is supposed that some of the input values x_{ij} or output values y_{rj} are not known exactly. It is only known that they lie within intervals $[x_{ij}^L, \ x_{ij}^U]$ and $[y_{rj}^L, \ y_{rj}^U]$, where $x_{ij}^L > 0, y_{rj}^L > 0$. In order to measure the efficiencies of the DMUs with uncertain inputs and outputs data, the following pair of linear programming models are presented for obtaining the lower and upper bounds on efficiency interval of each DMU (Azizi, 2013):

$$
\begin{aligned}
E_{kk}^L &= \max \sum_r u_r y_{rk}^L \\
s.t. \quad & \sum_i v_i x_{ik}^U = 1; \\
& \sum_i v_i x_{ij}^L - \sum_r u_r y_{rj}^U \geqslant 0; \quad j = 1, 2, \ \ldots, \ n; \\
& v_i, u_r \ \geqslant \ 0, \quad i = 1, 2, \ \ldots, \ m; \ r = 1, 2, \ \ldots, \ s.
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
E_{kk}^U &= \max \sum_r u_r y_{rk}^U \\
s.t. \quad & \sum_i v_i x_{ik}^L = 1; \\
& \sum_i v_i x_{ij}^L - \sum_r u_r y_{rj}^U \ \geqslant \ 0; \quad j = 1, 2, \ \ldots, \ n; \\
& v_i, \ u_r \ \geqslant \ 0, \quad i = 1, \ \ldots, \ m; \ r = 1, \ \ldots, \ s.
\end{aligned} \tag{2}
$$

In models (1) and (2), DMU_k is the DMU under evaluation, v_i ($i=1, \ldots, m$) and u_r ($r=1, \ldots, s$) are the weights assigned to the inputs and outputs, respectively. E_{kk}^L and

E_{kk}^U are the lower bound and upper bound of efficiency for DMU_k, respectively. Considering models (1) and (2), it is clear that $E_{kk}^L \leqslant E_{kk}^U$:

Theorem 1. Let E_{kk}^{*L}, E_{kk}^{*U} be the best relative efficiencies of models (1) and (2), respectively. Then $E_{kk}^{*L} \leqslant E_{kk}^{*U}$, besides, equality holds if and only if the original input and output data of DMU_k are reduced from interval value to exact value (Azizi, 2013).

Proof. Let v_i^*, u_r^* be the optimal solution to model (1). We define:

$$\beta_0 = \sum_i v_i^* x_{ik}^L, \quad \tilde{v}_i = \frac{v_i^*}{\beta_0}, \quad \tilde{u}_r = \frac{u_r^*}{\beta_0}.$$

Then:

$$\beta_0 = \sum_i v_i^* x_{ik}^L \leqslant \sum_i v_i^* x_{ik}^U = 1.$$

Thus:

$$0 < \beta_0 \leqslant 1,$$

$$\sum_i \tilde{v}_i x_{ik}^L = \sum_i \frac{v_i^*}{\beta_0} x_{ik}^L = \frac{1}{\beta_0} \sum_i v_i^* x_{ik}^L = \frac{1}{\beta_0} \cdot \beta_0 = 1,$$

$$\sum_i \tilde{v}_i x_{ij}^L - \sum_r \tilde{u}_r y_{rj}^U = \frac{1}{\beta_0}\left(\sum_i v_i^* x_{ik}^L - \sum_r u_r^* y_{rk}^U \right) \geqslant 0,$$

$$\tilde{v}_i = \frac{v_i^*}{\beta_0} \geqslant 0, \quad i = 1, \ldots, m, \quad \tilde{u}_r = \frac{u_r^*}{\beta_0} \geqslant 0, \quad r = 1, \ldots, s.$$

