Mathematics, Computers and Visual Arts: Some Applications of the Product-Delay Algorithm

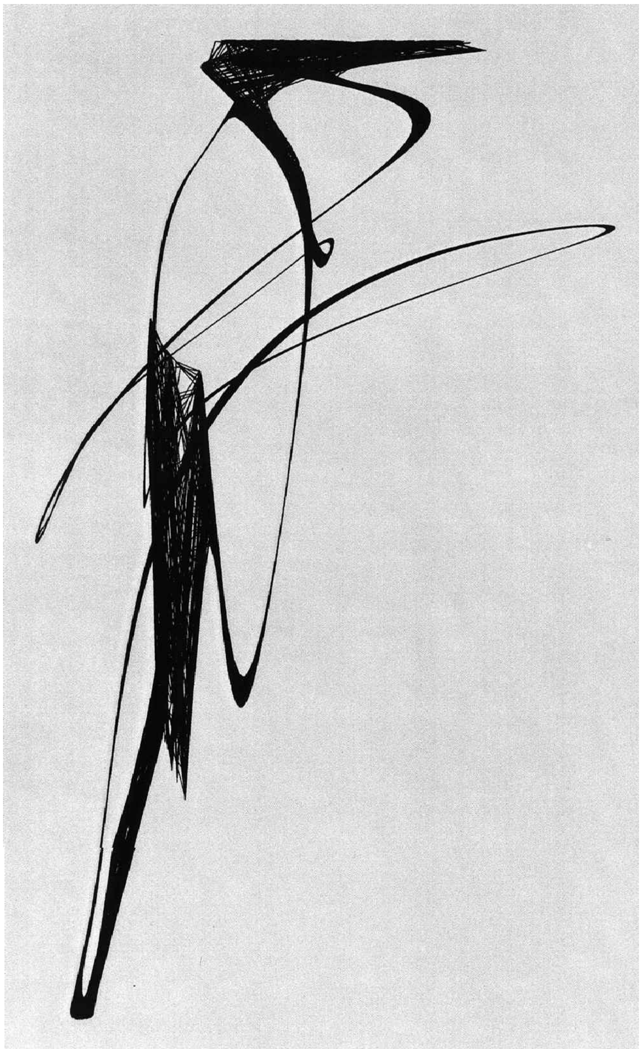
Asok K. Sen

ABSTRACT

The author experiments with a product-delay algorithm as a means of creating graphic designs on a computer. With the product-delay algorithm and a little imagination, it is possible to create a wide variety of artistic patterns, several examples of which are presented here.

During the past three decades there has been a great deal of interest in using computers for the creation of visual arts [1]. Many computer-based approaches have been developed to capture the imaginations of artists. Computers have also been used by artists and scientists to transform complex mathematical formulas into graphic designs. These mathematical formulations include various iteration schemes that are used to generate beautiful fractal patterns [2]. A myriad of interesting patterns can also be created by solving the systems of differential equations of nonlinear dynamics and chaos [3,4].

Fig. 1. Pattern created with the functions given by Equations (1), (2) and (3), with $\delta = 0.0017$. (© Asok K. Sen)



In my recent work [5,6], I have used a product-delay algorithm for creating graphic designs on a computer. The product-delay algorithm is based on the idea of Lissajous figures and may be described as follows. Two functions, $u(t)$ and $v(t)$, are appropriately chosen. These functions are multiplied to form another function, $x(t)$. It is convenient to think of t as time. The product function $x(t)$ is delayed or advanced by a fixed amount of time, yielding a function $y(t)$. These functions are evaluated from $t = 0$ to a final time, and the results are plotted in the $x - y$ plane. With the use of suitable functions, the $x - y$ displays exhibit interesting geometric patterns. The product-delay algorithm has been illustrated with sine and square waves [7], as well as with amplitude-modulated (AM) and frequency-modulated (FM) waves [8]. Using different types of waveforms, one can generate strikingly different patterns with this algorithm. It is also possible to create a wide variety of artistic patterns. In this article I will present a few examples of visual images that can be created with the product-delay algorithm and a little imagination.

The patterns presented below were created using different forms of AM and FM waves. Amplitude- and frequency-modulation processes are used extensively in radio communication [9]. They are also utilized in image processing technology [10], music and speech synthesis [11] and other applications.

All the computations reported here were performed using the software Maple V (Release 4) on a SUN Ultrasparc workstation. In Maple, the number of points at which a function is to be plotted is specified by prescribing a value for `numpoints` in the Plot command. Initially this set of points is equally spaced over the computational domain. Maple uses an adaptive sampling scheme that generates additional points in the neighborhoods where adjacent function values do not lie close to a straight line [12]. I used this adaptive sampling scheme to create the various patterns.

