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# Signal-to-noise-ratio maximisation for linear multi-antenna relay communications

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**Abstract:** The authors consider a linear relaying communication system using multiple antennas at a source, relay and destination. They maximise the end-to-end signal-to-noise-ratio (SNR) to find the relay transformation (RT) and the source transmit covariance (STC) matrices. For any given STC matrix, to maximise the SNR, they prove that the relay has to assign all its power in the direction of the dominant eigenmodes of the source–relay and the relay–destination channels. They also find the joint optimal rank-one matrices for the STC and RT. They prove that this solution also maximises the source–destination mutual information among all rank-one matrices. Furthermore, they find the optimal power budgets allocated to the source and relay that maximise the SNR under a constraint on the total transmit power of the system. This is a practical solution as only three positive quantities need to be communicated among the nodes to calculate these optimal power budgets. Interestingly, the authors computer simulations for multiple users reveal that the sum-rate significantly increases if users selfishly maximise their own SNRs using the proposed method, instead of maximising their own capacity. Thus, they conclude that the SNR-maximisation is spectrally more efficient as it consumes only the best subspaces and leaves the other subspaces free.

## 1 Introduction

Using multi-input–multi-output (MIMO) relays has recently attracted some attention, especially in the systems with multiple antennas at the source (S) and/or destination (D). Increasing the reliability of the communication link, coverage and capacity of the communication system with a fixed amount of power consumption are some of the benefits of using the MIMO relays [1, 2].

Two common protocols used for signal relaying are amplify and forward (AF) and decode and forward [3, 4]. We consider an AF (or non-regenerative) MIMO relay communication system, where the nodes are equipped with multiple antennas, the relay transmits a linearly transformed version of its received signal vector. Design of the relay transformation (RT) matrix has an important impact on the communication between the source and destination nodes. Considerable research has been done to design the RT matrix in order to improve the system performance in various aspects [4–9]. This transformation matrix in [5, 6] is designed to maximize the channel capacity. In [7], the RT matrix is proposed to increase the mutual information assuming a fixed amount of power budget at the relay station. Adjusting the quality-of-service (QoS) requirements in terms of the received SNRs in a relay system with multiple antennas is addressed in [8]. In [9], the authors maximize the received signal-to-noise-ratio (SNR) subject to power constraints and also zero-forcing and minimum-mean-squared-error (MMSE) criteria. In [10, 11], the problem of optimal MIMO relaying is addressed for two-way communication systems.

In this paper, we design the source transmit covariance (STC) and RT matrices by maximising the SNR. We find the optimal RT matrix for a given STC matrix, find the optimal STC matrix for a given RT matrix, and determine the joint optimal solution for STC and RT matrices. It is found that there always exist rank-one optimal STC and RT matrices. This implies that the SNR maximisation leads to consuming only the best subspaces of the channels with rank one and leaves the other subspaces free. This SNR maximisation, in return, allows other users to use the empty subspaces, such that their cross-interferences are minimised to achieve a higher rate compared with the case in which each user selfishly maximises its own capacity and uses more directions. We stress that the interference alignment [12–14] is not in the scope of this paper. The rank-one RT matrix that maximises the instantaneous SNR is not optimum, in the sense of the single user link capacity. However, for low SNRs or highly correlated channels, our results reveal that the single user capacity maximisation gives the same rank-one solutions. Furthermore, under a rank-one constraint, we prove that the maximisation of the source–destination (S–D) mutual information results in the same solution provided in this paper by the maximisation of the SNR. We also determine the optimal power allocation for the source and relay under a constraint on the total power transmitted in the system. This method is practical, as it requires only three positive quantities to be communicated among nodes to calculate the optimal power budgets. We use simulations to study the advantages of the proposed method for multiple source–relay–destination (S–R–D) triplets where each triplet maximises either its own capacity or SNR, without considering the other triplets. This selfish approach is practical since relays are unaware of the channel state information (CSI) of other triplets. The simulations reveal that the sum-rate is significantly higher by maximising individual SNRs than maximising individual capacities.

Throughout this paper, we show vectors and matrices with boldface lowercase and uppercase letters, respectively. Superscript $H$ is used to show the Hermitian transpose of a vector or a matrix, and $\mathrm{tr}(\cdot)$ denotes the trace of a square matrix. The $i$, $j$th entry of matrix $\boldsymbol{A}$ is shown by $[\boldsymbol{A}]_{i,j}$, and we use $\boldsymbol{A}^{1:k}$ to show a matrix consisting of arbitrary permutation of the first up to the $k$th columns of $\boldsymbol{A}$. $\mathbb{C}^{m\times n}$ represents the set of $m\times n$ dimensional complex matrices, the $k$-dimensional identity matrix is denoted by $\boldsymbol{I}_k$, and $E[\cdot]$ is for the stochastic expectation. For any matrix $\boldsymbol{X}$, we denote $\boldsymbol{R}_{\boldsymbol{X}} \overset{\text{def}}{=} \boldsymbol{X}\boldsymbol{X}^H$ and $\lambda_i^X$ as the $i$th largest eigenvalues of $\boldsymbol{R}_{\boldsymbol{X}}$. Finally, we use $\boldsymbol{A} \geq 0$ to show that $\boldsymbol{A}$ is a positive semi definite Hermitian matrix.

## 2 System model

We consider a two-hop quasi-static narrow-band relay communication system with $M$ transmit antennas at the source and $N$ receive antennas at the destination. The relay is equipped with $S$ receive and $L$ transmit antennas. We assume that the direct link between the source and destination is weak/negligible, and the transmission to the destination is performed in two hops. At the first hop, signal vector $\boldsymbol{s}\in\mathbb{C}^M$ is sent to the relay. The received signal at the relay is

$$\boldsymbol{r}_i = \boldsymbol{H}\boldsymbol{s} + \boldsymbol{n} \tag{1}$$

where $\boldsymbol{H}\in\mathbb{C}^{S\times M}$ is the channel response matrix between the source and relay, and $\boldsymbol{n}$ is the relay additive noise. In the second hop, the relay transmits a linear transformation of $\boldsymbol{r}_i$, which is $\boldsymbol{r}_o = \boldsymbol{W}\boldsymbol{r}_i$ to the destination, where $\boldsymbol{W}\in\mathbb{C}^{L\times S}$ is the RT matrix. The received signal at the destination after applying a combining matrix $\boldsymbol{E}^H$ is given by

$$\boldsymbol{y} = \boldsymbol{E}^H(\boldsymbol{G}^H\boldsymbol{r}_o + \boldsymbol{v}) = \boldsymbol{E}^H\boldsymbol{G}^H\boldsymbol{W}\boldsymbol{H}\boldsymbol{s} + \boldsymbol{E}^H\boldsymbol{G}^H\boldsymbol{W}\boldsymbol{n} + \boldsymbol{E}^H\boldsymbol{v} \tag{2}$$

where $\boldsymbol{G}^H\in\mathbb{C}^{N\times L}$ is the channel response between the relay and the destination, and $\boldsymbol{v}\in\mathbb{C}^N$ is the additive noise at the destination. We assume that $\boldsymbol{s}$, $\boldsymbol{v}$ and $\boldsymbol{n}$ are independent, zero-mean circularly symmetrical vectors with covariance matrices of $E[\boldsymbol{s}\boldsymbol{s}^H] = \boldsymbol{R}_s$, $E[\boldsymbol{v}\boldsymbol{v}^H] = \sigma_v^2\boldsymbol{I}_N$ and $E[\boldsymbol{n}\boldsymbol{n}^H] = \sigma_n^2\boldsymbol{I}_S$, respectively. Let us define the end-to-end instantaneous SNR of the system by

$$\begin{aligned}\mathrm{SNR} &\overset{\text{def}}{=} \frac{E_{\boldsymbol{s}}\left[\|\boldsymbol{E}^H\boldsymbol{G}^H\boldsymbol{W}\boldsymbol{H}\boldsymbol{s}\|^2\right]}{E_{\boldsymbol{n},\boldsymbol{v}}\left[\|\boldsymbol{E}^H\boldsymbol{G}^H\boldsymbol{W}\boldsymbol{n} + \boldsymbol{E}^H\boldsymbol{v}\|^2\right]}\\ &= \frac{\mathrm{tr}\left(\boldsymbol{R}_{\boldsymbol{Hs}}\boldsymbol{W}^H\boldsymbol{R}_{\boldsymbol{GE}}\boldsymbol{W}\right)}{\sigma_n^2\mathrm{tr}\left(\boldsymbol{W}^H\boldsymbol{R}_{\boldsymbol{GE}}\boldsymbol{W}\right) + \mathrm{tr}\left(\boldsymbol{R}_{\boldsymbol{E}}\right)\sigma_v^2}\end{aligned} \tag{3}$$

where $\boldsymbol{R}_{\boldsymbol{Hs}} \overset{\text{def}}{=} E_{\boldsymbol{s}}\left[\boldsymbol{H}\boldsymbol{s}\boldsymbol{s}^H\boldsymbol{H}^H\right] = \boldsymbol{H}\boldsymbol{R}_s\boldsymbol{H}^H$, $\boldsymbol{R}_{\boldsymbol{E}} = \boldsymbol{E}\boldsymbol{E}^H$ and $\boldsymbol{R}_{\boldsymbol{GE}} \overset{\text{def}}{=} \boldsymbol{G}\boldsymbol{E}\boldsymbol{E}^H\boldsymbol{G}^H$. The matrices $\boldsymbol{R}_{\boldsymbol{Hs}}$, $\boldsymbol{R}_{\boldsymbol{GE}}$ and noise variances $\sigma_n^2$, $\sigma_v^2$ are required to be measured/known at the relay and source in order to calculate the instantaneous SNR (3). However, the average matrices $\overline{\boldsymbol{R}_{\boldsymbol{Hs}}} = E_{\boldsymbol{s},\boldsymbol{H}}\left[\boldsymbol{H}\boldsymbol{s}\boldsymbol{s}^H\boldsymbol{H}^H\right] = E_{\boldsymbol{H}}\left[\boldsymbol{H}\boldsymbol{R}_s\boldsymbol{H}^H\right]$ and $\overline{\boldsymbol{R}_{\boldsymbol{GE}}} = E_{\boldsymbol{G}}\left[\boldsymbol{G}\boldsymbol{E}\boldsymbol{E}^H\boldsymbol{G}^H\right]$ are not random and depend on the array geometries. Therefore one may use the average matrices without any channel measurement in order to initially establish the communication link and fine-tune the solution once $\boldsymbol{R}_{\boldsymbol{Hs}}$ and $\boldsymbol{R}_{\boldsymbol{GE}}$ are measured. All the expressions written in this paper for the instantaneous SNR can be also employed for the average SNR (defined as the ratio of the signal power averaged over the distribution of $\boldsymbol{H}$ and $\boldsymbol{G}$ to the noise power). In other words, the average SNR is obtained by substituting $\boldsymbol{R}_{\boldsymbol{Hs}}$ and $\boldsymbol{R}_{\boldsymbol{GE}}$ in (3) with $\overline{\boldsymbol{R}_{\boldsymbol{Hs}}}$ and $\overline{\boldsymbol{R}_{\boldsymbol{GE}}}$, respectively. Maximisation of the average SNR leads to some SNR loss; however, this maximisation does not need the instantaneous CSI and significantly reduces the overhead of the channel estimation. Note that the average SNR is not the average of the instantaneous SNR over the channel matrices. We must also emphasise that the optimal design obtained using the average SNR criterion is only useful in the absence of the channel measurements. As soon as the communication link is established, the system shall acquire more accurate channel information and update the solutions maximising the instantaneous SNR to enhance the performance of the system.