Thus, $\tilde{v}_i, \tilde{u}_r$ are the feasible solution of model (2), and because E_{kk}^{*U} is the optimal solution of model (2), then:

$$\sum_r \tilde{u}_r y_{rk}^U \leqslant E_{kk}^{*U}.$$

Thus:

$$E_{kk}^{*L} = \sum_r u_r^* y_{rk}^L = \sum_r \tilde{u}_r \beta_0 \, y_{rk}^L \leqslant \beta_0 \sum_r \tilde{u}_r y_{rk}^U \leqslant \beta_0 E_{kk}^{*U} \leqslant E_{kk}^{*U}.$$

If the input-output number of DMU_k reduces to the exact value, the models (1) and (2) is the same. Thus the equal sign of the inequality holds. This completes the proof. ■

The models (1) and (2) show that the same situation is selected regardless of the upper bound or lower bound. Namely, all DMUs are in the optimal situation, and the worst and the optimal conditions for the evaluated DMU are served as the objective function of linear programming to determine the bound of the interval efficiency value. When the interval value reduces to the exact number, the bound is equal and the model becomes more logical reasoning.

3. The super efficiency model in DEA with interval data based on cross-evaluation

In this section, we propose the super efficiency IDEA model, the IDEA model based on cross-evaluation and the super efficiency model in DEA with interval data based on cross-evaluation.

3.1 The super efficiency IDEA model

The models (1) and (2) can effectively reduce the efficiency value range of DMUs. However, it can be seen from the result (Azizi, 2013) that the interval efficiency values of efficient DMUs are obviously crowded at 1, and cannot obtain the overall ranking order of DMUs. Thus, in order to rank all DMUs and make the comparability stronger, in this section, the super efficiency IDEA model is presented as follows:

$$\begin{aligned} \theta_{kk}^{L} &= \max \sum_{r} u_r y_{rk}^{L} \\ s.t. \quad & \sum_{i} v_i x_{ik}^{U} = 1; \\ \sum_{i} v_i x_{ij}^{L} - \sum_{r} u_r y_{rj}^{U} &\geqslant 0; \quad j = 1, 2, \ldots, n, j \neq k; \\ v_i, u_r &\geqslant 0, \quad i = 1, 2, \ldots, m; \ r = 1, 2, \ldots, s. \end{aligned} \tag{3}$$

$$\begin{aligned} \theta_{kk}^{U} &= \max \sum_{r} u_r y_{rk}^{U} \\ s.t. \quad & \sum_{i} v_i x_{ik}^{L} = 1; \\ \sum_{i} v_i x_{ij}^{L} - \sum_{r} u_r y_{rj}^{U} &\geqslant 0; \quad j = 1, 2, \ldots, n, \ j \neq k; \\ v_i, u_r &\geqslant 0, \quad i = 1, \ldots, m; \ r = 1, \ldots, s. \end{aligned} \tag{4}$$

By solving the models (3) and (4), the interval super efficiency scores are denoted by $\theta_{kk} = [\theta_{kk}^{L}, \theta_{kk}^{U}]$. To determine if a DMU is DEA efficient, based on the lower bound and upper bound of interval efficiency value for DMUs by solving models (3) and (4), the obtained result is a list of interval numbers and cannot be directly ranked.

In order to analyze the relative efficiency of each DMU, in this paper, for all evaluated units, the interval value $[\theta_{kk}^{L}, \theta_{kk}^{U}]$ $(k = 1, 2, \ldots, n)$ can be used to further discriminate them in three classes of efficiency as follows. Namely, when $\theta_{kk}^{L} \geqslant 1$, the DMU_k is called interval efficient in DEA; when $\theta_{kk}^{U} \geqslant 1$ and $\theta_{kk}^{L} < 1$, the DMU_k is called interval partial efficient in DEA; when $\theta_{kk}^{U} < 1$, the DMU_k is called interval non-efficient in DEA.

