The Topology of Wireless Communication

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This article studies the topological properties of wireless communication maps and their usability in algorithmic design. We consider the SINR model, which compares the received power of a signal at a receiver against the sum of strengths of other interfering signals plus background noise. To describe the behavior of a multistation network, we use the convenient representation of a reception map, which partitions the plane into reception zones, one per station, and the complementary region of the plane where no station can be heard. *SINR diagrams* have been studied in Avin et al. [2009] for the specific case where all stations use the same power. It was shown that the reception zones are convex (hence connected) and fat, and this was used to devise an efficient algorithm for the fundamental problem of point location. Here we consider the more general (and common) case where transmission energies are arbitrary (or nonuniform). Under that setting, the reception zones are not necessarily convex or even connected. This poses the algorithmic challenge of designing efficient point location techniques for the nonuniform setting, as well as the theoretical challenge of understanding the geometry of SINR diagrams (e.g., the maximal number of connected components they might have). Our key result exhibits a striking contrast between d- and $(d+1)$-dimensional maps for a network embedded in d-dimensional space. Specifically, it is shown that whereas the d-dimensional map might be highly fractured, drawing the map in one dimension higher "heals" the zones, which become connected (in fact, hyperbolically connected). We also provide bounds for the fatness of reception zones. Subsequently, we consider algorithmic applications and propose a new variant of approximate point location.

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1. INTRODUCTION

1.1. Background and Motivation

The use of wireless technology in communication networks is rapidly growing. This trend imposes increasingly heavy loads on the resources required by wireless networks.

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One of the main resources required for such communication is radio spectrum, which is limited by nature. Hence careful design of all aspects of the network is crucial to efficient utilization of its resources. Good planning of radio communication networks must take advantage of all its features, including both physical properties of the channels and structural properties of the entire network. While the physical properties of channels have been thoroughly studied, [Goldsmith 2005; Tse and Viswanath 2005], relatively little is known about the topology and geometry of the wireless network structure and their influence on performance issues.

There is a wide range of challenges in wireless communication for which better organization of the communication network may become useful. Specifically, understanding the topology of the underlying communication network may lead to more sophisticated algorithms for problems such as scheduling, topology control, and connectivity. We study wireless communication in free space; this is simpler than the irregular environment of radio channels in a general setting, which involves reflection and shadowing. We use the *signal to interference-plus-noise ratio (SINR)* model which is widely used by the electrical engineering community and is recently, explored by computer scientists as well. Let,

$$\mathrm{SINR}(s_i, p) = \frac{\psi_i \cdot \mathrm{dist}^{-\alpha}(s_i, p)}{\sum_{j \neq i} \psi_j \cdot \mathrm{dist}^{-\alpha}(s_j, p) + N}.$$

In this model, a receiver at point $p \in \mathbb{R}^d$ successfully receives a message from the sender s_i if and only if $\mathrm{SINR}(s_i, p) \geq \beta$, where N is the environmental noise, the constant $\beta \geq 1$ denotes the minimum SINR required for a message to be successfully received, α is the path-loss parameter, and $S = \{s_1, \ldots, s_n\}$ is the set of concurrently transmitting stations using power assignment ψ. Note that although $\beta < 1$ is also of practical interest, following Avin et al. [2012], we restrict attention to $\beta > 1$, which guarantees that the reception regions of different stations have no overlap and are hence easier to study from the topological point of view.

Within this context, we focus on one specific algorithmic challenge, namely, the point location problem, defined as follows. Given a query point p, it is required to identify which of the n transmitting stations is heard at p, if any, under interference from all other $n-1$ transmitting stations and background noise N. Obviously, one can directly compute $\mathrm{SINR}(s_i, p)$ for every $i \in \{1, \ldots, n\}$ in time $\Theta(n)$ and answer the preceding question accordingly. Yet, this computation may be too expensive (in time) if the query is asked for many different points p. Avin et al. [2009] initiate the study of the topology and geometry of wireless communication in the SINR model, and its application to the point location problem, in the relatively simple setting of *uniform powers*, namely, under the assumption that all stations transmit with the same power level. They show that in this setting, the SINR diagram assumes a particularly convenient form: the reception zones of all senders are convex and "fat." They later exploit these properties to devise an efficient data structure for point location queries, resulting in a logarithmic query-time complexity.