## 3 MIMO relay and source SNR optimisation

In this section, we design the STC and RT matrices by maximising either the instantaneous received SNR or the average SNR, subject to a power constraint at the source and relay nodes. As it was mentioned earlier, the instantaneous and average received SNR have the same formula, and depending on the available CSI, we can maximise either the instantaneous or average SNR. The average SNR calculation requires second order statistics of $\boldsymbol{H}$ and $\boldsymbol{G}$; whereas, the instantaneous SNR needs the instantaneous CSI. The STC and RT matrices ($\boldsymbol{R}_s$, $\boldsymbol{W}$) for maximising the instantaneous SNR in (3) are the solution to

$$\begin{aligned}[\boldsymbol{W}, \boldsymbol{R}_s] &= \arg\max_{\boldsymbol{W},\boldsymbol{R}_s}\ \mathrm{SNR}\\ &\text{s.t. } E[\|\boldsymbol{r}_o\|^2] \leq P_{\mathrm{r}},\ \ \mathrm{tr}(\boldsymbol{R}_s) \leq P_{\mathrm{s}},\ \ \boldsymbol{R}_s \geq 0\end{aligned} \tag{4}$$

where $E\left[\|\boldsymbol{r}_o\|^2\right]$ is the relay transmit power; $P_{\mathrm{r}}$ and $P_{\mathrm{s}}$ are the maximum power budgets for signal transmission from the relay and source, respectively. The relay's transmitted signal is

$$\boldsymbol{r}_o = \boldsymbol{W}\boldsymbol{r}_i = \boldsymbol{W}\boldsymbol{H}\boldsymbol{s} + \boldsymbol{W}\boldsymbol{n} \tag{5}$$

Hence, the transmit power $E\left[\|\boldsymbol{r}_o\|^2\right]$ is given by

$$E\left[\|\boldsymbol{r}_o\|^2\right] = \mathrm{tr}\left(\boldsymbol{W}\boldsymbol{R}_{\boldsymbol{Hs}}\boldsymbol{W}^H\right) + \sigma_n^2\mathrm{tr}\left(\boldsymbol{W}\boldsymbol{W}^H\right) \tag{6}$$

*Lemma 1:* The solution to (4) makes its first constraint active and (4) is equivalent to

$$\begin{aligned}[\boldsymbol{W}, \boldsymbol{R}_s] &= \arg\max_{\boldsymbol{W},\boldsymbol{R}_s}\mathrm{SNR}\\ &\text{s.t. } E[\|\boldsymbol{r}_o\|^2] = P_{\mathrm{r}},\ \mathrm{tr}\left(\boldsymbol{R}_s\right) \leq P_{\mathrm{s}},\ \ \boldsymbol{R}_s \geq 0\end{aligned} \tag{7}$$

*Proof:* Assume to the contrary that $\boldsymbol{W}$ is a solution to (4) for which the first constraint is inactive, that is, $P(\boldsymbol{W}) = \mathrm{tr}\left(\boldsymbol{W}\boldsymbol{R}_{\boldsymbol{Hs}}\boldsymbol{W}^H\right) + \sigma_n^2\mathrm{tr}\left(\boldsymbol{W}\boldsymbol{W}^H\right) < P_\mathrm{r}$. We consider the scaled solution $\gamma\boldsymbol{W}$ where $\gamma$ is a real constant. The maximum value for $\gamma$ is $\sqrt{P_\mathrm{r}/(P(\boldsymbol{W}))}$ for which the constraint is active, $P\left(\sqrt{P_\mathrm{r}/(P(\boldsymbol{W}))}\boldsymbol{W}\right) = P_\mathrm{r}$. Any larger value for $\gamma$ violates the constraint. For $\gamma\boldsymbol{W}$, the end-to-end SNR is

$$\mathrm{SNR}(\gamma\boldsymbol{W}) = \frac{\gamma^2\mathrm{tr}\left(\boldsymbol{R}_{\boldsymbol{Hs}}\boldsymbol{W}^H\boldsymbol{R}_{\boldsymbol{G}}\boldsymbol{W}\right)}{\gamma^2\sigma_n^2\mathrm{tr}\left(\boldsymbol{W}^H\boldsymbol{R}_{\boldsymbol{G}}\boldsymbol{W}\right) + N\sigma_v^2} \tag{8}$$

It is easy to check that SNR($\gamma\boldsymbol{W}$) is increasing in $\gamma^2$, and $(\partial\mathrm{SNR}(\gamma\boldsymbol{W}))/(\partial\gamma^2) \geq 0$. Thus, SNR($\gamma\boldsymbol{W}$) in (8) is maximised when $\gamma$ takes its maximum value, which makes the constraint $E\left[\|r_o\|^2\right] \leq P_\mathrm{r}$ in (4) active; and this contradicts our assumption that $P(\boldsymbol{W}) < P_\mathrm{r}$. □

Using (6) and (3) in (4), we have to obtain $\boldsymbol{W}$ and $\boldsymbol{R}_{\boldsymbol{s}}$ from

$$\begin{aligned}[\boldsymbol{W}, \boldsymbol{R}_{\boldsymbol{s}}] &= \arg\max_{\boldsymbol{W},\boldsymbol{R}_{\boldsymbol{s}}} \frac{\mathrm{tr}\left(\boldsymbol{R}_{\boldsymbol{Hs}}\boldsymbol{W}^H\boldsymbol{R}_{\boldsymbol{GE}}\boldsymbol{W}\right)}{\sigma_n^2\mathrm{tr}\left(\boldsymbol{W}^H\boldsymbol{R}_{\boldsymbol{GE}}\boldsymbol{W}\right) + \mathrm{tr}\left(\boldsymbol{R}_{\boldsymbol{E}}\right)\sigma_v^2} \\ &\text{s.t.}\begin{cases}\mathrm{tr}\left(\boldsymbol{W}\boldsymbol{R}_{\boldsymbol{Hs}}\boldsymbol{W}^H\right) + \sigma_n^2\mathrm{tr}\left(\boldsymbol{W}\boldsymbol{W}^H\right) = P_\mathrm{r} \\ \mathrm{tr}(\boldsymbol{R}_{\boldsymbol{s}}) \leq P_\mathrm{s},\ \ \boldsymbol{R}_{\boldsymbol{s}} \geq 0\end{cases}\end{aligned} \tag{9}$$

To better understand the solution to (9), we first study two related optimisation problems: (A) optimisation of $\boldsymbol{W}$ for a given $\boldsymbol{R}_{\boldsymbol{s}}$ and (B) optimisation of $\boldsymbol{R}_{\boldsymbol{s}}$ for a given $\boldsymbol{W}$. □

### 3.1 Optimal RT matrix given the STC matrix

The optimisation of $\boldsymbol{W}$ for a given $\boldsymbol{R}_{\boldsymbol{s}}$ is written as

$$\begin{aligned}\boldsymbol{W} &= \arg\max_{\boldsymbol{W}} \frac{\mathrm{tr}\left(\boldsymbol{R}_{\boldsymbol{Hs}}\boldsymbol{W}^H\boldsymbol{R}_{\boldsymbol{GE}}\boldsymbol{W}\right)}{\sigma_n^2\mathrm{tr}\left(\boldsymbol{W}^H\boldsymbol{R}_{\boldsymbol{GE}}\boldsymbol{W}\right) + \mathrm{tr}\left(\boldsymbol{R}_{\boldsymbol{E}}\right)\sigma_v^2} \\ &\text{s.t. } \mathrm{tr}\left(\boldsymbol{W}\boldsymbol{R}_{\boldsymbol{Hs}}\boldsymbol{W}^H\right) + \sigma_n^2\mathrm{tr}\left(\boldsymbol{W}\boldsymbol{W}^H\right) = P_\mathrm{r}\end{aligned} \tag{10}$$

*Theorem 1:* Any optima for the non-convex problem in (10) have the following singular value decomposition (SVD) with probability one

$$\boldsymbol{W} = \boldsymbol{U}_{\boldsymbol{GE}}\boldsymbol{P}_{\boldsymbol{G}}\boldsymbol{\Sigma}\boldsymbol{P}_{\boldsymbol{H}}^H V_{\boldsymbol{Hs}}^H \tag{11}$$

where $\boldsymbol{P}_{\boldsymbol{G}}$ and $\boldsymbol{P}_{\boldsymbol{H}}$ are $L \times L$, and $S \times S$ dimensional permutation matrices, respectively; and $\boldsymbol{\Sigma}$ is an $L \times S$ diagonal matrix with non-negative singular values $\boldsymbol{x} = [x_1, \ldots, x_{\min\{L,S\}}]^\mathrm{T}$ as its diagonal components. The matrices $\boldsymbol{V}_{\boldsymbol{Hs}}$ and $\boldsymbol{U}_{\boldsymbol{GE}}$ are obtained from the eigenvalue decompositions (EVDs) of $\boldsymbol{R}_{\boldsymbol{Hs}}$ and $\boldsymbol{R}_{\boldsymbol{GE}}$, as

$$\begin{aligned}\boldsymbol{R}_{\boldsymbol{Hs}} &= \boldsymbol{V}_{\boldsymbol{Hs}}\mathrm{diag}(\boldsymbol{\lambda}^{\boldsymbol{Hs}})\boldsymbol{V}_{\boldsymbol{Hs}}^H \\ \boldsymbol{R}_{\boldsymbol{GE}} &= \boldsymbol{U}_{\boldsymbol{GE}}\mathrm{diag}\left(\boldsymbol{\lambda}^{\boldsymbol{GE}}\right)\boldsymbol{U}_{\boldsymbol{GE}}^H\end{aligned} \tag{12}$$

where $\boldsymbol{V}_{\boldsymbol{Hs}} \in \mathbb{C}^{S\times S}$ and $\boldsymbol{U}_{\boldsymbol{GE}} \in \mathbb{C}^{L\times L}$ contain the eigenvectors of $\boldsymbol{R}_{\boldsymbol{Hs}}$ and $\boldsymbol{R}_{\boldsymbol{GE}}$, respectively; and

$$\boldsymbol{\lambda}^{\boldsymbol{Hs}} = \left[\lambda_1^{\boldsymbol{Hs}}, \ldots, \lambda_S^{\boldsymbol{Hs}}\right]^\mathrm{T}, \quad \boldsymbol{\lambda}^{\boldsymbol{GE}} = \left[\lambda_1^{\boldsymbol{GE}}, \ldots, \lambda_L^{\boldsymbol{GE}}\right]^\mathrm{T}$$

where $\lambda_1^{\boldsymbol{Hs}} \geq \cdots \geq \lambda_S^{\boldsymbol{Hs}} \geq 0$ and $\lambda_1^{\boldsymbol{GE}} \geq \cdots \geq \lambda_L^{\boldsymbol{GE}} \geq 0$.

*Proof:* See Appendix 1. □

For some special cases of $\boldsymbol{R}_{\boldsymbol{Hs}}$, $\boldsymbol{R}_{\boldsymbol{GE}}$ and relay format, the optimal $\boldsymbol{P}_{\boldsymbol{G}}$, $\boldsymbol{P}_{\boldsymbol{H}}$ and $\boldsymbol{\Sigma}$ are determined in [15] when $\boldsymbol{E} = \boldsymbol{I}_N$ and $\boldsymbol{R}_{\boldsymbol{s}} \propto \boldsymbol{I}_M$. In this paper, we find the optimal matrices in a general case. The optimal RT matrices used in [3, 6] to maximise the capacity are in the form of (11). The relaying matrix, reported in [16], which minimises the trace of the mean-squared-error matrix under some predefined SNR requirements, also has the structure of (11). In [17], the optimal RT matrices for a multi-hop MIMO relay system are proposed using a linear MMSE receiver to guarantee a predetermined QoS criteria. The optimal RT matrices for this approach also are in the form of (11). In [7], it is proved that the RT matrix (11) maximises the mutual information for a given available transmission power at the relay station. It is also shown in [18, 19] that (11) is the optimal structure for Schur-convex objectives.

Substituting (11) in (10) leads to

$$\begin{aligned}&\max_{\boldsymbol{\Sigma},\boldsymbol{P}_{\boldsymbol{G}},\boldsymbol{P}_{\boldsymbol{H}}} \frac{\mathrm{tr}(\boldsymbol{P}_{\boldsymbol{H}}^H\mathrm{diag}(\boldsymbol{\lambda}^{\boldsymbol{Hs}})\boldsymbol{P}_{\boldsymbol{H}}\boldsymbol{\Sigma}^H\boldsymbol{P}_{\boldsymbol{G}}^H\mathrm{diag}(\boldsymbol{\lambda}^{\boldsymbol{GE}})\boldsymbol{P}_{\boldsymbol{G}}\boldsymbol{\Sigma})}{\mathrm{tr}\left(\boldsymbol{\Sigma}^H\boldsymbol{P}_{\boldsymbol{G}}^H\mathrm{diag}(\boldsymbol{\lambda}^{\boldsymbol{GE}})\boldsymbol{P}_{\boldsymbol{G}}\boldsymbol{\Sigma}\right) + \mathrm{tr}(\boldsymbol{R}_{\boldsymbol{E}})\sigma_v^2/\sigma_n^2} \\ &\text{s.t. } \mathrm{tr}(\boldsymbol{\Sigma}\boldsymbol{P}_{\boldsymbol{H}}^H\mathrm{diag}(\boldsymbol{\lambda}^{\boldsymbol{Hs}})\boldsymbol{P}_{\boldsymbol{H}}\boldsymbol{\Sigma}^H) + \sigma_n^2\mathrm{tr}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^H) = P_\mathrm{r}\end{aligned} \tag{13}$$

Since $\boldsymbol{P}_{\boldsymbol{G}}^H\mathrm{diag}(\boldsymbol{\lambda}^{\boldsymbol{GE}})\boldsymbol{P}_{\boldsymbol{G}} = \mathrm{diag}\left(\boldsymbol{\lambda}_{\boldsymbol{g}}^{\boldsymbol{GE}}\right)$ and $\boldsymbol{P}_{\boldsymbol{H}}^H\mathrm{diag}(\boldsymbol{\lambda}^{\boldsymbol{Hs}})\boldsymbol{P}_{\boldsymbol{H}} = \mathrm{diag}(\boldsymbol{\lambda}_{\boldsymbol{h}}^{\boldsymbol{Hs}})$ are diagonal where $\boldsymbol{\lambda}_{\boldsymbol{g}}^{\boldsymbol{GE}} = \boldsymbol{P}_{\boldsymbol{G}}\boldsymbol{\lambda}^{\boldsymbol{GE}}$ and $\boldsymbol{\lambda}_{\boldsymbol{h}}^{\boldsymbol{Hs}} = \boldsymbol{P}_{\boldsymbol{H}}\boldsymbol{\lambda}^{\boldsymbol{Hs}}$, we can write (13) as

$$\begin{aligned}&\max_{\boldsymbol{x},\boldsymbol{g},\boldsymbol{h}} \frac{\sum_{i=1}^{\min\{S,L\}}\lambda_{h_i}^{\boldsymbol{Hs}}\lambda_{g_i}^{\boldsymbol{GE}}x_i^2}{\sum_{i=1}^{\min\{S,L\}}\lambda_{g_i}^{\boldsymbol{GE}}x_i^2 + \mathrm{tr}\left(\boldsymbol{R}_{\boldsymbol{E}}\right)\sigma_v^2/\sigma_n^2} \\ &\text{s.t.} \sum_{i=1}^{\min\{S,L\}}\left(\lambda_{h_i}^{\boldsymbol{Hs}} + \sigma_n^2\right)x_i^2 = P_\mathrm{r}\end{aligned} \tag{14}$$

where $\boldsymbol{g}$ and $\boldsymbol{h}$ are the arbitrary permutations over $(1, \ldots, L)$ and $(1, \ldots, S)$, respectively. To solve this optimisation problem, we have to determine both $\boldsymbol{x}$ and proper permutations. The following theorem determines the power allocation strategy for optimal relaying.