Moreover, since interval evaluation values of some DMUs are actually reduced to an exact value when there are parts of DMUs with crisp data. Thus, the optimistic coefficient α based on compromise rule is introduced to represent the optimistic degree of DM. In general, let $\alpha = 0.5$, the super efficiency for any DMU_j $(j = 1, 2, \ldots, n)$ is then calculated as:

$$\theta_{kk} = \alpha \theta_{kk}^{L} + (1 - \alpha)\, \theta_{kk}^{U}. \tag{5}$$

Then the overall ranking of all DMUs with imprecise data based on super efficiency IDEA is obtained.

3.2 The IDEA model based on cross-evaluation

Based on the above discussions, the IDEA model can be a good evaluation tool for dealing with imprecise data. However, in order to maximize its own efficiency ratio, the DMU under evaluation heavily weighs the inputs and outputs of the favorable DMUs and ignores those of the others, which may not provide persuasive solutions. In this section, an IDEA method based on cross-efficiency evaluation is introduced to overcome this shortcoming.

First, the cross-efficiency method in DEA uses peer evaluation instead of self-evaluation.

Using the set of favorable weights $\left(u_{1k}^{*}, \ \ldots, \ u_{sk}^{*}, v_{1k}^{*}, \ \ldots, \ v_{mk}^{*}, E_{kk}^{*}\right)$ for DMU_k, the efficiency for any DMU_j $(j = 1, 2, \ldots, n)$ is then computed as follows:

$$E_{kj} = \frac{\sum_{r=1}^{s} u_{rk}^{*} y_{rj}}{\sum_{i=1}^{m} v_{ik}^{*} x_{ij}}, \quad k, j = 1, 2, \ \ldots, \ n. \tag{6}$$

Here E_{kj} denotes the relative efficiency of DMU_j with the optimal weights for inputs and outputs of DMU_k.

Then the cross-efficiency value of DMU_j $(j = 1, 2, \ldots, n)$ is defined as the average of the relative efficiencies in all DMUs:

$$\overline{E}_j = \frac{1}{n} \sum_{k=1}^{n} E_{kj}, \quad j, k = 1, 2, \ \ldots, \ n, \tag{7}$$

which measures the average efficiency based on all DMUs' preferable weights and shows that how the DMU_j associated with the column are rated by the rest of the DMU_k.

It can be noted that optimal weights solved by the models (1) and (2) are generally not unique. As a result, the efficiency value E_{kj} defined in (6) is generated arbitrarily. This is the shortcoming of cross-efficiency measure and can confuse the DMs. To eliminate this limitation, Doyle and Green (1994) proposed the well-known aggressive and benevolent models to determine the optimal weights. The principle is that it selects a set of weights which not only minimize or maximize the efficiency of a specific DMU under evaluation but also minimize or maximize the efficiencies of all other DMU_j in some sense. In this paper, we merely establish an aggressive model based on interval data.

Consider two DMUs DMU_k and DMU_j, under the constraint that the self-evaluation interval value $[E_{kk}^{L}, E_{kk}^{U}]$ of DMU_k keeps unchanged, the aggressive efficiency value of DMU_j with respect to DMU_k is computed by models (8) and (9):

$$\begin{aligned}
E_{kj}^{L} &= \min \textstyle\sum_r u_r y_{rj}^{L} \\
s.t. \quad & \textstyle\sum_r u_r y_{rj}^{U} \leqslant \sum_i v_i x_{ij}^{L}; \quad 1 \leqslant j \leqslant n; \\
& \textstyle\sum_i v_i E_{kk}^{L} x_{ik} \leqslant \sum_r u_r y_{rk} \leqslant \sum_i v_i E_{kk}^{U} x_{ik}; \\
& \textstyle\sum_i v_i x_{ij}^{U} = 1; \\
& v_i \geqslant 0, \ u_r \geqslant 0, \quad i = 1, 2, \ \ldots, \ m, \quad r = 1, 2, \ \ldots, \ s;
\end{aligned} \tag{8}$$