Asok K. Sen (mathematician, teacher), Dept. of Mathematical Sciences, Indiana University, 402 N. Blackford Street, Indianapolis, IN 46202, U.S.A. E-mail: <asen@math.iupui.edu>.

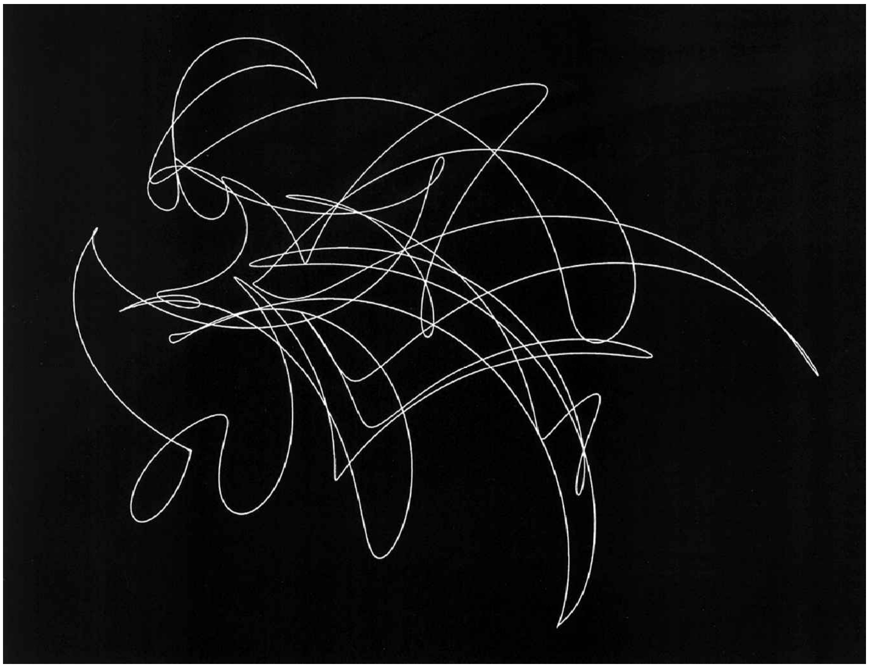
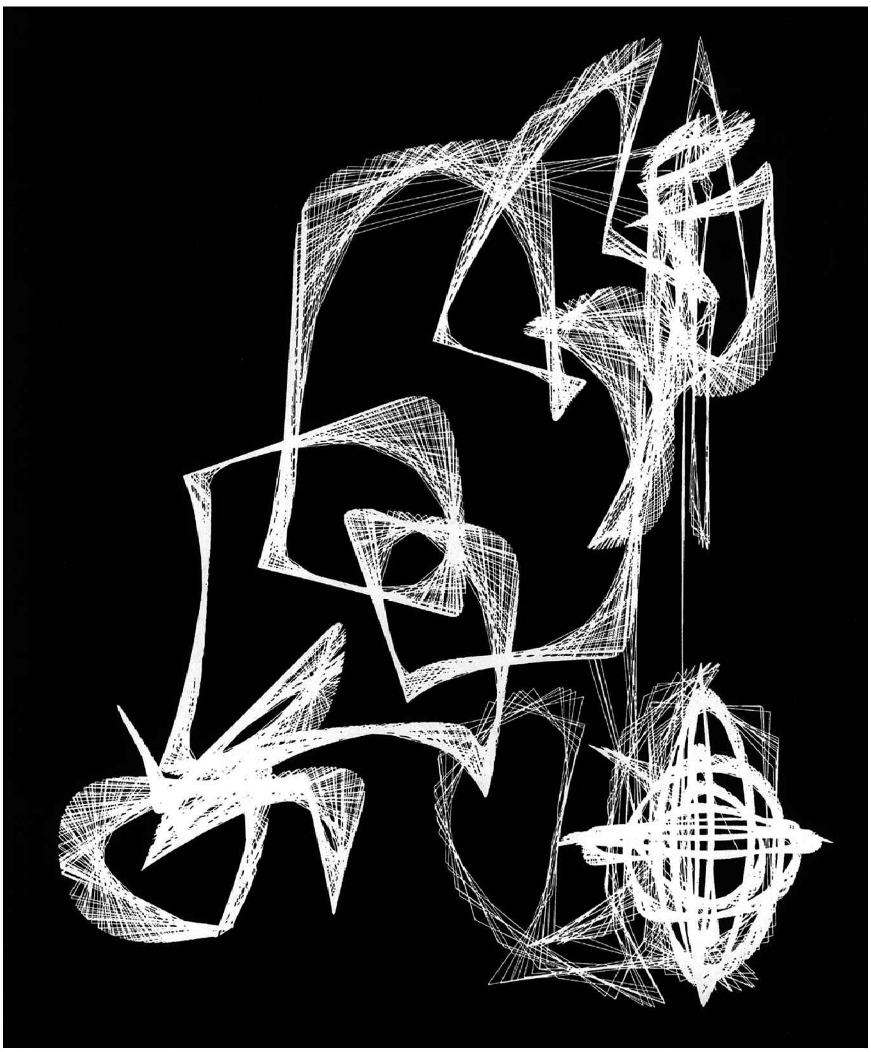