*Theorem 2:* A solution to (14) is the dominant mode excitation (DME) as follows

$$\begin{cases} x_1^2 = \cdots = x_F^2 = \dfrac{P_\mathrm{r}}{F\left(\lambda_1^{\boldsymbol{Hs}} + \sigma_n^2\right)} \\ x_{F+1} = \cdots = x_{\min\{L,S\}} = 0\end{cases} \tag{15}$$

where $F = \min\left\{F_1^{\boldsymbol{Hs}}, F_1^{\boldsymbol{GE}}\right\}$ shows the multiplicity of the dominant mode $\left(\lambda_1^{\boldsymbol{Hs}}, \lambda_1^{\boldsymbol{GE}}\right)$, and $F_1^{\boldsymbol{Hs}}$ and $F_1^{\boldsymbol{GE}}$ show the multiplicity of the dominant eigenvalue of $\boldsymbol{R}_{\boldsymbol{Hs}}$ and $\boldsymbol{R}_{\boldsymbol{GE}}$, respectively. The solution is not unique and has the



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following general form

$$\begin{cases} \sum_{i=1}^{F} x_i^2 = \dfrac{P_r}{\lambda_1^{\boldsymbol{Hs}} + \sigma_n^2} \\ x_{F+1} = \cdots = x_{\min\{L,S\}} = 0 \end{cases} \tag{16}$$

*Proof:* Lagrangian for (14) can be written as

$$\mathcal{L} = \mathcal{C} + \sum_{i=1}^{\min\{S,L\}} \eta\Big(\Big(\lambda_{h_i}^{\boldsymbol{Hs}} + \sigma_n^2\Big)x_i^2 - P_\mathrm{r}\Big) \tag{17}$$

where

$$\mathcal{C} = \frac{\sum_{i=1}^{\min\{S,L\}} \lambda_{h_i}^{\boldsymbol{Hs}} \lambda_{g_i}^{\boldsymbol{GE}} x_i^2}{\sum_{i=1}^{\min\{S,L\}} \lambda_{g_i}^{\boldsymbol{GE}} x_i^2 + \mathrm{tr}(\boldsymbol{R_E})\sigma_v^2/\sigma_n^2}$$

is the objective function in (14) and $\eta$ is the Lagrange multiplier associated with the equality constraint in (14). Setting $\partial\mathcal{L}/\partial x_m = 0$ leads to

$$x_m\left(\frac{\lambda_{h_m}^{\boldsymbol{Hs}} \lambda_{g_m}^{\boldsymbol{GE}} - \lambda_{g_m}^{\boldsymbol{GE}}\mathcal{C}}{\sum_{i=1}^{\min\{S,L\}} \lambda_{g_i}^{\boldsymbol{GE}} x_i^2 + \mathrm{tr}(\boldsymbol{R_E})\sigma_v^2/\sigma_n^2} + \eta\Big(\lambda_{h_m}^{\boldsymbol{Hs}} + \sigma_n^2\Big)\right) = 0 \tag{18}$$

Note that, here, the Lagrange multiplier method only provides the first-order necessary conditions for the optimal solution. From (18), we should either have $x_m = 0$ or

$$\Big(\lambda_{h_m}^{\boldsymbol{Hs}} - \mathcal{C}\Big)\lambda_{g_m}^{\boldsymbol{GE}} + \eta\Big(\lambda_{h_m}^{\boldsymbol{Hs}} + \sigma_n^2\Big) \times \left(\frac{\mathrm{tr}\big(\boldsymbol{R_E}\big)\sigma_v^2}{\sigma_n^2} + \sum_{i=1}^{\min\{S,L\}} \lambda_{g_i}^{\boldsymbol{GE}} x_i^2\right) = 0 \tag{19}$$

Let $\mathcal{J}$ be the set of indices of non-zero elements of $\boldsymbol{x}$, for which (19) is valid. Multiplying both sides of (19) by $x_m^2$ and taking the summation over all $m \in \mathcal{J}$, we can easily obtain

$$\eta\left(\sum_{i\in\mathcal{J}} \lambda_{g_i}^{\boldsymbol{GE}} x_i^2 + \mathrm{tr}\big(\boldsymbol{R_E}\big)\frac{\sigma_v^2}{\sigma_n^2}\right) = -\frac{\mathrm{tr}\big(\boldsymbol{R_E}\big)\sigma_v^2}{\sigma_n^2 P_\mathrm{r}}\mathcal{C} \tag{20}$$

Substituting (20) in (19), for all $m \in \mathcal{J}$, we must have

$$\mathcal{C} = \frac{\lambda_{h_m}^{\boldsymbol{Hs}} \lambda_{g_m}^{\boldsymbol{GE}}}{\lambda_{g_m}^{\boldsymbol{GE}} + \mathrm{tr}\big(\boldsymbol{R_E}\big)\Big(\lambda_{h_m}^{\boldsymbol{Hs}} + \sigma_n^2\Big)\sigma_v^2/\big(P_\mathrm{r}\sigma_n^2\big)} \tag{21}$$

The right-hand side of (21) is an increasing function of $\lambda_{h_m}^{\boldsymbol{Hs}}$ and $\lambda_{g_m}^{\boldsymbol{GE}}$. Therefore to maximise the objective function $\mathcal{C}$, we should only take the largest eigenvalues, which are $\lambda_1^{\boldsymbol{Hs}}$ and $\lambda_1^{\boldsymbol{GE}}$. This simply means that we must have $x_m = 0$ for $m > F$. Under this condition, we must have $\sum_{i=1}^{F} x_i^2 = P_\mathrm{r}/\big(\lambda_1^{\boldsymbol{Hs}} + \sigma_n^2\big)$ to satisfy the power constraint. □

Furthermore, the maximum achievable SNR is given by

$$\sup_{\substack{\boldsymbol{W} \\ \text{s.t. } E[\|\boldsymbol{r}_o\|^2] \le P_\mathrm{r}}} \mathrm{SNR} = \frac{P_\mathrm{r}\lambda_1^{\boldsymbol{Hs}}\lambda_1^{\boldsymbol{GE}}}{\sigma_n^2 P_\mathrm{r}\lambda_1^{\boldsymbol{GE}} + \mathrm{tr}\big(\boldsymbol{R_E}\big)\sigma_v^2\big(\lambda_1^{\boldsymbol{Hs}} + \sigma_n^2\big)} \tag{22}$$

*Theorem 3:* reveals that all the power must be invested in the direction of the dominant eigenvectors of the source–relay (S–R) and relay–destination (R–D) channels' correlation matrices. Thus, using Theorem 2, an optimal RT matrix for (15) is

$$\boldsymbol{W}_\mathrm{o} = \sqrt{\frac{P_\mathrm{r}}{F\big(\lambda_1^{\boldsymbol{Hs}} + \sigma_n^2\big)}}\boldsymbol{U}_{\boldsymbol{GE}}^{1:F}\big(\boldsymbol{V}_{\boldsymbol{Hs}}^{1:F}\big)^H \tag{23}$$

where columns of $\boldsymbol{V}_{\boldsymbol{Hs}}^{1:F}$ consist of any permutations of the first $F$ eigenvectors of $\boldsymbol{R_{Hs}}$ (eigenvectors are corresponding to the dominant eigenvalue of $\boldsymbol{R_{Hs}}$), and $\boldsymbol{U}_{\boldsymbol{GE}}^{1:F}$ is defined similarly. A special solution is obtained in [20] for the case that only one data stream is sent from the source to the destination. There, using some proper beamforming at both the source and destination nodes, it is shown that the DME with $F = 1$ gives the RT matrix that maximises the SNR.

*Remark 1:* If we maximise the instantaneous SNR to design the RT matrix, then we have $F = 1$ with probability one. In this case, the optimal relaying matrix is unique. In addition, the solution to (14) is given by $x_i^2 = P_\mathrm{r}\delta(i-1)/\big(\lambda_1^{\boldsymbol{Hs}} + \sigma_n^2\big)$, $\lambda_{h_1}^{\boldsymbol{Hs}} = \lambda_1^{\boldsymbol{Hs}}$, and $\lambda_{g_1}^{\boldsymbol{GE}} = \lambda_1^{\boldsymbol{GE}}$ where $\delta(.)$ is the Kronecker delta. However, the solution to (14) is not unique when $F > 1$. In this case, one can allocate arbitrary powers to the dominant modes, provided that the power constraint in (16) is satisfied. For instance, similar to the case $F = 1$, we may allocate all the power to the first dominant mode, or equally distribute the power to all the dominant modes as in (15).

### 3.2 Optimal STC matrix given the RT matrix

The optimisation problem in this case can be written as

$$\boldsymbol{R_s} = \arg\max_{\boldsymbol{R_s}} \frac{\mathrm{tr}\big(\boldsymbol{R_{Hs}}\boldsymbol{W}^H\boldsymbol{R_{GE}}\boldsymbol{W}\big)}{\sigma_n^2\mathrm{tr}\big(\boldsymbol{W}^H\boldsymbol{R_{GE}}\boldsymbol{W}\big) + \mathrm{tr}\big(\boldsymbol{R_E}\big)\sigma_v^2} \quad \text{s.t.} \quad \mathrm{tr}\big(\boldsymbol{R_s}\big) \le P_\mathrm{s},\ \boldsymbol{R_s} \ge 0 \tag{24}$$

Only the numerator of the objective function of (24) depends on $\boldsymbol{R_s}$. Thus, (24) leads to

$$\boldsymbol{R_s} = \arg\max_{\boldsymbol{R_s}} \mathrm{tr}\big(\boldsymbol{R_s}\boldsymbol{H}^H\boldsymbol{W}^H\boldsymbol{GEE}^H\boldsymbol{G}^H\boldsymbol{WH}\big) \quad \text{s.t.} \quad \mathrm{tr}(\boldsymbol{R_s}) \le P_\mathrm{s},\ \boldsymbol{R_s} \ge 0 \tag{25}$$

As $\boldsymbol{R_s}$ and $\boldsymbol{H}^H\boldsymbol{W}^H\boldsymbol{GEE}^H\boldsymbol{G}^H\boldsymbol{WH}$ are Hermitian positive semi-definite matrices, we have

$$\begin{aligned} \mathrm{tr}\big(\boldsymbol{R_s}\boldsymbol{H}^H\boldsymbol{W}^H\boldsymbol{GEE}^H\boldsymbol{G}^H\boldsymbol{WH}\big) &\le \sum_{i=1}^{M} \lambda_i^{\boldsymbol{E}^H\boldsymbol{G}^H\boldsymbol{WH}}\lambda_i^{\boldsymbol{s}} \\ &\le \lambda_1^{\boldsymbol{E}^H\boldsymbol{G}^H\boldsymbol{WH}}\mathrm{tr}(\boldsymbol{R_s}) \\ &\le \lambda_1^{\boldsymbol{E}^H\boldsymbol{G}^H\boldsymbol{WH}}P_\mathrm{s} \end{aligned} \tag{26}$$

The upper bound in (26) is achieved using $\boldsymbol{R_s} = P_s \boldsymbol{q}\boldsymbol{q}^H$ where $\boldsymbol{q}$ is the unitary dominant eigenvector of $\boldsymbol{H}^H\boldsymbol{W}^H\boldsymbol{G}\boldsymbol{E}\boldsymbol{E}^H\boldsymbol{G}^H\boldsymbol{W}\boldsymbol{H}$, that is, the source allocates all its power to the direction of the dominant mode of the equivalent channel. The SNR improvement using this optimal solution, compared with $\boldsymbol{R_s} = P_s/M\boldsymbol{I}_M$, is

$$10\log_{10}\left(\frac{M\lambda_1^{\boldsymbol{E}^H\boldsymbol{G}^H\boldsymbol{W}\boldsymbol{H}}}{\sum_{m=1}^{M}\lambda_m^{\boldsymbol{E}^H\boldsymbol{G}^H\boldsymbol{W}\boldsymbol{H}}}\right) \le 10\log_{10}(M)\text{dB}$$

For example, changing $\boldsymbol{R_s}$ results in no improvement in the SNR when all the eigenvalues of $\boldsymbol{R}_{\boldsymbol{E}^H\boldsymbol{G}^H\boldsymbol{W}\boldsymbol{H}}$ are equal.