$$
\begin{aligned}
E_{kj}^U &= \min \textstyle\sum_r u_r y_{rj}^U \\
s.t.\quad & \textstyle\sum_r u_r y_{rj}^U \leqslant \sum_i v_i x_{ij}^L; \quad 1 \leqslant j \leqslant n; \\
& \textstyle\sum_i v_i E_{kk}^L x_{ik} \leqslant \sum_r u_r y_{rk} \leqslant \sum_i v_i E_{kk}^U x_{ik}; \\
& \textstyle\sum_i v_i x_{ij}^L = 1; \\
& v_i \geqslant 0, \quad u_r \geqslant 0, \qquad i = 1, 2, \ldots, m, \quad r = 1, 2, \ldots, s;
\end{aligned} \tag{9}
$$

here E_{kk}^L, E_{kk}^U denote the lower bound and upper bound of efficiency value for DMU_k, respectively. Assume that the relative efficiency values are uniformly distributed over $[E_{kj}^L, E_{kj}^U]$, and E_{kj}^L, E_{kj}^U denote the lower bound and upper bound of evaluation of DMU_j with respect to DMU_k.Then, for DMU_j ($j = 1, 2, \ldots, n$), the efficiency rated by DMU_k lies in $[E_{kj}^L, E_{kj}^U]$. Because the self-evaluation value E_{kk} is an interval number, and the upper bound and lower bound are obtained on the condition that other DMUs are in the most favorable situation and the evaluated DMU is in the most favorable and unfavorable situation. Therefore, the efficiency value of the evaluated DMU at the arbitrary point in the interval of input and output is still in the interval range of E_{kk}. According to the discussions in the above sections, if a possible weight set satisfies the conditions of models (8) and (9), the cross-efficiency obtained by this weight set should be feasible. Due to a linear constraint space in models (8) and (9) and the convex objective function, the cross-efficiency for DMU_j to be computed is an interval number, sometimes maybe reduce to a real number.

In the models (8) and (9), it is shown that the proposed models do not need extra variable changes and use a fixed, unified production frontier for computation of the efficiency intervals of the DMUs with interval input and output data DMU_j seeks to minimize its efficiency values, respectively under the same condition of maintaining the self-evaluation efficiency interval value of DMU_k unchanged. And the same situations are also selected without upper or lower limitation. Namely, all DMUs are in the optimal situation, and the worst and optimal conditions for the evaluated DMU are served as the objective function of linear programming to determine the bound of the interval efficiency. When the interval value reduces to a certainty number, the lower bound is equal to the upper bound.

By solving the models (8) and (9), all DMUs' interval cross-efficiency values is obtained, as shown in Table I.

Table I is an interval cross-efficiency matrix for n DMUs. Unlike the traditional one, all the values in the matrix are not real but interval numbers. It can be seen that the elements in the diagonal are the special cases that $[E_{kj}^L, E_{kj}^U] = [E_{kk}^L, E_{kk}^U]$ for any $k = j = 1, 2, \ldots, n$.

Table I.
An interval cross-efficiency matrix for n DMUs

	DMU_j				
DMU_k	1	2	3	[…]	n
1	$[E_{11}^L, E_{11}^U]$	$[E_{12}^L, E_{12}^U]$	$[E_{13}^L, E_{13}^U]$	[…]	$[E_{1n}^L, E_{1n}^U]$
2	$[E_{21}^L, E_{21}^U]$	$[E_{22}^L, E_{22}^U]$	$[E_{23}^L, E_{23}^U]$	[…]	$[E_{24}^L, E_{24}^U]$
3	$[E_{31}^L, E_{31}^U]$	$[E_{32}^L, E_{32}^U]$	$[E_{33}^L, E_{33}^U]$	[…]	$[E_{34}^L, E_{34}^U]$
[…]	[…]	[…]	[…]	[…]	[…]
n	$[E_{n1}^L, E_{n1}^U]$	$[E_{n2}^L, E_{n2}^U]$	$[E_{n3}^L, E_{n3}^U]$	[…]	$[E_{nn}^L, E_{nn}^U]$

