Antenna Array Design as Inference

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Abstract. This paper describes the use of the Bayesian inference framework for the design of linear antenna arrays. The principal advantage of using the Bayesian inference framework for array design is that it makes possible the automatic determination of the number of radiators required to meet given design requirements. The inference framework achieves this by making accessible to the array designer powerful computational tools developed for the simultaneous solution of parameter estimation and model selection problems.

Keywords: Model Based Design, Design Optimization, Unequally Spaced Arrays

INTRODUCTION

This paper describes our initial work to develop the use of the Bayesian inference framework for design. The principal advantage of using the Bayesian inference framework for design is that it makes possible the automatic determination of the design complexity required to meet given design requirements.

Our efforts so far have been applied to the problem of designing antenna arrays to achieve a designated far-field radiation pattern that is bounded by upper and lower limits at each of a set of given radiation angles. This much-studied design problem is traditionally treated as an optimization problem where the error between the desired and achieved pattern is to be minimized. The drawback of the optimization approach to design is that it only works for systems with predetermined complexity. For antenna arrays, this means that the number of radiating elements must be chosen in advance of the optimization. Since an array with a large number of elements can always achieve the required error, designers typically specify arrays with more elements than required. This problem can be overcome by using an inference framework rather than an optimization framework for design.

In the inference framework, the error between the desired and achieved pattern is assigned a probability density function with width determined from design requirements. This error *pdf* defines the likelihood of a candidate array design. Similarly, the parameters of the individual array elements (e.g. current amplitude and phase, position in space) are assigned *pdfs*. The assigned *pdfs* for the parameters must reflect constraints on the parameters but can also be used to indicate the designer's preferences. A broad *pdf* for the number of array elements is also assigned. In the inference approach to design, a posterior *pdf* is defined as proportional to the product of the likelihood, the parameter *pdfs*, and the *pdf* for the number of elements. Using sampling techniques, variate sets are drawn from this posterior distribution. Each sample is a design for an array. Using

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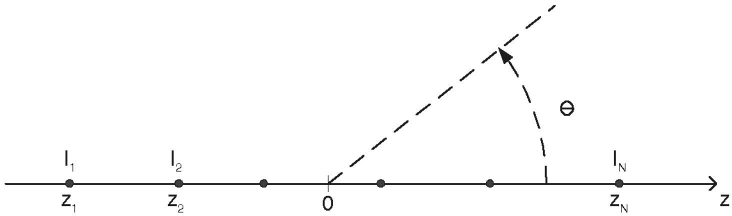

FIGURE 1. Linear array of N isotropic point radiators.

a reasonable number of these variate sets, the distribution for the number of array elements can be estimated. The number of array elements that maximizes the distribution is then chosen for the design and, of all the variate sets with the same chosen number of elements, the variate set with the minimum error is selected as the final design. The quantitative Ockham's razor implicit in the Bayesian inference approach guarantees that the number of array elements chosen is the correct compromise between design complexity and design performance for the design requirements imposed. Initial numerical results for an array design problem taken from the literature support the design as inference approach.

LINEAR ANTENNA ARRAYS

Figure 1 illustrates a linear antenna array of N isotropic radiators positioned on the z-axis. For this array, the far-field power-pattern expressed in dB

$$U(\theta) = 10\log|AF(\theta)|^2 \text{ for } 0 \leq \theta \leq \pi \tag{1}$$

is determined using the array factor

$$AF(\theta) = \sum_{n=1}^{N} I_n \exp[j2\pi(z_n/\lambda)\cos\theta] \tag{2}$$

where $j = \sqrt{-1}$, and λ is the free space wavelength at the time-harmonic operating frequency [1]. In Figure 1 and the expression above, z_n and I_n are the position and complex driving current of the n^{th} antenna element. The angle θ indicates the direction to the far-field observation point.

Linear arrays are commonly used to obtain a broadside power pattern. For a broadside pattern, $U(\theta)$ is maximum at $\theta = \pi/2$. A broadside pattern can be obtained using a linear array with N pairs of radiators positioned so that $z_{-n} = -z_n$ for $1 \leq n \leq N$ and with currents $I_{-n} = I_n = a_n$ where a_n is a real valued constant. For this broadside array, the array factor can be expressed as

$$AF(\theta) = 2\sum_{n=1}^{N} a_n \cos[2\pi(z_n/\lambda)\cos\theta]. \tag{3}$$

Here we will consider only the design of broadside arrays with N pairs of radiators.