In (24) or (25) the relay power constraint is not considered. If we consider the relay power constraint, the optimisation problem can be written as

$$\begin{aligned}\boldsymbol{R_s} &= \arg\max_{\boldsymbol{R_s}} \operatorname{tr}(\boldsymbol{R_s}\boldsymbol{A})\\ &\text{s.t.}\begin{cases}\operatorname{tr}(\boldsymbol{R_s}\boldsymbol{B}) = c\\ \operatorname{tr}(\boldsymbol{R_s}) = P_s, \quad \boldsymbol{R_s} \ge 0\end{cases}\end{aligned} \tag{27}$$

where $\boldsymbol{A} \overset{\text{def}}{=} \boldsymbol{H}^H\boldsymbol{W}^H\boldsymbol{G}\boldsymbol{E}\boldsymbol{E}^H\boldsymbol{G}^H\boldsymbol{W}\boldsymbol{H}$, $\boldsymbol{B} \overset{\text{def}}{=} \boldsymbol{H}^H\boldsymbol{W}^H\boldsymbol{W}\boldsymbol{H}$ and $c \overset{\text{def}}{=} P_r - \sigma_n^2\operatorname{tr}(\boldsymbol{W}\boldsymbol{W}^H)$ are some known parameters. The above optimisation problem is a semi-definite program [21], and this convex problem can be solved by using efficient numeric methods, for example, see [22].

### 3.3 Joint optimal STC and RT matrices

Here, we jointly optimise $\boldsymbol{W}$ and $\boldsymbol{R_s}$ as (9). Note that the criteria in (9) is maximised with respect to $\boldsymbol{W}$ for a given $\boldsymbol{R_s}$, and the maximum achievable SNR is given in (22). Thus, we only need to maximise (22) with respect to $\boldsymbol{R_s}$. Note that for any function $f(a, b)$ and under appropriate constraints, we have

$$\arg\max_{a,b} f(a,b) = \arg\max_{a}\{\max_{b} f(a,b)\}.$$

In (22), $\lambda_1^{\boldsymbol{Hs}}$ is the only term which depends on $\boldsymbol{R_s}$. Since the SNR criterion in (22) is an increasing function of $\lambda_1^{\boldsymbol{Hs}}$, the optimal $\boldsymbol{R_s}$ shall maximise $\lambda_1^{\boldsymbol{Hs}}$. Obviously, under the constraint $\operatorname{tr}(\boldsymbol{R_s}) \le P_s$, the upper bound for $\lambda_1^{\boldsymbol{Hs}}$ is

$$\lambda_1^{\boldsymbol{Hs}} \le \operatorname{tr}(\boldsymbol{R_{Hs}}) = \operatorname{tr}(\boldsymbol{R_s}\boldsymbol{R}_{\boldsymbol{H}^H}) \le \lambda_1^{\boldsymbol{H}}\operatorname{tr}(\boldsymbol{R_s}) \le \lambda_1^{\boldsymbol{H}}P_s \tag{28}$$

Therefore, using (22) and (28), the following is the maximum achievable SNR for (9) (see (29))

This upper bound is achieved when the relay received signal belongs to the subspace of the dominant eigenvalue of $\boldsymbol{R_H}$ (where $\boldsymbol{R_H} = \boldsymbol{H}\boldsymbol{H}^H$), and the source and relay allocate all their transmit powers to the directions of dominant eigenvectors of $\boldsymbol{R}_{\boldsymbol{H}^H}$ (where $\boldsymbol{R}_{\boldsymbol{H}^H} = \boldsymbol{H}^H\boldsymbol{H}$) and $\boldsymbol{R_{GE}}$, respectively. The functions in (22) and (29) are similar, where substituting $\lambda_1^{\boldsymbol{Hs}}$ for $P_s\lambda_1^{\boldsymbol{H}}$ in (22), we obtain (29). Note that the optimisation problem in (24) is different from the one in this section. The optimal $\boldsymbol{R_s}$ in Section 3.2 depends on a given relay matrix $\boldsymbol{W}$; whereas, the optimal $\boldsymbol{R_s}$ in this section only depends on the R–D channel matrix, as the relay matrix is also optimised. Furthermore, the optimisation problem in this section has an additional constraint on the relay power compared with the problem in (24).

Let us compare the maximum achievable SNRs in (22) using the uniform power allocation $\boldsymbol{R_s} = P_s/M\boldsymbol{I}_M$ with the achievable SNR using $\boldsymbol{R_s}^{\text{opt}}$ given in (29). We define the SNR improvement as

$$\mathcal{J} \overset{\text{def}}{=} 10\log_{10}\left(\frac{\text{SNR}(\boldsymbol{R_s}^{\text{opt}})}{\text{SNR}(P_s/M\boldsymbol{I}_M)}\right)$$

where $\text{SNR}(\boldsymbol{R_s}^{\text{opt}})$ is the achievable SNR of the system provided in (29). For $\boldsymbol{R_s} = P_s/M\boldsymbol{I}_M$, we can use $\lambda_1^{\boldsymbol{Hs}} = P_s\lambda_1^{\boldsymbol{H}}/M$ in (22) to find the expression of the achievable $\text{SNR}(P_s/M\boldsymbol{I}_M)$. Using this result and (29), we can express $\mathcal{J}$ as

$$\begin{aligned}\mathcal{J} &= 10\log_{10}(M)\\ &+ 10\log_{10}\left(1 - \frac{M-1}{M}\left(1 + \frac{\text{SNR}_{\text{rd}} + \operatorname{tr}(\boldsymbol{R_E})}{\operatorname{tr}(\boldsymbol{R_E})\,\text{SNR}_{\text{sr}}}\right)^{-1}\right)\end{aligned} \tag{30}$$

where $\text{SNR}_{\text{rd}} = P_r\lambda_1^{\boldsymbol{GE}}/\sigma_v^2$ and $\text{SNR}_{\text{sr}} = P_s\lambda_1^{\boldsymbol{H}}/\sigma_n^2$ are the link SNRs for the R–D and the S–R, respectively. Therefore we conclude that an improvement of up to $10\log_{10}(M)$dB in the SNR is achieved by optimising the STC matrix compared with using uniform power allocation $\boldsymbol{R_s} = P_s/M\boldsymbol{I}_M$. Furthermore, the amount of this improvement tends to $10\log_{10}(M)$dB as

$$\frac{\operatorname{tr}(\boldsymbol{R_E}) + \text{SNR}_{\text{rd}}}{\operatorname{tr}(\boldsymbol{R_E})\,\text{SNR}_{\text{sr}}} \to \infty$$

Therefore the source optimisation contributes more improvement if the R–D link quality is better compared with that of the S–R.

*Remark 2:* Following the same procedure as Section 3.3, we can obtain the optimal combining matrix at the destination by maximising (29) over $\boldsymbol{E}$ as

$$\max_{\boldsymbol{E}} \frac{\lambda_1^{\boldsymbol{GE}}}{\operatorname{tr}(\boldsymbol{R_E})} \tag{31}$$

where the above problem can easily be obtained by dividing the numerator and denominator of (29) by $\lambda_1^{\boldsymbol{GE}}$. Similar to (28) we have

$$\lambda_1^{\boldsymbol{GE}} \le \operatorname{tr}(\boldsymbol{R_{GE}}) = \operatorname{tr}(\boldsymbol{R_E}\boldsymbol{R}_{\boldsymbol{G}^H}) \le \lambda_1^{\boldsymbol{G}}\operatorname{tr}(\boldsymbol{R_E}) \tag{32}$$

Consequently, $\lambda_1^{\boldsymbol{GE}}/\operatorname{tr}(\boldsymbol{R_E}) \le \lambda_1^{\boldsymbol{G}}$ and this upper bound is achieved when $\boldsymbol{R_E} \propto \boldsymbol{e}\boldsymbol{e}^H$ where $\boldsymbol{e}$ is the unit-length dominant eigenvector of $\boldsymbol{R}_{\boldsymbol{G}^H}$.

*Remark 3:* The above results show that there exists a rank-one optimal solution for $\boldsymbol{R_s}$ and $\boldsymbol{W}$. In addition, for the

$$\sup_{\substack{\boldsymbol{R_s},\boldsymbol{W}\\ \text{s.t.}\quad E[\|\boldsymbol{r}_o\|^2] = P_r,\ \operatorname{tr}(\boldsymbol{R_s}) \le P_s,\ \boldsymbol{R_s} \ge 0}} \text{SNR} = \frac{P_rP_s\lambda_1^{\boldsymbol{H}}\lambda_1^{\boldsymbol{GE}}}{\sigma_n^2P_r\lambda_1^{\boldsymbol{GE}} + \operatorname{tr}(\boldsymbol{R_E})\sigma_v^2(P_s\lambda_1^{\boldsymbol{H}} + \sigma_n^2)} \tag{29}$$



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instantaneous matrices $\boldsymbol{R_{GE}}$ and $\boldsymbol{R_{Hs}}$, the largest eigenvalues often have a multiplicity of one. For example, for complex Wishart matrices (Rayleigh fading), the multiplicity is one with probability one. This implies that the SNR maximisation requires the use of the signal subspaces with rank one. As a result, each user consumes only the best one-dimensional (1D) subspaces and does not transmit (receive) any power in (from) other directions. Consequently, in the context of a system with multiple users, other users and relays can make use of other subspaces. In a multi-user environment, the SNR maximisation is spectrally more efficient than the maximisation of the capacity. As in the former method, by aligning the subspaces allocated to different users, their cross-interferences are reduced [14]. Furthermore, for high noise power or when the channels' elements are highly correlated, the capacity optimisation also results in the same rank-one solutions (the maximisation of the SNR criterion also becomes optimum in terms of the capacity of a single user). Using the proposed optimal rank-one matrices, the link between the source and destination is equivalent to a single-input–single-output system. Therefore the SNR in (3) is closely related to the error rate performance.

*Theorem 4:* Maximising the mutual information between $\boldsymbol{y}$ and $\boldsymbol{s}$ over all rank-one matrices $\boldsymbol{W}$ for $\boldsymbol{E} = \boldsymbol{I}_N$ gives the same solution as maximising the SNR, without such a rank constraint.