Then, in similar way, the optimistic coefficient α based on compromise rule is also introduced to represent the optimism degree of DM. Using these interval values matrixes, the cross-efficiency for any DMU_j ($j=1, 2, \ldots, n$) is then calculated as:

$$E_{kj} = \alpha E_{kj}^L + (1-\alpha)E_{kj}^U. \tag{10}$$

3.3 The super efficiency model in DEA with interval data based on cross-evaluation

Nevertheless, as the above discussions, although cross-efficiency evaluation with interval data takes into account of the comprehensive efficiency of input-output for all DMUs, it has some following limitations. First, there may exist the phenomenon $E_{kj}=[1, 1]=1$ for some DMUs and the DMs cannot judge the preference relation among the DMUs. Second, the cross-evaluation overweighs the advantage of the evaluated DMU and omits its disadvantage.

To handle the above mentioned problems, this paper develops an approach by combining the interval super efficiency model proposed in Section 3.1 and interval cross-evaluation model proposed in Section 3.2, which eliminates the phenomenon of $E_{kj}=[1, 1]=1$ for the efficient DMU. Thus, the evaluation value of efficient DMU is greater than 1 and all the non-diagonal elements of cross-efficiency matrix are different interval value for different order of DMU, which greatly reduces the situation $E_{kj}=[1, 1]=1$ for efficient DMUs in the established cross-efficiency matrix.

Then, the models (8) and (9) are transformed into the following models:

$$\begin{aligned} E_{kj}^L &= \min \textstyle\sum_r u_r y_{rj}^L \\ s.t. \quad & \textstyle\sum_r u_r y_{rj}^U \leqslant \sum_i v_i x_{ij}^L; \quad j \neq k,\ 1 \leqslant j \leqslant n; \\ & \textstyle\sum_i v_i \theta_{kk}^L x_{ik} \leqslant \sum_r u_r y_{rk} \leqslant \sum_i v_i \theta_{kk}^U x_{ik}; \\ & \textstyle\sum_i v_i x_{ij}^U = 1; \\ & v_i \geqslant 0,\ u_r \geqslant 0, \quad i = 1, 2, \ldots, m, \quad r = 1, 2, \ldots, s; \end{aligned} \tag{11}$$

$$\begin{aligned} E_{kj}^U &= \min \textstyle\sum_r u_r y_{rj}^U \\ s.t. \quad & \textstyle\sum_r u_r y_{rj}^U \leqslant \sum_i v_i x_{ij}^L; \quad j \neq k,\ 1 \leqslant j \leqslant n; \\ & \textstyle\sum_i v_i \theta_{kk}^L x_{ik} \leqslant \sum_r u_r y_{rk} \leqslant \sum_i v_i \theta_{kk}^U x_{ik}; \\ & \textstyle\sum_i v_i x_{ij}^L = 1; \\ & v_i \geqslant 0,\ u_r \geqslant 0, \quad i = 1, 2, \ldots, m, \quad r = 1, 2, \ldots, s; \end{aligned} \tag{12}$$

In a word, the steps of the proposed approach are summarized as follows.

Step 1: the super efficiency interval value $[\theta_{kk}^L, \theta_{kk}^U]$ of self-evaluation θ_{kk} is obtained by models (3) and (4), and the super efficiency value θ_{kk} is calculated by formula (5).

Step 2: the interval cross-efficiency value $[E_{kj}^L, E_{kj}^U]$ of each DMU is solved by models (11) and (12).

Step 3: the interval cross-evaluation matrix E is obtained by cross-evaluation interval value.

Step 4: the cross-evaluation value E_{kj} for DMU_j ($j=1, 2, \ldots, n$) is computed by formula (10).

Step 5: the ultimate cross-efficiency score $\overline{E}_j$ of DMU_j ($j=1, 2, \ldots, n$) is defined as the average of these efficiencies E_{kj} by formula (7). The bigger the $\overline{E}_j$, and the better the DMU_j.