ARRAY DESIGN

In the antenna array design problem, we seek to find the values of the array parameters $\{N, \mathbf{X}_N\} = \{N, z_1, a_1, z_2, a_2, \cdots, z_N, a_N\}$ so that the power pattern conforms, in some sense, to a desired power pattern defined at a finite number of angles θ_m where $0 \leq \theta_m \leq \pi$ for $1 \leq m \leq M$.

There are a number of methods that could be used to specify the desired power pattern and the desired degree of compliance. For example, the upper and lower envelopes of the desired power pattern could be given for every θ_m. Here, these envelopes are denoted by $U_U(\theta_m)$ and $U_L(\theta_m)$. Typically, a normalized pattern is specified; hence, the desired upper envelope of the power pattern should have a peak value of 0 dB. Values of θ_m, where the designer is indifferent to the minimum value of the power pattern, can be accommodated by setting $U_L(\theta_m)$ equal to a large negative number. It may be necessary to set $U_L = U_U$ = 0 dB at the pattern peak to achieve a normalized pattern.

ASSIGNING THE LIKELIHOOD

In inference problems, the likelihood is determined from the sampling distribution for the data. In turn, the sampling distribution is found from the assigned error distribution by writing data equal to model plus error. In the following, we define the data and assign the error distribution so that the array design problem is isomorphic to the inference problem where

$$d_m = g(N, \mathbf{X}_N, \theta_m) + e_m \text{ for } 1 \leq m \leq M. \tag{4}$$

The model for the power pattern $g(N, \mathbf{X}_N, \theta)$ is determined using the array factor for the pattern type of interest. For a broadside array

$$g(N, \mathbf{X}_N, \theta) = 10 \log \left[2 \sum_{n=1}^{N} a_n \cos\left[2\pi (z_n/\lambda) \cos\theta\right] \right]^2 . \tag{5}$$

In the design problem, there is no observed data; however, for our purposes we can define the "data" at θ_m as the average of the upper and lower envelopes so that

$$d_m \triangleq [U_{max}(\theta_m) + U_{min}(\theta_m)]/2. \tag{6}$$

The error distribution used here is given by

$$p(e_m|\sigma_m, \Delta_m) = \begin{cases} \frac{1}{2(\sigma_m+\Delta_m)} \exp\left[-\frac{|e_m|-\Delta_m}{\sigma_m}\right] & \text{for } |e_m| > \Delta_m \\ \frac{1}{2(\sigma_m+\Delta_m)} & \text{otherwise} \end{cases} \tag{7}$$

where

$$\Delta_m = [U_{max}(\theta_m) - U_{min}(\theta_m)]/2. \tag{8}$$

Here, we assume that all probability distributions are conditioned on the prior information, and to simplify the notation we suppress the conditioning on the prior information.

In the context of antenna design, the value of σ_m is assigned to indicate the desired degree of compliance between the achieved power pattern and the desired power pattern at θ_m.

Using $e_m = d_m - g(N, \mathbf{X}_N, \theta_m)$ in $p(e_m|\sigma_m, \Delta_m)$ yields the sampling distribution:

$$p(d_m|\sigma_m,\Delta_m,N,\mathbf{X}_N) =$$

$$\begin{cases} \frac{\exp[-\{|d_m - g(N,\mathbf{X}_N,\theta_m)| - \Delta_m\}/\sigma_m]}{2(\sigma_m+\Delta_m)} & \text{for } |d_m - g(N,\mathbf{X}_N,\theta_m)| > \Delta_m \\ \frac{1}{2(\sigma_m+\Delta_m)} & \text{otherwise} \end{cases} . \quad (9)$$