*Proof:* The mutual information between $\boldsymbol{y}$ and $\boldsymbol{s}$ is [6]

$$\mathcal{I}(\boldsymbol{y};\boldsymbol{s}) = \log\left(\det\left(\boldsymbol{I} + \frac{1}{\sigma_n^2}\boldsymbol{R_{Hs}}\left(\boldsymbol{I} - \boldsymbol{S}^{-1}\right)\right)\right) \tag{33}$$

where $\boldsymbol{S} \stackrel{\text{def}}{=} \boldsymbol{I}_N + \sigma_n^2/\sigma_v^2 \boldsymbol{W}^H \boldsymbol{R_G} \boldsymbol{W}$. Here, we consider the power constraints $\text{tr}(\boldsymbol{R_s}) = P_\text{s}$ and $E\left[\|\boldsymbol{r}_o\|^2\right] = P_\text{r}$ and maximise $\mathcal{I}(\boldsymbol{y};\boldsymbol{s})$ over the space of rank-one matrices $\boldsymbol{W}$ as follows

$$\begin{aligned}&\max_{\boldsymbol{R_s},\boldsymbol{W}} \mathcal{I}(\boldsymbol{y};\boldsymbol{s})\\ &\text{s.t.}\ \text{Rank}(\boldsymbol{W}) = 1,\ E\left[\|\boldsymbol{r}_o\|^2\right] = P_\text{r},\ \text{tr}\left(\boldsymbol{R_s}\right) = P_\text{s}\end{aligned} \tag{34}$$

Using the SVD $\boldsymbol{W} = \beta \boldsymbol{a}\boldsymbol{b}^H$ and $\boldsymbol{b}^H\boldsymbol{b} = \boldsymbol{a}^H\boldsymbol{a} = 1$, it is easy to write

$$\mathcal{I}(\boldsymbol{y};\boldsymbol{s}) = \log\left(1 + \frac{\boldsymbol{a}^H\boldsymbol{R_G}\boldsymbol{a}\beta^2/\sigma_v^2}{1 + \boldsymbol{a}^H\boldsymbol{R_G}\boldsymbol{a}\beta^2\sigma_n^2/\sigma_v^2}\boldsymbol{b}^H\boldsymbol{R_{Hs}}\boldsymbol{b}\right) \tag{35}$$

Thus, problem (34) is equivalent to

$$\begin{aligned}&\max_{\boldsymbol{R_s},\boldsymbol{a},\boldsymbol{b},\beta^2} \frac{\boldsymbol{b}^H\boldsymbol{R_{Hs}}\boldsymbol{b}}{1 + \left(\boldsymbol{a}^H\boldsymbol{R_G}\boldsymbol{a}\beta^2\sigma_n^2/\sigma_v^2\right)^{-1}}\\ &\text{s.t.}\ \begin{cases}\text{tr}(\boldsymbol{R_s}) = P_\text{s},\ \boldsymbol{b}^H\boldsymbol{b} = 1,\ \boldsymbol{a}^H\boldsymbol{a} = 1\\ \beta^2\left(\boldsymbol{b}^H\boldsymbol{R_{Hs}}\boldsymbol{b} + \sigma_n^2\right) = P_\text{r}\end{cases}\end{aligned} \tag{36}$$

Using the last constraint in the numerator of (36), we can rewrite the problem as

$$\begin{aligned}&\max_{\boldsymbol{R_s},\boldsymbol{a},\boldsymbol{b},\beta^2} \frac{P_\text{r} - \beta^2\sigma_n^2}{1 + \left(\boldsymbol{a}^H\boldsymbol{R_G}\boldsymbol{a}\beta^2\sigma_n^2/\sigma_v^2\right)^{-1}}\\ &\text{s.t.}\ \begin{cases}\text{tr}(\boldsymbol{R_s}) = P_\text{s},\ \boldsymbol{b}^H\boldsymbol{b} = 1,\ \boldsymbol{a}^H\boldsymbol{a} = 1\\ \beta^2 = P_\text{r}\left(\boldsymbol{b}^H\boldsymbol{R_{Hs}}\boldsymbol{b} + \sigma_n^2\right)^{-1}\end{cases}\end{aligned} \tag{37}$$

In (37), to find $\boldsymbol{a}$, we only need to maximise $\boldsymbol{a}^H\boldsymbol{R_G}\boldsymbol{a}$ under the constraint $\boldsymbol{a}^H\boldsymbol{a} = 1$. Thus, $\boldsymbol{a}$ must be the dominant eigenvector of $\boldsymbol{R_G}$. Since the objective function in (37) is a decreasing function of $\beta^2$, we shall choose the minimum value for $\beta^2$. Using $\beta^2 = P_\text{r}\left(\boldsymbol{b}^H\boldsymbol{R_{Hs}}\boldsymbol{b} + \sigma_n^2\right)^{-1}$, the minimum $\beta^2$ is achieved when $\boldsymbol{b}$ is the dominant eigenvector of $\boldsymbol{R_{Hs}}$. Consequently, the optimal RT matrix, $\boldsymbol{W}$, is identical to (23) for $F = 1$. By substituting the optimal values for $\boldsymbol{a}$, $\boldsymbol{b}$ and $\beta^2$ in (36), it can be checked that the objective function is increasing in $\lambda_1^{\boldsymbol{Hs}}$. Thus, similar to Section 3.3, maximising $I(\boldsymbol{y};\boldsymbol{s})$ under a rank-one constraint for $\boldsymbol{W}$ requires the source to allocate all its transmit power to the direction of dominant eigenvector of $\boldsymbol{R_{H^H}}$. □

## 4 Optimum power budget allocation

Using a total available transmit power $P_\text{T}$, we aim to optimise the transmit powers $P_\text{s}$ and $P_\text{r}$ from the source and relay, respectively, by maximising the end-to-end SNR

$$\begin{aligned}&\max_{\boldsymbol{W},P_\text{s},P_\text{r}} \text{SNR}\\ &\text{s.t.}\ E[\|\boldsymbol{r}_o\|^2] \le P_\text{r},\ P_\text{s} + P_\text{r} = P_\text{T},\ P_\text{s} \ge 0,\ P_\text{r} \ge 0\end{aligned} \tag{38}$$

The solution to (38) provides the highest achievable SNR for a bounded amount of interference for other neighbouring users since the amount of interference contribution that this source–relay pair produces for other users is related to the total transmitting power $P_\text{T}$. Assuming $\boldsymbol{R_s} = P_\text{s}/M\boldsymbol{I}_M$ and using (22), we simplify (38) as

$$\begin{aligned}&\max_{P_\text{s},P_\text{r}}\left(\frac{P_\text{s}}{M\sigma_n^2}\right)\left(\frac{P_r\lambda_1^{\boldsymbol{H}}\lambda_1^{\boldsymbol{GE}}}{P_\text{r}\lambda_1^{\boldsymbol{GE}} + N\sigma_v^2/\sigma_n^2\left(P_\text{s}\lambda_1^{\boldsymbol{H}}/M + \sigma_n^2\right)}\right)\\ &\text{s.t.}\ \ P_\text{s} + P_\text{r} = P_\text{T},\ P_\text{s} \ge 0,\ P_\text{r} \ge 0\end{aligned} \tag{39}$$

Note that when we replace $\sigma_n^2$ in (29) with $M\sigma_n^2$, we obtain the objective function of (39). Thus, the results of this section can easily be applied to the system with the optimal STC matrix.

Substituting $P_\text{r} = P_\text{T} - P_\text{s}$ in (39), the optimal $P_\text{s}$ can be computed from

$$P_{\text{s}\circ} = \arg\max_{P_\text{s}} \mathcal{Q}(P_\text{s}),\quad \text{s.t.}\ 0 \le P_\text{s} \le P_\text{T} \tag{40}$$

where

$$\mathcal{Q}(P_\text{s}) \stackrel{\text{def}}{=} \frac{P_\text{s}P_\text{T} - P_\text{s}^2}{P_\text{s}\left(N\lambda_1^{\boldsymbol{H}}\sigma_v^2/\left(M\sigma_n^2\right) - \lambda_1^{\boldsymbol{GE}}\right) + P_\text{T}\lambda_1^{\boldsymbol{GE}} + N\sigma_v^2}$$

The second derivative of $\mathcal{Q}(P_s)$ with respect to $P_s$ is

$$\frac{\partial^2 \mathcal{Q}(P_s)}{\partial P_s^2} = \frac{-2M^2 N\sigma_v^2/\sigma_n^2\left(P_T\lambda_1^{\boldsymbol{H}} + M\sigma_n^2\right)\left(P_T\lambda_1^{\boldsymbol{GE}} + N\sigma_v^2\right)}{\left(P_s N\sigma_v^2/\sigma_n^2\lambda_1^{\boldsymbol{H}} + M\lambda_1^{\boldsymbol{GE}}\left(P_T - P_s\right) + MN\sigma_v^2\right)^3} \tag{41}$$

It is seen that for all $P_s \in [0, P_T]$, we have $\partial^2\mathcal{Q}(P_s)/\partial P_s^2 \le 0$. In addition, we have $\mathcal{Q}(0) = \mathcal{Q}(P_T) = 0$. Thus, the 'Rolle's theorem' (which is a special case of the 'mean value theorem') implies that there exists a $P_s \in [0, P_T]$ satisfying $\partial\mathcal{Q}(P_s)/\partial P_s = 0$. There are two roots for $\partial\mathcal{Q}(P_s)/\partial P_s$, where only one of them satisfies $\partial^2\mathcal{Q}(P_s)/\partial P_s^2 \le 0$. As a result, the optimal power budgets are

$$P_{s\circ} = \frac{MP_T\lambda_1^{\boldsymbol{GE}} + MN\sigma_v^2 - \sqrt{\Delta}}{M\lambda_1^{\boldsymbol{GE}} - N\sigma_v^2\lambda_1^H/\sigma_n^2} \tag{42}$$

$$P_{r\circ} = P_T - P_{s\circ} \tag{43}$$

where $\Delta \stackrel{\text{def}}{=} \mathrm{MN}\left(P_T\lambda_1^{\boldsymbol{GE}} + N\sigma_v^2\right)\left(P_T\lambda_1^{\boldsymbol{H}} + M\sigma_n^2\right)\sigma_v^2/\sigma_n^2$.

For the case $\lambda_1^{\boldsymbol{H}}/\left(M\sigma_n^2\right) = \lambda_1^{\boldsymbol{GE}}/\left(N\sigma_v^2\right)$, the above optimal solutions are undefined. However, for such a case, it is easy to show that we must have $P_{s\circ} = P_{r\circ} = P_T/2$. We can rewrite (42) as

$$P_{s\circ} = \frac{1 + \rho_{rd}P_T - \sqrt{\Delta'}}{\rho_{rd} - \rho_{sr}} \tag{44}$$

where $\Delta' \stackrel{\text{def}}{=} \left(1 + P_T\rho_{rd}\right)\left(1 + P_T\rho_{sr}\right)$, $\rho_{sr} \stackrel{\text{def}}{=} \lambda_1^{\boldsymbol{H}}/\left(M\sigma_n^2\right)$ and $\rho_{rd} \stackrel{\text{def}}{=} \lambda_1^{\boldsymbol{GE}}/\left(N\sigma_v^2\right)$. This equation shows that the source or relay does not need to know $\lambda_1^{\boldsymbol{H}}$, $\lambda_1^{\boldsymbol{GE}}$, $\sigma_n^2$, $\sigma_v^2$, $M$, $N$ and the optimal powers can be determined as a function of only three parameters $\left(\rho_{sr}, \rho_{sr}, P_T\right)$. In other words, the optimal powers are determined as a function of the link qualities of the best modes and the total power. In addition (44) reveals that when both link qualities are poor, the equal power allocation is close to optimal and we obtain $P_{s\circ} \to P_T/2$ as $(\rho_{sr}, \rho_{rd}) \to (0, 0)$.

*Lemma 2:* Optimum source power budget is a decreasing function of the S–R channel quality $\rho_{sr}$ and is an increasing function of the R–D channel quality $\rho_{rd}$.

*Proof:* From (42) we obtain

$$\frac{\partial P_{s\circ}}{\partial \rho_{sr}} = \frac{\left(\rho_{rd}P_T + 1\right)\left(2\sqrt{\Delta'} - P_T(\rho_{rd} + \rho_{sr}) - 2\right)}{2\left(-\rho_{rd} + \rho_{sr}\right)^2\sqrt{\Delta'}} = \frac{-\left(\rho_{rd}P_T + 1\right)P_T^2}{2\left(2\sqrt{\Delta'} + P_T(\rho_{rd} + \rho_{sr}) + 2\right)\sqrt{\Delta'}} \le 0$$

Similarly, we can prove $\partial P_{s\circ}/\partial\rho_{rd} \ge 0$. Since $P_{s\circ} = P_T - P_{r\circ}$, we have $\partial P_{r\circ}/\partial\rho_{sr} \ge 0$ and $\partial P_{r\circ}/\partial\rho_{rd} \le 0$. Thus, the optimal relay power budget, $P_{ro}$, is decreasing in $\rho_{rd}$ and is increasing in $\rho_{sr}$.

This is intuitively expected since as one of the link qualities degrades, we expect that the allocated power to that link be increased to compensate for the degradation. □

## 5 Simulation results

For computer simulations, we generate the channel matrices $\boldsymbol{H}$ and $\boldsymbol{G}^H$ to be zero-mean circularly symmetric complex Gaussian (ZM-CSCG) with correlation matrices [23, 24]

$$\left[E[\boldsymbol{HH}^H]\right]_{i,j} = \mathcal{J}_0\left(|i-j|d_H\right) \quad \text{for} \quad i, j = 1, \ldots, S$$

$$\left[E[\boldsymbol{GG}^H]\right]_{i,j} = \mathcal{J}_0\left(|i-j|d_G\right) \quad \text{for} \quad i, j = 1, \ldots, L$$

where $\mathcal{J}_0(\cdot)$ is the zero-order Bessel function of the first kind, $d_G$ and $d_H$ are proportional with the carrier frequency and with the transmit and receive antenna separation vectors at relay, respectively. Note that when $\boldsymbol{R_s} = P_s/M\boldsymbol{I}_M$, the received total powers over these channels are equal to the transmit powers. The additive noise vectors in the assumed system are considered to be spatially and temporally white ZM-CSCG. The matrices used in this section are: (i) the conventional AF transform $\boldsymbol{W} \propto \boldsymbol{I}$ [7]; (ii) matched filter (MF) $\boldsymbol{W} \propto \boldsymbol{GH}^H$ [25]; (iii) MMSE matrix [26]; (iv) modified analogue relaying (MAR) matrix [3] or the best unitary excitation (BUE) matrix [15]; (v) water-filling (WF) in the direction of the channels' eigenmodes [6]; (via) DME, when the F-CSI is available at the relay, and an equal power is allocated to the source and relay; (vib) DME, when the relay has the F-CSI, and the optimal powers are allocated to the source and relay; (vic) DME, when the relay has the P-CSI (the relay only knows $E[\boldsymbol{HR_sH}^H]$ and $E[\boldsymbol{GEE}^H\boldsymbol{G}^H]$), and the source and relay have an equal transmit power budget; (vid) similar to (vic), but with the optimal power allocated to the source and relay.