4. An numerical example

In this section, we apply our proposed models to a data set related to 29 public secondary schools in Greece. The data set is taken from the paper by Smirlis *et al.* (2006). The DEA inputs are budget, facilities index, and level of education, and the DEA outputs are admission, average mark, and excellent students. Due to the deficits in students' records and lack of suitable bookkeeping facilities, some schools are unable to provide the required data for some inputs and outputs exactly. The data information is shown in Table II. In this paper, all models are solved by Lingo 11.0 software.

By solving the interval super efficiency DEA models (3) and (4) for the data set in Table II, the interval relative efficiencies of each DMU are calculated and shown in Table III. And the super efficiency values of all DMUs are obtained by formula (5) and also shown in Table III. Meantime, the values obtained from IDEA models (1) and (2) (Azizi, 2013) are shown in the last column of Table III. Thus, the values obtained from models (3) and (4) can be compared with the values obtained from models (1) and (2). As it can be seen in Table III, when we use a definite, fixed production frontier for

Schools (DMUs)	x_{1j}	x_{2j}	x_{3j}	y_{1j}	y_{2j}	y_{3j}
1	23,940	6	52.17	19	14.7	10
2	25,450	5	76.43	38	14.7	14
3	24,000	4	43.00	34	15.0	4
4	26,500	7	[43, 43.7]	29	14.3	4
5	31,200	6	43.7	48	14.0	11
6	32,600	5	76.43	36	[14, 16]	17
7	[31,586; 42,124]	5	52.21	73	[14, 16]	18
8	35,600	5	93.67	40	15.7	22
9	19,160	4	96.17	33	15.1	38
10	42,800	4	43.8	62	[16, 18]	13
11	42,840	7	82.43	78	14.5	27
12	41,000	4	75.17	62	13.6	27
13	45,980	7	81.96	70	15.2	28
14	51,000	7	76.43	59	15.5	15
15	52,200	2	43.20	76	15.7	25
16	56,000	7	54.71	56	13.2	26
17	56,700	7	75.17	59	13.3	33
18	58,140	4	37.79	78	14.8	34
19	[52,031; 62,569]	4	59.40	96	[16, 18]	18
20	60,100	7	78.67	95	15.5	35
21	60,040	7	47.56	83	14.5	23
22	63,450	7	58.86	76	14.2	49
23	61,110	7	56.24	98	11.0	33
24	61,820	7	68.12	85	14.4	33
25	65,000	5	58.86	47	13.4	36
26	64,050	7	76.22	68	14.9	46
27	74,600	6	39.00	111	15.3	39
28	76,650	5	56.24	124	13.2	48
29	82,470	4	[68.12, 81.96]	100	13.7	37

Table II. The original data of 29 public secondary schools in Greece

Schools (DMUs)	Super efficiency IDEA models (3) and (4) θ_{kk}^L	θ_{kk}^U	θ_{kk} ($\alpha = 0.5$)	Ranking	IDEA models (1) and (2) E_{kk}^L	E_{kk}^U
1	1.0576	1.0706	1.0641	12	1.0000	1.000
2	1.0513	1.1397	1.0955	10	1.0000	1.000
3	1.1858	1.2101	1.19795	4	1.0000	1.000
4	0.9194	0.946	0.9327	18	1.0705	1.0768
5	0.9174	1.0064	0.9619	16	1.0084	1.0084
6	0.8411	0.9319	0.8865	23	1.0117	1.1428
7	1.1688	1.1792	1.1740	6	1.0000	1.0000
8	0.9069	0.9241	0.9155	19	1.0257	1.0257
9	1.3731	1.3731	1.3731	1	1.0000	1.0000
10	1.0355	1.1927	1.1141	9	1.0000	1.0096
11	0.8969	0.9243	0.9106	20	1.0171	1.0171
12	1.0494	1.0611	1.05525	13	1.0000	1.0000
13	0.8626	0.9502	0.9064	22	1.0438	1.0438
14	0.6255	0.7242	0.67485	29	1.1613	1.1613
15	0.8966	0.9168	0.9067	21	1.0307	1.0307
16	0.7479	0.7751	0.7615	28	1.2241	1.2241
17	0.7622	0.7753	0.76875	27	1.1691	1.1691
18	1.1779	1.2116	1.19475	5	1.0000	1.0000
19	1.1135	1.1377	1.1256	8	1.0000	1.0000
20	1.0389	1.0556	1.04725	14	1.0000	1.0000
21	0.8438	0.8998	0.8718	24	1.1554	1.1554
22	1.1608	1.1608	1.1608	7	1.0000	1.0000
23	0.9094	0.9665	0.93795	17	1.0858	1.0858
24	0.8104	0.8779	0.84415	25	1.0889	1.0889
25	0.8379	0.8379	0.8379	26	1.1047	1.1047
26	1.0276	1.0443	1.03595	15	1.0000	1.0000
27	1.3245	1.3245	1.3245	2	1.0000	1.0000
28	1.2056	1.2641	1.23485	3	1.0000	1.0000
29	1.0756	1.0844	1.08	11	1.0000	1.0000