Noting that $L(N,\mathbf{X}_N) \propto p(\mathbf{D}|\boldsymbol{\sigma},\boldsymbol{\Delta},N,\mathbf{X}_N)$, and treating the errors as independent yields the following expression for the likelihood:

$$L(N,\mathbf{X}_N) = \exp\left[-\sum_{m=1}^{M} Q_m/\sigma_m\right] \quad (10)$$

where

$$Q_m = \begin{cases} |d_m - g(N,\mathbf{X}_N,\theta_m)| - \Delta_m & \text{for } |d_m - g(N,\mathbf{X}_N,\theta_m)| > \Delta_m \\ 0 & \text{otherwise} \end{cases} . \quad (11)$$

Rewriting the expression above as

$$Q_m = \begin{cases} U_{min} - g(N,\mathbf{X}_N,\theta_m) & \text{for } g(N,\mathbf{X}_N,\theta_m) < U_{min}(\theta_m) \\ g(N,\mathbf{X}_N,\theta_m) - U_{max} & \text{for } g(N,\mathbf{X}_N,\theta_m) > U_{max}(\theta_m) \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

reveals that Q_m is the distance by which the model exceeds the desired power pattern range at θ_m.

It is interesting to note the relationship between the likelihood defined by (10) and the "fitness function" used in the design as optimization framework. The fitness function is given by the expression

$$F(\mathbf{X}_N) = \sum_{m=1}^{M} Q_m. \quad (13)$$

In the optimization framework for design, the value of N is chosen by the designer and then a value of $\mathbf{X}_N$ for which $F(\mathbf{X}_N)$ is less than a predetermined threshold is sought. Comparing (10) and (13), it is clear that the fitness function is equal to minus the log likelihood for the case where $\sigma_m = 1$ for all m. This relationship is due to construction.

THE POSTERIOR DISTRIBUTION

Because knowledge of $\mathbf{D}$ and $\boldsymbol{\Delta}$ is the same as knowledge of $\mathbf{B}_{max}$ and $\mathbf{B}_{min}$, the posterior distribution can be denoted as $p(N,\mathbf{X}_N|\mathbf{B}_{max},\mathbf{B}_{min},\boldsymbol{\sigma})$. From Bayes' rule,

$$p(N,\mathbf{X}_N|\mathbf{B}_{max},\mathbf{B}_{min},\boldsymbol{\sigma}) \propto p(N,\mathbf{X}_N)L(N,\mathbf{X}_N). \quad (14)$$

Here, the prior for the array parameters is expressed as

$$p(N,\mathbf{X}_N) = p(N)p(a_1)p(a_2)\dots p(a_N)p(z_1)p(z_2)\dots p(z_N) \tag{15}$$

where the priors for the amplitudes $p(a_1), p(a_2), \dots, p(a_N)$ are assigned identical distributions and likewise the priors for the positions $p(z_1), p(z_2), \dots, p(z_N)$ are assigned identical distributions.

To test the design as inference approach, the program BayeSys [2] was used to draw K samples from the posterior distribution for the array parameters. These samples are $\{N^k, \mathbf{X}_{N^k}\}$ for $1 \leq k \leq K$. Each of the K samples drawn using BayeSys is a design for the array. These samples form a Monte Carlo approximation to the posterior distribution so that

$$p(N,\mathbf{X}_N|\mathbf{B}_{max},\mathbf{B}_{min},\boldsymbol{\sigma}) \approx \frac{1}{K}\sum_{k=1}^{K} \varepsilon(N-N^k)\delta(\mathbf{X}_N - \mathbf{X}^k_{N^k}) \tag{16}$$

where δ is the Dirac delta function and

$$\varepsilon(N-N^k) = \begin{cases} 1 & \text{for } \; N-N^k = 0 \\ 0 & \text{otherwise} \end{cases} . \tag{17}$$

Equation (16) can be used to approximate the expected values of functions under the posterior distribution. Using this property yields the following:

$$\langle N \rangle \approx \frac{1}{K}\sum_{k=1}^{K} N^k \tag{18}$$

and

$$p(N|\mathbf{B}_{max},\mathbf{B}_{min},\boldsymbol{\sigma}) \approx \frac{1}{K}\sum_{k=1}^{K} \varepsilon(N-N^k). \tag{19}$$

EXAMPLE RESULTS

An array design problem considered in [3, 4, 5, 6, 7] is used to illustrate array design as inference. This design is for a broadside array based on pairs of radiators. Because the array factor in this case is symmetric about $\theta = 90^\circ$, we need to consider only the range of $0 \leq \theta_m \leq 90^\circ$. For this design, we use 101 equally spaced values of θ_m.