We have to stress that when the relay has the F-CSI, the matrices $\boldsymbol{U_{GE}}$ and $\boldsymbol{V_{Hs}}$ used in (23) and the $\lambda_1^{\boldsymbol{GE}}$, $\lambda_1^{\boldsymbol{H}}$ used in (42) and (43) are the eigenvectors and the corresponding eigenvalues of $\boldsymbol{GEE}^H\boldsymbol{G}^H$ and $\boldsymbol{HR_sH}^H$, respectively. Also, in the case where the P-CSI is available at the relay, $\left(\boldsymbol{U_{GE}}, \lambda_1^{\boldsymbol{GE}}\right)$ and $\left(\boldsymbol{V_{Hs}}, \lambda_1^{\boldsymbol{Hs}}\right)$ are the eigenvectors and eigenvalues of $E[\boldsymbol{GEE}^H\boldsymbol{G}^H]$ and $\mathrm{E}[\boldsymbol{HR_sH}^H]$, respectively. In our simulations, for the relaying schemes (i) to (vib), the relay has the F-CSI, and for the schemes (vic) and (vid), the P-CSI is available at the relay (in this case, the RT matrix is designed to maximise the average SNR).

Here, we briefly compare the computational complexity (CC) of various relaying schemes used in this section, and we show that the CC of the proposed DME is much less than that of the MMSE, WF and BUE (or MAR) methods. We mainly consider the computationally demanding operations, such as the matrix inversion, multiplication and EVD. Note that for computing the MF and MMSE matrices, we must have $M = N$, and the WF matrix in [6] is derived only for $S = L$. The MF matrix only requires the calculation of $\boldsymbol{GH}^H$ that needs $LSN$ complex multiplications. The MMSE matrix (see (13) in [26]) is computationally more expensive as it requires the calculation of $\boldsymbol{GH}^H$, $\boldsymbol{HH}^H$, $\boldsymbol{GG}^H$ and the inversion of $S \times S$ and $L \times L$ dimensional matrices. Moreover, to obtain the MMSE relay matrix, we need to calculate the multiplication of three $L \times L$, $L \times S$ and $S \times S$ dimensional matrices and calculate the roots of a $2M$th-order polynomial [26]. For the WF, BUE and DME relay matrices, the calculation of $\boldsymbol{R_{GE}}$ and $\boldsymbol{R_{Hs}}$ is required. Since the singular vectors of the WF and BUE matrices are the eigenvectors of $\boldsymbol{R_{GE}}$ and $\boldsymbol{R_{Hs}}$, the EVD of these $L \times L$ and $S \times S$ matrices are required for the WF and BUE methods (see [27, Section 5.4.5] for the order of the CC). The CC of the WF matrix is higher than that of




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the BUE, as in the WF, the singular values of the relay matrix are the solutions of the water filling equation [6, see (49) and (50)]. The bisection or Newton's methods may be used to obtain the singular values of the WF matrix. Finally, the proposed DME only requires the dominant left singular vectors of $\boldsymbol{GE}$ and $\boldsymbol{HR_s}$. Thus, its CC is significantly less than all other methods (except for the MF method) since computing the dominant singular vector of a matrix can be performed using some efficient approaches, for example, the power method [27, Section 7.3.1].

In all the simulations except for Fig. 6, we have $M = N = L = S$. We use $\boldsymbol{E} = \boldsymbol{I}_N$ and $\boldsymbol{R_s} = P_s/M\boldsymbol{I}_M$ when these two matrices are not optimally designed. The mean of the instantaneous SNR and the instantaneous capacity averaged over 7000 independent channel realisations are shown in Figs. 1 and 2 as a function of $d_H$ when $d_G = d_H$, $P_T = 2$, and $M = 4$. In this simulation, various noise powers at the relay and destination are assumed. The results confirm that the DME matrix (23) gives the maximum SNR, and it is seen that the proposed method for assigning the power budget to the relay and the source significantly increases not only the SNR, but also the system capacity. The study of the system capacity and SNR reveals that the DME matrix with the P-CSI is a proper relaying approach when the channels are highly correlated (when $d_H$ and $d_G$ are small). It is also seen that the capacity obtained with the DME is near to that with the WF when the channels are highly correlated or the noise variance is high.

Fig. 3 shows the effect of the power budget allocation and the STC matrix. As expected, the best performance is achieved by the joint optimisation of the power budget and matrices. However, the performance gain from this optimisation depends on the noise power and channels conditions. For $\sigma_n^2 = 4$, $\sigma_v^2 = 0.25$, we observe an improvement of about 5.4 dB for the STC design. This is because $(N + \mathrm{SNR_{rd}})/(N\mathrm{SNR_{sr}})$ becomes a large number, and as discussed in Section 3.3, we expect the performance improvement to be around $10\log_{10}(M) = 6$ dB.

Fig. 4 shows the empirical probability density function (pdf) of the instantaneous SNR for the two sets of the noise variances and uncorrelated channel matrices, $E[\boldsymbol{HH}^H] = E[\boldsymbol{GG}^H] = \boldsymbol{I}_4$. This figure reveals that the achieved SNR (which is a function of the random channels) exhibits a log-normal distribution. We observe that the optimisation of both $\boldsymbol{R_s}$ and the power budgets results in a significant shift to the distribution of the achieved SNR. Note that the widths of these distributions are almost equal.

In previous simulations, $M = 4$; Fig. 5 shows the effect of the number of antenna on the pdf of the achievable SNR of

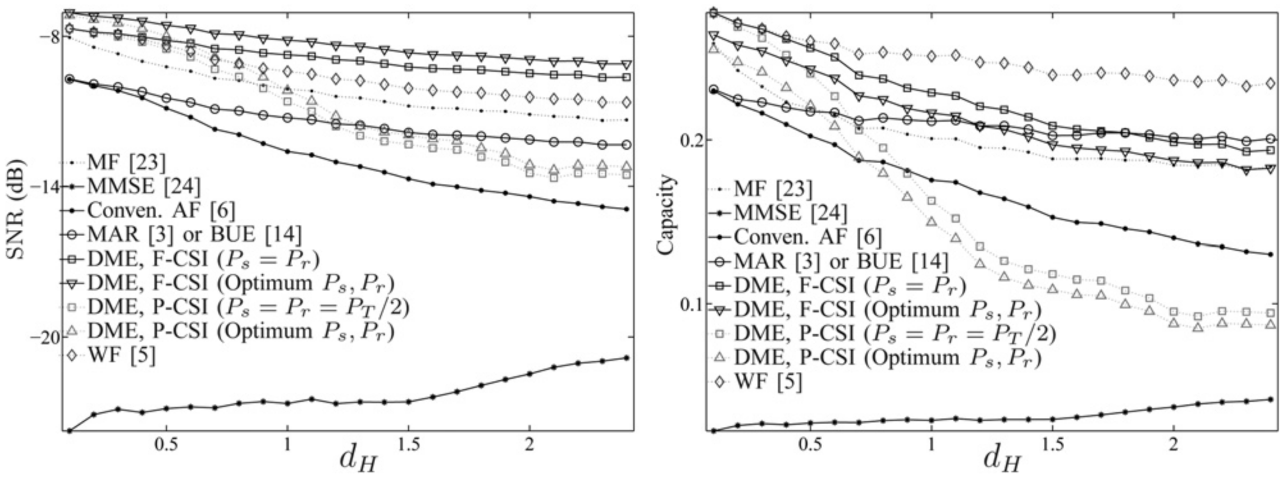


**Fig. 1** *Average of the instantaneous SNR as a function of $d_H$ (left) and average of the instantaneous capacity as a function of $d_H$ (right) for $\sigma_n^2 = 6$ dB and $\sigma_v^2 = -6$ dB*

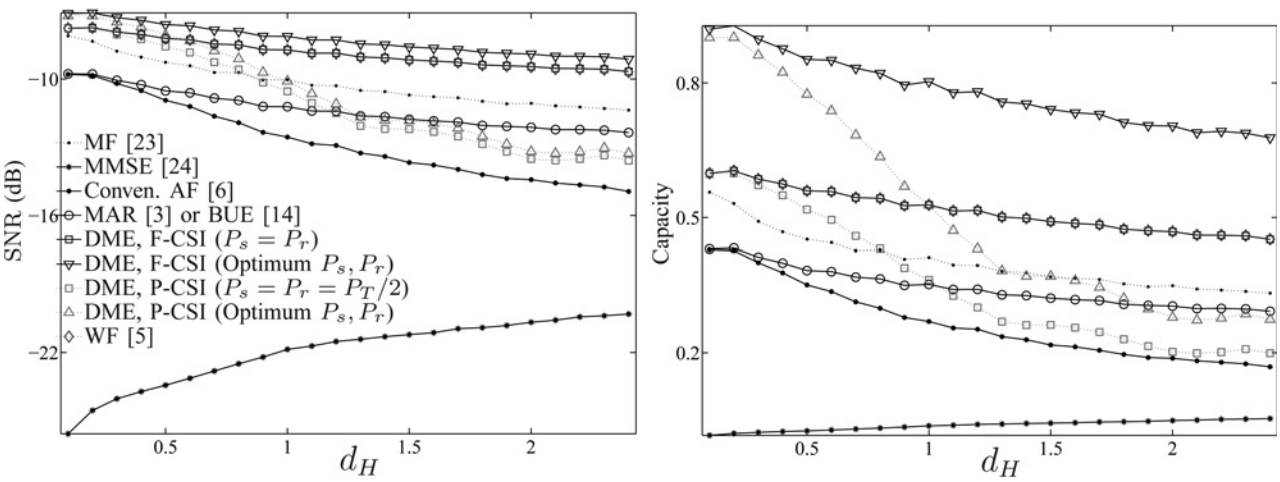


**Fig. 2** *Average of the instantaneous SNR as a function of $d_H$ (left) and average of the instantaneous capacity as a function of $d_H$ (right) for $\sigma_n^2 = -6$ dB and $\sigma_v^2 = 6$ dB*

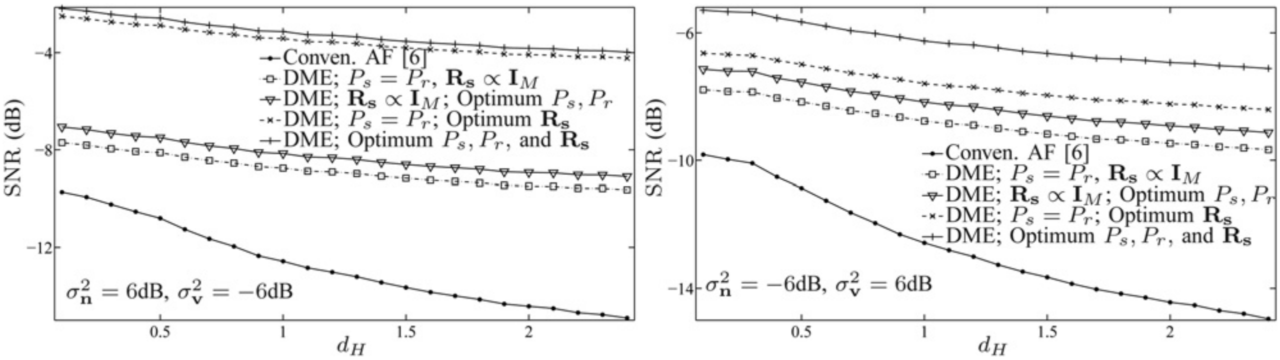


**Fig. 3** *Average of the instantaneous SNR as a function of $d_H$ for various $\sigma_n^2$ and $\sigma_v^2$ when the DME with the F-CSI is used*