Table III. Efficiency intervals for the 29 schools

interval super efficiency models (3) and (4) of these 29 DMUs, only 15 DMUs are recognized as DEA efficient. They together determine an efficiency frontier. The ranking of all DMUs is shown in the fifth columns of Table III. From Table III, we can note that the interval values of efficient DMUs are all greater than 1, the result of models (3) and (4) is more reasonable than that of models (1) and (2). The orders obtained by the proposed method can really represent the true orders of the alternatives.

Then, we use models (11) and (12) to derive the schools' interval cross-efficiency scores based on the results of models (3) and (4). These cross-evaluation values forms the interval cross-evaluation fuzzy matrix E, and the elements on the main diagonal are the self-evaluation results when DMU_k evaluates itself according to models (3) and (4). There are only the first ten rows and ten columns of the interval cross-efficiency matrix shown in Table IV due to the space constraints.

The final cross-efficiency scores of schools are calculated by using formulas (10) and (7) and shown in Table V. And then, the derived overall ranking are shown in the last column of Table V. It can be seen from Table V that the ranking is the same as the order obtained by super efficiency IDEA models (3) and (4), which directly suggests that the ranking obtained by the proposed models (11) and (12) in this paper can represent the true orders of the alternatives.

Table IV. The partial interval cross-efficiency matrix by models (11) and (12)

