Stochastic-Conceptual Models Applied to Number Theory

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Abstract. Concepts are defined as couples (O, A) of sets O and A: the object O (a set of none, or one or more elements) is assigned to the set A of these elements' (common) attributes. The objects change according to the sequence of attributes. Only couples of objects and attributes, that is concepts, are adequate for our world. The connections and links we need in databases and multimedia are expressed, naturally, by concepts, since concepts have been proved to dispose the order of a lattice (more complex and rich than linear and hierarchical ones).

The lattice can be created by two algebraic operations: "intersection" as the multiplication and "symmetric-difference (!)" as the addition (!). There are, also, two other operations: the "union" and the "complement of a concept". Intersection and union (which cannot play the role neither of the addition nor of the multiplication) express *similarities*, while the other two operations express *dissimilarities*. The operation "complement of a concept" expresses the different, the uncommon, the variety. The symmetric-difference of two concepts has been proved to be a "*distance*" between them (in the mathematical sense!).

We must not always see the natural numbers with their linear order (1,2,...,n,n+1,...), but is rather better to give them a more complex structure: the structure of a lattice. We need three operations: "union" of two numbers, "intersection" of two numbers and "complement" of a number.

Research conclusion: the conceptual distance of (O_1, A_1) and (O_2,A_2) and is always a prime number! (conceptual means, from the point of view of the characteristic "divisibility" we are examining now and not the Euclidean or any other distance). This is the unique way the prime numbers are generated: not by unions and intersections(which express similarities), but by distances(differences)! ...

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DEFINITION AND ROLE OF THE CONCEPTUAL DISTANCE

After the seven definitions given in the Mathematical Structure of Concepts (ICNAAM2010 Proceedings), we define now the "conceptual distance". **Definition**. We call distance d(X,Y) of two sets X and Y, the non-negative integer expressing the number of elements of the set $X \underset{\cdot}{+} Y$, that is of their symmetric-difference (in symbols $n(X \underset{\cdot}{+} Y)$).

So, $d(X,Y) = n(X \underset{\cdot}{+} Y)$.

The three known properties of a distance hold:

$d(X,Y)= n(X \underset{\cdot}{+} Y) \geq 0$ and $d(X,X) = n(X \underset{\cdot}{+} X)= n(\Phi)=0$

$d(X,Y)= n(X \underset{\cdot}{+} Y) = n(X \underset{\cdot}{+} Y) = d(Y,X)$, since $X \underset{\cdot}{+} Y = Y \underset{\cdot}{+} X$

$d(X,Y) + d(Y,Z) = n(X \underset{\cdot}{+} Y) + n(Y \underset{\cdot}{+} Z)$.

We observe that $(X \underset{\cdot}{+} Y) \underset{\cdot}{+} (Y \underset{\cdot}{+} Z) = X \underset{\cdot}{+} Z$. Generally, $A \underset{\cdot}{+} B=(A \cup B) - (A \cap B)$, but $n(A \underset{\cdot}{+} B) = [n(A) + n(B) - n(A \cap B)] - n(A \cap B) = n(A) + n(B) -2n(A \cap B)$.

So, $n(X \underset{\cdot}{+} Z) = n(X \underset{\cdot}{+} Y)+ n(Y \underset{\cdot}{+} Z) - 2n\,[(X \underset{\cdot}{+} Y) \cap (Y \underset{\cdot}{+} Z)] \Rightarrow n(X \underset{\cdot}{+} Y) + n(Y \underset{\cdot}{+} Z) \geq n(X \underset{\cdot}{+} Z)$.

Consequently, the third property is valid.