Fig. 2. Pattern generated by the functions given by Equations (3) through (5), with δ = 0.0012. (© Asok K. Sen)

Fig. 3. Pattern generated from Equations (3) through (5), with an offset of magnitude unity introduced into Equations (4) and (5), the frequency of the square wave carrier in Equation (4) changed to 50, and Equation (3) used with δ = 0.00085. (© Asok K. Sen)



VISUAL EXAMPLES

For the first example I will consider a pair of AM and FM waves given by the following functions.

$$u(t) = 1 + [1 + \sin(2\pi.1000t)]\sin(2\pi.1500t) \qquad (1)$$

$$v(t) = 2 + \sin[2\pi.1500t + 0.5\tan(2\pi.500t)]. \qquad (2)$$

The AM waveform in Equation (1) consists of a sine wave carrier of frequency 1500, modulated by a sine wave of frequency 1000 with 100% modulation. Note that a tangent function is used in Equation (2) for frequency modulation. Both $u(t)$ and $v(t)$ have non-zero offsets, equal to 1 and 2, respectively. The functions given by Equations (1) and (2) are used in conjunction with the functions

$$x(t) = u(t)v(t),\ y(t) = x(t - \delta) \qquad (3)$$

with $\delta = 0.0017$ and evaluated from $t = 0$ to $t = 0.25$ with 250 numpoints. The resulting $x - y$ display is shown in Fig. 1. This figure has the appearance of a bird perched on the branch of a tree.

In the next example the following functions are used as $u(t)$ and $v(t)$.

$$u(t) = [1 + 0.5\sin(2\pi.5000t)]\sin(2\pi.500t)/|\sin(2\pi.500t)| \qquad (4)$$

$$v(t) = \sin(2\pi.500t) + 0.25\sin[2\pi.1000t + 2.5\sin(2\pi.5000t)]. \qquad (5)$$

The function $u(t)$ represents an AM wave in which a square wave carrier of frequency 500 is modulated by a sine wave of frequency 5000 with 50% modulation. The function $v(t)$ is a linear combination of a sine wave and an FM wave. Consider Equations (3), (4) and (5) with the delay in Equation (3) chosen as $\delta = 0.0012$. The pattern shown in Fig. 2 is created by evaluating these functions from $t = 0$ to $t = 0.05$ with 20000 numpoints. This figure has the appearance of an insect on a twig.

For the final example, consider the following modifications of the set of Equations (3), (4) and (5). An offset of magnitude unity is introduced in both Equations (4) and (5), the frequency of the square wave carrier in Equation (4) is changed to 50, and Equation (3) is used with $\delta = 0.00085$. Figure 3 results from evaluating these functions from $t = 0$ to $t = 0.25$ with 250 numpoints.

CONCLUDING REMARKS

I have presented a few examples of visual art that can be created with the product-delay algorithm. By selecting functions and parameters, I am able to generate a rich variety of artistic patterns with this algorithm. By virtue of its simplicity, the algorithm can be programmed easily and quickly on a computer.

Acknowledgments

I would like to thank Marc Frantz, John Hicks, Richard Patterson and José Ramos for stimulating discussions.

References and Notes

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Glossary

Amplitude Modulation—an amplitude-modulated (AM) waveform consists of a carrier wave whose amplitude is varied by a modulating wave. When both the carrier and modulating waves are sinusoidal, an AM waveform is customarily expressed as

$$w(t) = a_c\,[1 + \alpha \sin(2\pi f_m t)]\,\sin(2\pi f_c t). \qquad (6)$$

Here a_c represents the carrier amplitude, and f_c and f_m are, respectively, the frequencies of the carrier and modulating waves; a is referred to as the modulation index. Both the carrier and modulating waves may have other shapes, such as square, triangular, etc.

Frequency Modulation—in a frequency-modulated (FM) waveform, the frequency of the carrier wave is controlled by the modulating wave. With a sine wave carrier and a sine wave modulation, an FM waveform may be described by the function

$$w(t) = a_c \sin[2\pi f_c t + \beta \sin(2\pi f_m t)]. \qquad (7)$$

As before, the frequencies of the carrier and modulating waves are denoted by f_c and f_m, respectively. The symbol β is called the index of frequency modulation.

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