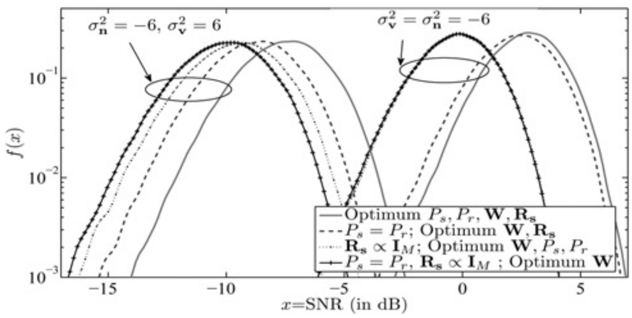


**Fig. 4** *Empirical pdf of $x = 10\log_{10}$SNR in dB for the uncorrelated channels when $\sigma_n^2 = -6$ dB, $M = N = L = S = 4$, $P_T = 2$ and the optimal relay matrix (DME) is used*

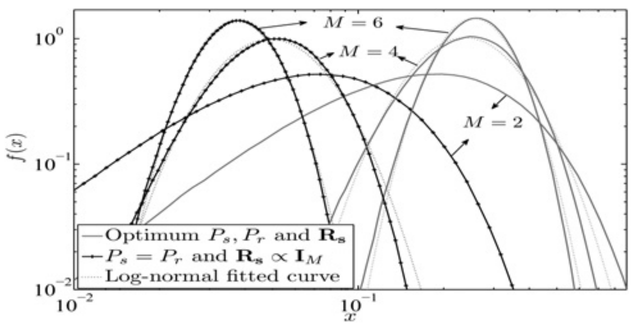


**Fig. 5** *Empirical pdf of the instantaneous $x = $SNR for the uncorrelated channels when $\sigma_n^2 = 10$ dB, $\sigma_v^2 = -10$ dB, $M = N = L = S$, $P_T = 2$ and the optimal relay matrix (DME) is used*

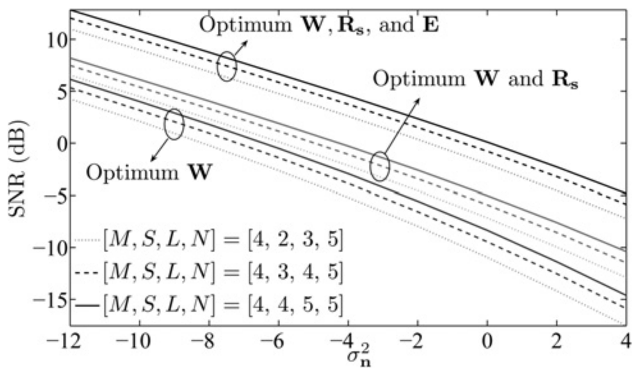


**Fig. 6** *Average of the instantaneous SNR of the DME with F-CSI as a function of $\sigma_n^2$ for various number of antennas at the nodes, when all the channel matrices are uncorrelated and $\sigma_n^2 = \sigma_v^2$.*

the system for uncorrelated channels, $M = N = L = S$, $\sigma_n^2 = 10$ dB and $\sigma_v^2 = -10$ dB. This figure shows that the optimization of both $\boldsymbol{R_s}$ and the optimal power allocation results in a shift in the pdf of the SNR, which is equivalent to 4.05, 6.81 and 8.35 dB improvement in the mean of the achieved SNR for $M = 2$, $M = 4$ and $M = 6$, respectively. Obviously, this improvement is more than $10\log_{10}(M)$dB. As the number of antennas increases, the proposed optimisation methods result in more significant improvement. Note that for the large values of $(N + \text{SNR}s_{\text{rd}})/(N\text{SNR}_{\text{sr}})$, we have shown that the source optimisation shall lead to approximately $10\log_{10}(M)$dB improvement in the SNR; for this simulation, 2.93, 5.88 and 7.56 dB improvements in the mean of the SNR are achieved, respectively, for $M = 2$, $M = 4$ and $M = 6$. We also observe that the empirical pdf of the achieved SNR accurately fits a log-normal distribution for large $M$, such as $M = 4$ and 6. Fig. 5 also shows that the widths of these estimated pdfs decrease as the number of antenna increases. Particularly, in a jointly optimised system, the mean of $10\log_{10}$(SNR) is increasing in $M$; whereas, its variance is decreasing. This simply means that the joint optimisation of the involved variables results in the significant improvement of the outage probability performance of the system.

Fig. 6 shows the average of the instantaneous SNR of the DME against $\sigma_n^2$ using the F-CSI for unequal number of antennas at the different nodes where channel matrices are uncorrelated, $\sigma_n^2 = \sigma_v^2$. In all the other simulations, we have $M = S = L = N$. This figure shows the amount of the improvement gained from the nodes optimisation in the DME method. For example, the achievable SNR is increased by more than 3 dB by optimising the source. This figure reveals that the achievable SNR of the system is an increasing function of the number of the relay's antennas. For instance, the achievable SNR increases almost 2 dB when the number of relay's antennas is increased from $[S = 2, L = 3]$ to $[S = 4, L = 5]$.

We now evaluate the proposed method in a multi-user environment for $K$ multi-antenna S–R–D triplets in the system. For all nodes, we consider the same number of antenna elements, $M_i = L_i = S_i = N_i = M$. We assume that all the additive noise vectors have the same power, and equal power is transmitted from all sources and relays; $\sigma_{v,i}^2 = \sigma_{n,i}^2 = \sigma_v^2 = \sigma_n^2$ and $P_{s,i} = P_{r,i} = P_s = P_r = 1$ for all $i = 1, \ldots, K$. All the elements of all the channel matrices are generated as independent of each other with ZM-CSCG distribution. The relays have only the knowledge of their own forward and backward channels, and they either use the DME or WF matrix for the relaying. We assume that the S–R channels are orthogonal. This orthogonality can be



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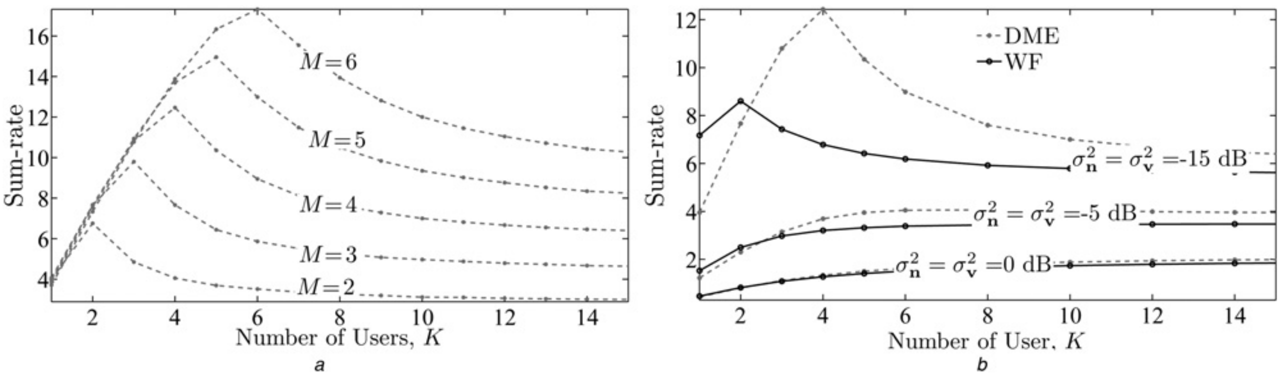


**Fig. 7** *Average of the sum-rate as a function of the number of the users K when all the channels are uncorrelated, $\sigma_n^2 = \sigma_v^2$, and all the nodes have M antennas*

*a* Performance of the DME for $\sigma_n^2 = -15$ dB
*b* Performance of the DME and WF for $M = 4$

achieved by assigning different subcarriers in orthogonal frequency division multiplexing systems or assigning orthogonal codes to the sources and relays. As a result, each relay amplifies the signal corresponding to only one source (there is no cross-interference at the relays). However, the cross-interferences at the destinations are taken into account. We assume that all signals have Gaussian distribution and express the sum-rate as

$$C = \sum_{i=1}^{K} C_i = \sum_{i=1}^{K} \log_2 \det\left(\boldsymbol{I}_N + \boldsymbol{R}_{\boldsymbol{G}_{i,i}^H \boldsymbol{W}_i \boldsymbol{H}_{i,i} \boldsymbol{s}_i} \boldsymbol{R}_{I,i}^{-1}\right)$$

where $\boldsymbol{s}_i$ is the transmitted signal from the $i$th source with $E\left[\boldsymbol{s}_i \boldsymbol{s}_i^H\right] = P_s/M\boldsymbol{I}_M$, $\boldsymbol{W}_i$ is the RT matrix at the $i$th relay, $\boldsymbol{H}_{i,j}$ is the channel matrix between the $j$th source and $i$th relay, $\boldsymbol{G}_{i,j}^H$ is the channel matrix between the $j$th relay and $i$th destination with $E\left[\boldsymbol{H}_{i,j}\boldsymbol{H}_{k,l}^H\right] = \delta(i-k)\delta(j-l)\boldsymbol{I}_M$, $\mathrm{E}\left[\boldsymbol{G}_{i,j}\boldsymbol{G}_{k,l}^H\right] = \delta(i-k)\delta(j-l)\boldsymbol{I}_M$ for $i, j, k, l = 1, \ldots, K$, and

$$\boldsymbol{R}_{I,i} = \sigma_v^2 \boldsymbol{I}_N + \sigma_n^2 \sum_{j=1}^{K} \boldsymbol{R}_{\boldsymbol{G}_{i,j}^H \boldsymbol{W}_j} + \frac{P_s}{M} \sum_{j=1, j \neq i}^{K} \boldsymbol{R}_{\boldsymbol{G}_{i,j}^H \boldsymbol{W}_j \boldsymbol{H}_{j,j}}$$

Fig. 7*a* shows the average of the sum-rate over 5000 independent simulation runs as a function of the number of users $K$ in the system. The proposed rank-one DME relaying matrix is used in this simulation for $\sigma_v^2 = \sigma_n^2 = -15$ dB. This figure shows that the sum-rate increases as the number of users $K$ increases for $K \leq M$. However, for $K > M$, because of the impact of the cross interferences, the sum-rate decreases as $K$ increases. We conclude that in this simulation, the best number of triplets is the number of antennas. Fig. 7*b* shows the sum-rate of two systems in terms of the number of triplets and for different noise levels and $M = 4$. In the first system, each relay, without considering the other triplets, maximises its own link capacity by using the WF matrix. In the second system, the proposed DME is used at each relay ignoring the other triplets and the individual link SNRs are maximised. We observe that at low SNRs, the performances of these systems are similar. This is because the WF matrix tends to the DME matrix as SNR reduces. For a single user $K = 1$, the first system is designed to be optimal and outperforms the second, particularly for small noise variances. Interestingly, the second system significantly outperforms the first one in a multi-user environment. This is because the DME gives a rank-one matrix, that is, each relay emits power only in a 1D subspace. As a result, significantly fewer cross-interferences among triplets are produced using the DME. Therefore maximising the link SNR is a better method than maximising the link capacity in the multi-user scenario where triplets are unaware of each other and selfishly attempt to improve their own link.

## 6 Conclusion

In this paper, we considered a relay communication system equipped with multiple antennas. The relay forwards a linear transformation of its received signal to the destination. We optimised the STC and relaying weights by maximising the end-to-end SNR of the system, assuming different constraints on the transmitted powers. We have proved that for a given STC matrix, to maximise the SNR, the relay has to assign all its power in the direction of the dominant eigenvectors of the S–R and R–D channel matrices. Given the RT matrix, we derived the optimal STC matrix. Finally, we have jointly optimised the STC and RT matrices to maximise the SNR. By optimising the STC matrix, we obtain about $10\log_{10}(M)$ dB improvement in the SNR. Moreover, we proved that there always exist rank-one matrices for the STC and RT that maximise the SNR. Using this optimal solution, the signals are transmitted only over the best channel subspaces, and the other directions are left empty, which could be utilised by other users. We showed that the maximisation of the mutual information (between the source and destination) over all rank-one relaying matrices gives the same solution obtained by the SNR maximisation without such a rank constraint. We also found the optimal power allocation among the source and relay, given a total transmit power budget. The optimal allocated budgets can be calculated only as a function of the total available power and the link qualities, where the channel link quality is proportional to the ratio of the channel's dominant eigenvalue to the noise power at the output of the channel. Using computer simulation, we evaluated the effect of the STC and RT matrices on the end-to-end SNR and capacity of the system. We observed that when the elements of the channel matrices are highly correlated, the average and instantaneous SNR maximisation leads to a similar result. At low SNRs and for the correlated channels, the SNR maximisation is optimal in terms of the capacity of a single user and is spectrally efficient. In the multi-user scenario, interestingly, we observed that the link SNR maximisation results in a significant increase in the sum-rate compared with the link capacity maximisation, where users selfishly try to optimise their own link qualities.