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Let's go, now, to the concepts. We can take $d(O_1, O_2)=n\ (O_1 \dot{+} O_2)$, which is a distance between objects, but it does not say many things, since it is quantitative but not qualitative: two sets of objects may have many different elements, coming from the same homogenous population (Biometry, Psychometry, students' and teachers' evaluation ....). Besides, we are not working with objects or attributes, but with both of them, that is concepts. The symmetric-difference $O_1 \dot{+} O_2$ of the objects, has the icon $(A_1 \dot{+} A_2)^c$. So, if we want the real distance of O_1 and O_2, we must check $(A_1 \dot{+} A_2)^c$. $d(A_1,A_2)=n(A_1 \dot{+} A_2)=n(\Omega') - n((A_1 \dot{+} A_2)^c)$, where Ω' is the set of all attributes (in our certain application). So, $n((A_1 \dot{+} A_2)^{\circ})=n(\Omega') - d(A_1,A_2)$. $n(\Omega')$ is a constant. Consequently, if the distance of the attributes is increasing, $n((A_1 \dot{+} A_2)^c)$ is decreasing and the distance of the objects is, accordingly, decreasing. The explanation comes naturally: if we have a large range of attributes, this range can fit only to a small range of objects (larger intension, smaller extension). It is the same with statistical analysis by zones.

How can we succeed to have small distance between two objects? When $d(A_1, A_2)$ is large, or, equivalently, when $n(A_1 \dot{+} A_2)$ is large, which refers to the intersection $(A_1 \cap A_2)$ and "the area out of $A_1 \cup A_2$", that is the set $(A_1 \cup A_2)^c$. This second set is a fuzzy factor in the definition or comparison of concepts.

If it is large, then $d(O_1, O_2)$ is small, maybe $d(O_1, O_2) \approx 0$, which means, not that $O_1=O_2$ (*on the contrary!...*), but that O_1 and O_2 *can be easily confused and considered as equal*! In many applications, we do not know exactly O_1 , O_2 , A_1 , A_2, but only their common elements and their differences. Instead of "fishing" (*stochastically!...*) in the "area" of $(A_1 \cup A_2)^c$, is better to try to maximize $A_1 \cap A_2$ (*except if we take the risk ...*).

The more attributes two objects have in common, the more they are alike, or almost equal, that is $d(O_1, O_2) \approx 0$. In the same time, we try to minimize or, at least, keep below a certain level, the "area" $(A_1 \cup A_2)^c$ (since, we can never extinguish this fuzzy factor). It is exactly what happens in Statistics with the mistakes of I or II kind and the levels of significance.

The symmetric-difference shows us the dissimilarities and the intersection the similarities among our objects. Two concepts (O_1, A_1), (O_2, A_2) and D their symmetric-difference are located on the same level and, consequently, *there is no order among them*. What does the symmetric - difference say? When we "go out of" O, nothing is sure for the attributes of O^c. We can be sure only for a finite number of objects, but we cannot be sure for all the others. This means that, for foreign sets of objects, we cannot be sure for their common attributes: except of the known ones $A_1 \cap A_2$, there may exist other unknown (because they belong to the complement of $A_1 \cup A_2$). Consequently, enumeration, *in this area, is* impossible. (Sotiropoulos, ICNAAM2010 Proceedings).

At this point I would like to emphasize the connection of the fuzzy nature of concepts with what Prof. Ivar Ekeland declares in his book "Mathematics and the unexpected" (The University of Chicago Press, 1988). I cite some phrases: "the limits of the capabilities of computations", "the separation of the quantitative estimation from the qualitative *understanding*", "even inside the most exact and ambitious mathematical models, there exists enough place for the unexpected", "a new type of model that will show us the possibilities of the future, without foreseeing just one that will happen", "curves that looked like clearly and exactly sketched, can be resolved in a fuzzy area", "*we have passed from identity to resemblance*", "a deterministic system can show random behaviour, if part of the information is unknown" and many others of the same kind.

APPLICATION TO NUMBER THEORY (PRIME NUMBERS)

How natural numbers are generated? Is there a deterministic rule by which they are generated, or they appear stochastically (at least, some of them)? We must not always see the natural numbers with their linear order (1,2,...,n,n+1,...), but is rather better to give them a more complex structure: the structure of a lattice. We need three operations: "union" of two numbers, "intersection" of two numbers and "complement" of a number.

A way to do that is to define as "union" the Least Common Multiple (LCM) and the "intersection" as the Maximum Common Divisor (MCD). But, what is the "complement" of a number? If we have no complement, we have no symmetric-difference, consequently, we have no distance(...), all the natural numbers seem to be ordered by these two operations(but, are, really, all?). Besides, the symmetric-difference plays the role of "addition" in the Algebra Boole which comes from the Lattice Structure (Order). So, something is missing... If we go from the natural numbers to the integers [by using the new characteristic "minus" (-)], then we could say that the complement of x is

–x and the distance of x and y is $|x-y|$, but –x has to do with the linearity of the integers and not with the characteristic of divisibility (like LCM and MCD). In a way, -x expresses quantity but not quality.