Schools (DMUs)	1	2	3	4	5	6	7	8	9	10
1	[1.0576, 1.0706]	[1.0949, 1.0952]	[1.1836, 1.1941]	[0.9149, 0.9279]	[0.8779, 0.8874]	[0.8001, 0.8212]	[1.1711, 1.1809]	[0.9239, 0.9313]	[1.3224, 1.3358]	[1.1144, 1.1252]
2	[1.0423, 1.0681]	[1.0513, 1.1397]	[1.1915, 1.1977]	[0.9015, 0.9081]	[0.8668, 0.8698]	[0.9072, 0.9117]	[1.1645, 1.1721]	[0.8882, 0.9189]	[1.3316, 1.3427]	[1.1036, 1.1129]
3	[1.0572, 1.0599]	[1.0939, 1.0996]	[1.1858, 1.2101]	[0.9136, 0.9293]	[0.8773, 0.9011]	[0.9493, 0.9592]	[1.1778, 1.1809]	[0.9072, 0.9491]	[1.3305, 1.3442]	[1.1262, 1.1933]
4	[1.0596, 1.0611]	[1.0736, 1.0988]	[1.1722, 1.1938]	[0.9194, 0.9460]	[0.8427, 0.8989]	[0.8313, 0.8467]	[1.1636, 1.1757]	[0.9162, 0.9212]	[1.3259, 1.3301]	[1.0771, 1.1812]
5	[1.0498, 1.0627]	[1.0835, 1.1011]	[1.1831, 1.1999]	[0.9198, 0.9289]	[0.9174, 1.0064]	[0.8162, 0.8196]	[1.1842, 1.1874]	[0.9414, 0.9488]	[1.3165, 1.3267]	[1.1224, 1.1779]
6	[1.0565, 1.0616]	[1.0767, 1.0955]	[1.1767, 1.1955]	[0.9162, 0.9275]	[0.8779, 0.9114]	[0.8411, 0.9319]	[1.1713, 1.1779]	[0.9493, 0.9524]	[1.3374, 1.3480]	[1.0509, 1.1661]
7	[1.0529, 1.0588]	[1.0819, 1.1032]	[1.1912, 1.1964]	[0.9031, 0.9109]	[0.9563, 0.9757]	[0.8328, 0.8532]	[1.1688, 1.1792]	[0.8613, 0.8787]	[1.3201, 1.3344]	[1.1012, 1.1171]
8	[1.0547, 1.0662]	[1.0734, 1.0875]	[1.1628, 1.1818]	[0.8989, 0.9088]	[0.9312, 0.9464]	[0.8144, 0.8239]	[1.1663, 1.1718]	[0.9069, 0.9241]	[1.3424, 1.3512]	[1.0931, 1.1291]
9	[1.0432, 1.0554]	[1.0898, 1.1076]	[1.1724, 1.1939]	[0.8814, 0.9103]	[0.8911, 0.9004]	[0.8735, 0.8996]	[1.1705, 1.1793]	[0.8326, 0.8556]	[1.3731, 1.3731]	[1.1728, 1.2018]
10	[1.0517, 1.0623]	[1.0691, 1.0939]	[1.1812, 1.1875]	[0.9012, 0.9151]	[0.9679, 0.9731]	[0.8979, 0.9121]	[1.1696, 1.1718]	[0.8912, 0.9211]	[1.3356, 1.3462]	[1.0355, 1.1927]

Schools (DMUs)	Cross-efficiency score	Ranking
1	1.0582	12
2	1.0948	10
3	1.1914	4
4	0.9276	18
5	0.9565	16
6	0.8814	23
7	1.1709	6
8	0.9138	19
9	1.3319	1
10	1.1074	9
11	0.9048	20
12	1.0515	13
13	0.8974	22
14	0.6744	29
15	0.8999	21
16	0.7546	28
17	0.7631	27
18	1.1889	5
19	1.1224	8
20	1.0444	14
21	0.8627	24
22	1.1599	7
23	0.9323	17
24	0.8392	25
25	0.8217	26
26	1.0332	15
27	1.3101	2
28	1.2611	3
29	1.0753	11

Table V. Results from models (11) and (12) based on interval super efficiency

Based on the above discussions, we can see that the proposed approach can present a complete rank. We also note that the rankings can represent the true orders of the alternatives. The method has two main advantages. First, using the combination of cross-evaluation IDEA with super efficiency IDEA, each DMU is rated not only under its own evaluation but also under the evaluation of the others. This incorporation creates a unique ordering among the DMU in practice. Second, cross-evaluation can eliminate unrealistic weighting schemes that might be used by the DMUs.

5. Conclusions

In some practical situations, the outputs and inputs of DMUs are not known exactly due to the uncertainty and complexity. However, the existing classical IDEA methods can only classify them as efficient or inefficient, or cannot reflect the true order of all DMUs. In view of these drawbacks, in this paper, we provide an approach to rank each DMU with interval data. The proposed model utilizes the combination of super efficiency and cross-evaluation to evaluate DMUs with interval data. In addition, this approach can avoid the fact that the original DEA model can only distinguish the performance of efficient DMUs from inefficient ones, but cannot discriminate between the efficient DMUs. Finally, in order to prove the effectiveness of the proposed approach, a numerical example is illustrated. And we also make a comparison of the results using super efficiency IDEA model with that of the cross evaluation-based measure of super efficiency in DEA with interval data, during which we observe that the method is efficient to solve multiple attribute decision making problems with interval data.

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