## 7 Acknowledgment

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## 9 Appendix

### 9.1 Appendix 1: Proof of Theorem 1

*Proof:* We define the Lagrangian function for (10) as

$$\mathcal{L} = \frac{\operatorname{tr}\left(\boldsymbol{R_{Hs}}\boldsymbol{W}^H\boldsymbol{R_{GE}}\boldsymbol{W}\right)}{\operatorname{tr}\left(\boldsymbol{W}^H\boldsymbol{R_{GE}}\boldsymbol{W}\right)+\operatorname{tr}\left(\boldsymbol{R_E}\right)\sigma_v^2/\sigma_n^2} + \mu\left(\operatorname{tr}\left(\boldsymbol{W}\boldsymbol{R_{Hs}}\boldsymbol{W}^H\right)+\sigma_n^2\operatorname{tr}\left(\boldsymbol{W}\boldsymbol{W}^H\right)-P_{\mathrm{r}}\right) \tag{45}$$

where $\mu$ is the Lagrange multiplier for the constraint in (10). Setting $\partial L/\partial \boldsymbol{W}^H = 0$ leads to

$$\left(a_1\boldsymbol{R_{GE}}+\mu\boldsymbol{I}_L\right)\boldsymbol{W}\boldsymbol{R_{Hs}} = \left(a_2\boldsymbol{R_{GE}}-\mu\sigma_n^2\boldsymbol{I}_L\right)\boldsymbol{W} \tag{46}$$

where $a_1 = \left(\operatorname{tr}\left(\boldsymbol{W}^H\boldsymbol{R_{GE}}\boldsymbol{W}\right)+\operatorname{tr}\left(\boldsymbol{R_E}\right)\sigma_v^2/\sigma_n^2\right)^{-1}$ and

$$a_2 = \frac{\operatorname{tr}\left(\boldsymbol{R_{Hs}}\boldsymbol{W}^H\boldsymbol{R_{GE}}\boldsymbol{W}\right)}{\left(\operatorname{tr}\left(\boldsymbol{W}^H\boldsymbol{R_{GE}}\boldsymbol{W}\right)+\operatorname{tr}\left(\boldsymbol{R_E}\right)\sigma_v^2/\sigma_n^2\right)^2}$$

Note that the equality constraint in (10) is not linear, and consequently, the optimisation problem is not convex. Thus, (46) is a necessary condition for the optimal $\boldsymbol{W}$, and the optimal relay matrix must satisfy (46). Using the EVD of $\boldsymbol{R_{Hs}}$ and $\boldsymbol{R_{GE}}$ in (12), we can express (46) as

$$\left(a_1\operatorname{diag}\left(\boldsymbol{\lambda^{GE}}\right)+\mu\boldsymbol{I}_L\right)\widetilde{\boldsymbol{W}}\operatorname{diag}\left(\boldsymbol{\lambda^{Hs}}\right) = \left(a_2\operatorname{diag}\left(\boldsymbol{\lambda^{GE}}\right)-\mu\sigma_n^2\boldsymbol{I}_L\right)\widetilde{\boldsymbol{W}}$$

where $\widetilde{\boldsymbol{W}} \overset{\text{def}}{=} \boldsymbol{U}_{\boldsymbol{GE}}^H\boldsymbol{W}\boldsymbol{V}_{\boldsymbol{Hs}}$. Thus, for $i = 1, \ldots, L$ and $j = 1, \ldots, S$ we have

$$\left(\left(a_1\lambda_i^{\boldsymbol{GE}}+\mu\right)\lambda_j^{\boldsymbol{Hs}} - a_2\lambda_i^{\boldsymbol{GE}}+\mu\sigma_n^2\right)[\widetilde{\boldsymbol{W}}]_{i,j} = 0 \tag{47}$$

Using the above equation, we prove that

(1) Each column of $\widetilde{\boldsymbol{W}}$ contains no more than one non-zero element

if $[\widetilde{\boldsymbol{W}}]_{m,n} \neq 0$, we prove that $[\widetilde{\boldsymbol{W}}]_{k,n} = 0$, $\forall k \in \{1, \ldots, S\} \backslash \{m\}$. We assume that $[\widetilde{\boldsymbol{W}}]_{m,n}$ and $[\widetilde{\boldsymbol{W}}]_{k,n}$ are nonzero. From



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$[\widetilde{\boldsymbol{W}}]_{m,n} \neq 0$ and (47) we have

$$\left(a_1\lambda_m^{\boldsymbol{GE}} + \mu\right)\lambda_n^{\boldsymbol{Hs}} - a_2\lambda_m^{\boldsymbol{GE}} + \mu\sigma_n^2 = 0 \tag{48}$$

which leads to $\mu = \left(\lambda_m^{\boldsymbol{GE}}\left(a_2 - a_1\lambda_n^{\boldsymbol{Hs}}\right)\right)/\left(\lambda_n^{\boldsymbol{Hs}} + \sigma_n^2\right)$. From $\left[\widetilde{\boldsymbol{W}}\right]_{k,n} \neq 0$ and (47), we obtain

$$\left(a_1\lambda_k^{\boldsymbol{GE}} + \mu\right)\lambda_n^{\boldsymbol{Hs}} - a_2\lambda_k^{\boldsymbol{GE}} + \mu\sigma_n^2 = 0 \tag{49}$$

Substituting $\mu = \left(\lambda_m^{\boldsymbol{GE}}\left(a_2 - a_1\lambda_n^{\boldsymbol{Hs}}\right)\right)/\left(\lambda_n^{\boldsymbol{Hs}} + \sigma_n^2\right)$ in (49) leads to

$$\left(\lambda_k^{\boldsymbol{GE}} - \lambda_m^{\boldsymbol{GE}}\right)\left(a_1\lambda_n^{\boldsymbol{Hs}} - a_2\right) = 0 \tag{50}$$

The probability of $\left(\lambda_k^{\boldsymbol{GE}} - \lambda_m^{\boldsymbol{GE}}\right) = 0$ is zero. Thus, $a_1\lambda_n^{\boldsymbol{Hs}} - a_2 = 0$ with probability one and we must have

$$\lambda_n^{\boldsymbol{Hs}} = \frac{a_2}{a_1} = \frac{\operatorname{tr}\left(\operatorname{diag}\left(\boldsymbol{\lambda}^{\boldsymbol{Hs}}\right)\widetilde{\boldsymbol{W}}^H \operatorname{diag}\left(\boldsymbol{\lambda}^{\boldsymbol{GE}}\right)\widetilde{\boldsymbol{W}}\right)}{\operatorname{tr}\left(\widetilde{\boldsymbol{W}}^H \operatorname{diag}(\boldsymbol{\lambda}^{\boldsymbol{GE}})\widetilde{\boldsymbol{W}}\right) + \operatorname{tr}(\boldsymbol{R_E})\sigma_v^2/\sigma_n^2}$$

with probability one. There may be some $\widetilde{\boldsymbol{W}}$ which satisfy (47) with more than one nonzero element in the $n$th column. However, we show that such a matrix is a non-optimal extremum. From $\lambda_n^{\boldsymbol{Hs}} = a_2/a_1$, we have $\mu = \left(\lambda_m^{\boldsymbol{GE}}\left(a_2 - a_1\lambda_n^{\boldsymbol{Hs}}\right)\right)/\left(\lambda_n^{\boldsymbol{Hs}} + \sigma_n^2\right) = 0$. Using $\mu = 0$ in (47), we have $a_1\lambda_i^{\boldsymbol{GE}}\left(\lambda_j^{\boldsymbol{Hs}} - \lambda_n^{\boldsymbol{Hs}}\right)\left[\widetilde{\boldsymbol{W}}\right]_{i,j} = 0$ for all $i$, $j$. The probability of $\lambda_j^{\boldsymbol{Hs}} - \lambda_n^{\boldsymbol{Hs}} = 0$ is zero for $j \neq n$ and $a_1\lambda_i^{\boldsymbol{GE}} \neq 0$; thus, we must have $[\widetilde{\boldsymbol{W}}]_{i,j} = 0$ for all $j \neq n$, that is, all columns of $\widetilde{\boldsymbol{W}}$ are zero excluding the $n$th column. For such a matrix, we can rewrite (10) as

$$\begin{aligned} &\max_{\{[\widetilde{\boldsymbol{W}}]_{i,n}\}} \frac{\lambda_n^{\boldsymbol{Hs}} \sum_{i=1}^{\min\{S,L\}} \lambda_i^{\boldsymbol{GE}}\left|[\widetilde{\boldsymbol{W}}]_{i,n}\right|^2}{\sum_{i=1}^{\min\{S,L\}} \lambda_i^{\boldsymbol{GE}}\left|[\widetilde{\boldsymbol{W}}]_{i,n}\right|^2 + \operatorname{tr}\left(\boldsymbol{R_E}\right)\sigma_v^2/\sigma_n^2} \\ &\text{s.t. } \left(\lambda_n^{\boldsymbol{Hs}} + \sigma_n^2\right)\sum_{i=1}^{\min\{S,L\}}\left|[\widetilde{\boldsymbol{W}}]_{i,n}\right|^2 = P_{\mathrm{r}} \end{aligned} \tag{51}$$

which is equivalent to the following optimisation problem

$$\begin{aligned} &\max_{\{|[\widetilde{\boldsymbol{W}}]_{i,n}|^2\}} \sum_{i=1}^{\min\{S,L\}} \lambda_i^{\boldsymbol{GE}}\left|[\widetilde{\boldsymbol{W}}]_{i,n}\right|^2 \\ &\text{s.t. } \left(\lambda_n^{\boldsymbol{Hs}} + \sigma_n^2\right)\sum_{i=1}^{\min\{S,L\}}\left|[\widetilde{\boldsymbol{W}}]_{i,n}\right|^2 = P_{\mathrm{r}} \end{aligned} \tag{52}$$

The solution for this linear programming problem is $\left|[\widetilde{\boldsymbol{W}}]_{i,n}\right|^2 = P_{\mathrm{r}}\delta(i-1)/\left(\lambda_n^{\boldsymbol{Hs}} + \sigma_n^2\right)$. Thus, except for the (1, $n$)th element, all the other elements of $\widetilde{\boldsymbol{W}}$ are zero, which is in contradiction with the assumption that $[\widetilde{\boldsymbol{W}}]_{m,n}$ and $[\widetilde{\boldsymbol{W}}]_{k,n}$ are non-zero.

(2) Each row of $\widetilde{\boldsymbol{W}}$ contains no more than one non-zero element

if $[\widetilde{\boldsymbol{W}}]_{m,n} \neq 0$, we prove that $[\widetilde{\boldsymbol{W}}]_{m,q} = 0$, $\forall q \in \{1, \ldots, S\} \backslash \{n\}$. Assume that $[\widetilde{\boldsymbol{W}}]_{m,n} \neq 0$ and $[\widetilde{\boldsymbol{W}}]_{m,q} \neq 0$. Since $[\widetilde{\boldsymbol{W}}]_{m,n}$ and $[\widetilde{\boldsymbol{W}}]_{m,q}$ are non-zero, from (47), we conclude (48) and

$$\left(a_1\lambda_m^{\boldsymbol{GE}} + \mu\right)\lambda_q^{\boldsymbol{Hs}} - a_2\lambda_m^{\boldsymbol{GE}} + \mu\sigma_n^2 = 0 \tag{53}$$

By subtracting (48) from (53) we have

$$\left(a_1\lambda_m^{\boldsymbol{GE}} + \mu\right)\left(\lambda_q^{\boldsymbol{Hs}} - \lambda_n^{\boldsymbol{Hs}}\right) = 0 \tag{54}$$

Since the probability that $\lambda_q^{\boldsymbol{Hs}} = \lambda_n^{\boldsymbol{Hs}}$ is zero, from (54), we have $\mu = -a_1\lambda_m^{\boldsymbol{GE}}$ with probability one. Substituting this $\mu$ in (48) leads to $\left(\sigma_n^2 a_1 + a_2\right)\lambda_m^{\boldsymbol{GE}} = 0$, which is impossible since $\sigma_n^2 a_1 + a_2 > 0$ and $\lambda_m^{\boldsymbol{GE}} \neq 0$. Therefore $\left[\widetilde{\boldsymbol{W}}\right]_{m,n}$ and $\left[\widetilde{\boldsymbol{W}}\right]_{m,q}$ cannot be non-zero, simultaneously.

Thus, $\widetilde{\boldsymbol{W}}$ has up to one non-zero element in each column and each row. Therefore by permutations of the columns and rows of $\widetilde{\boldsymbol{W}}$, we can convert it to a diagonal matrix $\boldsymbol{\Sigma}$. Therefore we have $\widetilde{\boldsymbol{W}} = \boldsymbol{P_G}\boldsymbol{\Sigma}\boldsymbol{P_H}^H$, where $\boldsymbol{P_H}$ and $\boldsymbol{P_G}$ are the permutation matrices. Finally, using the definition of $\widetilde{\boldsymbol{W}}$, we have $\boldsymbol{W} = \boldsymbol{U_{GE}}\widetilde{\boldsymbol{W}}\boldsymbol{V_{Hs}}^H$ and (11). □