The array power pattern to be synthesized is required to possess deep nulls in particular directions to eliminate interference coming from these directions. Design requirements for the array are as follows: The normalized power pattern is to have a main beam at 90° with a -3 dB beam width of 7.4°, a 20° beam width at the -40 dB side-lobe level, and -55 dB nulls from 30° to 40° from the center of the main beam. The total length of the array must be less than 9.5λ.

To attain these requirements, we made the following assignments:

$$p(N) = \text{Uniform}(1,20),$$

$$p(z_n/\lambda) = \text{Uniform}(0.25,4.75),$$

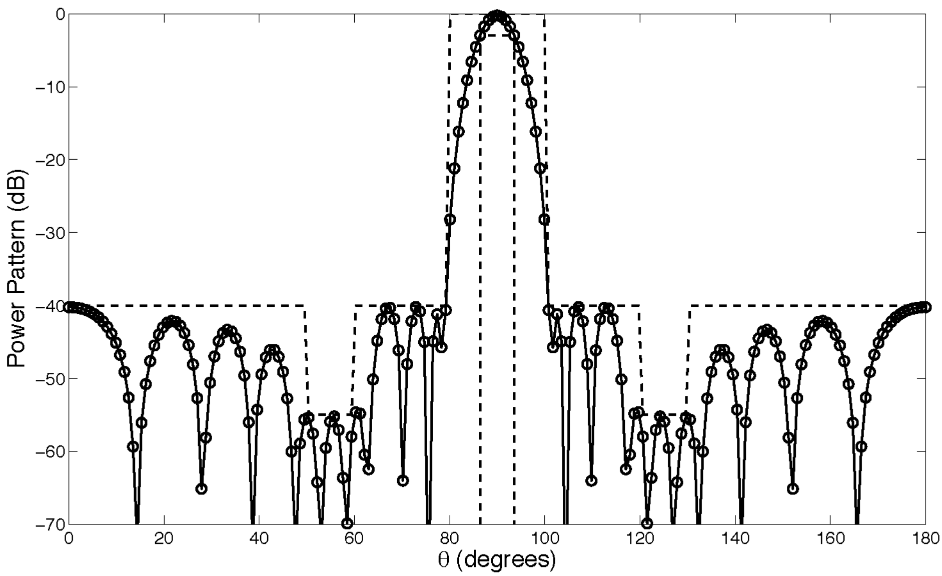

FIGURE 2. Power pattern of a broadside array designed to have a -55 dB null 30 to 40 degrees from the main beam. U_{max} and U_{min} are illustrated with dashed lines.

$$p(a_n) = \text{Uniform}(-0.5, 0.5),$$

and

$$\sigma_m = \begin{cases} 1/50 \text{ dB} & \text{for } 0^\circ < \theta_m < 86.3^\circ \\ 1/250 \text{ dB} & \text{for } 86.3^\circ < \theta_m < 90^\circ \end{cases}.$$

Figure 2 illustrates the upper and lower power pattern envelopes used to enforce the power pattern shape requirements. Also illustrated in Figure 2 is the power pattern obtained using the design as inference framework. Table 1 demonstrates that $N = 8$ is the most probable value of the number of radiator pairs needed to satisfy the design requirements. Hence, the array design chosen from the samples generated using BayeSys is the sample with the highest likelihood among samples with $N = 8$.

CONCLUSION

The design as inference framework shows great promise in the area of linear array design. Unlike the optimization framework for design that is commonly employed, the

TABLE 1. Estimated posterior probability for the number of radiator pairs for the broadside array with a pattern null. For this case, $\langle N \rangle = 8.11$.

N	8	9	10	11
$p(N \mid \mathbf{B}_{max}, \mathbf{B}_{min}, \boldsymbol{\sigma})$	0.904082	0.085714	0.009439	0.000765

inference framework is able to determine the number of radiators required to achieve the design requirements. The ability to automatically determine the design complexity based on the design requirements is the primary advantage of using the inference framework for design, and we look forward to using the inference framework for design to our advantage in other design application areas.

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