At this point, we understand that we must use characteristics and express numbers as concepts[that is, couples (O,A) with their four operations, as defined, e.g., in Ref. 17 below]. Every number is an isolated object o and A is its set of attributes [according to the one or more characteristic(s) we use]. By using the four operations, we get other numbers, sets O of numbers and so on (as results of the application of the Mathematical Theory of Concepts). In this way, the set of integer numbers becomes a network (especially, a lattice), not just a line, and unfolds the whole of its richness (in characteristics). Moreover, we can produce classes of numbers, new numbers, classes of classes of numbers, new kinds of numbers and so on. Numbers exist if the appropriate concepts exist. If we have concepts, we get the corresponding numbers. Of course, this happens with every object in our world ("real" or "imaginary"...), not only with the object "number".

We define:

1. as set A of attributes of a number o, the set of all its divisors, 2. as A_i UA_j of two sets A_i and A_j of attributes, their usual set-theoretic union, 3. as the complement A^c of a set A of attributes, the set of all numbers, less than o and different from 1, which are not divisors of o (e.g., if o={30}, then A={1,2,3,6,5,10,15,30} and A^c={4,7,8,9,11,12,13,14,16,17,18,19, 20,21, 22,23,24,25,26,27, 28,29}). Then, it is proved that: 1. (o_1, A_1) $\dot{\cup}$ (o_2,A_2) gives the MCD of the two numbers o_1 and o_2 or, rather, the class of the MCD, 2. (o_1, A_1) $\underset{\bullet}{\cap}$ (o_2,A_2) gives the LCM of the two numbers o_1 and o_2 or, rather, the class of the LCM, 3. the symmetric-difference of (o_1, A_1) and (o_2,A_2) gives the conceptual distance of (o_1, A_1) and (o_2,A_2) and is always a prime number! (conceptual means, from the point of view of the characteristic "divisibility" we are examining now and not the Euclidean or any other distance). This is the unique way the prime numbers are generated: not by unions and intersections (which express similarities), but by distances (differences)! ...

Important results:

We cannot find a mathematical formula in order to compute the prime numbers. We can make a computer program, or just an algorithm (like the famous one of the Greek mathematician and philosopher Eratosthenes), checking sequentially the natural numbers and thus finding sequentially the prime numbers (if runtime is enough...), but there is no formula returning the prime numbers! The primes are the non-ordered elements of the system of concepts (see FIGURE 1. Ref. 17 below), consequently we cannot use the other elements to compute the primes..... (the contrary is true: we use just the primes and the usual multiplication between them to construct the others. The non-ordered cannot be computed and, consequently, cannot be forseen... They are the naughty children of the system! Of course, we must not forget that, in our lattice structure, multiplication is the intersection and, especially, in the lattice of the natural numbers (as defined above), intersection is the LCM.

For the definition of the symmetric-difference (conceptual distance, prime number), we need the union of two concepts (natural numbers, in our case), the intersection of two concepts and the complement of a concept. But the complement is not a deterministic function, because of the *fuzzy factor* (see the first part of this work and No 17 of the References). In the above example, A^c={4,7,8,9,11,12,13,14,16,17,18,19, 20,21, 22,23,24,25,26,27, 28,29}, that is absolutely defined. Then, where is the fuzziness? This happens, because in the above definition 3 of the complement, we had imposed the restriction "less than o". Without this restriction, we go in "the outer area", where the non-divisors of o are of infinite number!... Consequently, we have infinite number of complements, of symmetric-differences, of prime numbers. So, we have obtained a stochastic way to find the prime numbers, but not a deterministic formula or rule.

As the Greek mathematician Euclides has proved, prime numbers are infinite and ever increasing. We understand, now, that: as they increase, they have less frequency of appearance (the possibility to find a prime is decreasing). Really, as the numbers are increasing (in the structure of the lattice), the possibility to find a non-divisor, not already used, becomes smaller.

CONCLUSION

"The music of the prime numbers comes from the discrete echoes of a piano and not from the continuous echoes of a violin"!

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