In actual wireless communication systems, however, many wireless communication devices can modify their transmission power [Chiang et al. 2008]. Moreover, it has been demonstrated convincingly that allowing transmitters to use different power levels increases the efficiency of various communication patterns in terms of resource utilization (particularly, energy consumption and communication time). Hence, it is important to develop both a deep understanding of the underlying structural properties and suitable algorithmic techniques for handling various communication-related problems in nonuniform wireless networks as well. In particular, it may be useful to develop algorithms for solving the problem of point location in such networks.

Unfortunately, it turns out that once we turn to the more general case of nonuniform wireless networks, the picture becomes more involved, and the topological features of the SINR diagram are more complicated than in the uniform case. In particular, simple examples (with as few as five stations, as illustrated later on) show that the reception zone of a station is not necessarily connected, and therefore is not convex. Other "nice" features of the problem in the uniform setting, such as fatness, are no longer satisfied as well. Subsequently, algorithmic design problems become more difficult. In particular, the point location problem becomes harder and cannot be solved directly via the techniques developed in Avin et al. [2009] for the uniform case.

In this article, we aim to improve our understanding of the topological and geometric structure of the reception zones of SINR diagrams in the general (nonuniform) case. The difficulty in point location with variable power follows from several independent sources. First, one must overcome the fact that the number of connected cells is not always known (and there are generally several connected cells). A second problem is that the shape of each connected cell is no longer as simple as in the uniform case. Yet another problem is the possibility of singularity points on the boundaries of the reception zones. (Typically, those problems become harder in higher dimensions, but as seen later, this is not always the case for wireless networks.)

Nevertheless, we manage to establish several properties of SINR diagrams in nonuniform networks that are slightly weaker than convexity but are still useful for tackling our algorithmic problems, such as enjoying hyperbolic convexity. To illustrate these properties, let us take a look at the simplest example where a problem already occurs. When we look at two stations in one dimension, the reception zones might not be connected. Surprisingly, when we look at the same example in two dimensions (instead of one), the reception zones of both stations become connected. As shown later on, this is no coincidence. Moreover, when we examine closely the two-dimensional case, we see that the reception zones are no longer convex but actually hyperbolic convex (as opposed to non convex in the one dimensional case). We use this strategy of adding a dimension to the original problem and moving from Euclidean geometry to hyperbolic geometry to solve the point location problem.

1.2. Contributions

In this article, we aim toward gaining better understanding of SINR diagrams with nonuniform power. Better characterization of reception maps has a theoretical as well as practical motivation. The starting point of our work is the following observation: in a nonuniform setting, reception zones are neither convex nor fat. In addition, they are not connected. The loss of these "niceness" properties, previously established for the uniform power setting [Avin et al. 2009], appears even for the presumably simple case, where all stations are aligned on a line.

This raises several immediate questions. The first is a simple "counting" question that has strong implications on our algorithmic question: What is the maximal number of reception cells that may occur in an SINR diagram of a wireless network on n transmitters. The second question has a broader scope: Are there any "niceness" properties that can be established in a nonuniform setting. Specifically, we aim toward finding other (weaker but still useful) forms of convexity that are satisfied by cells in nonuniform reception maps. Apart from their theoretical interest, these questions are also of considerable practical significance, as obviously, having reception zones with some form of convexity might ease the development of protocols for various design and communication tasks. For the general setting where stations are embedded in $\mathbb{R}^d$, the problem of bounding the number of maximal connected components seems to be harder, even for $d = 2$. We are able to show that the number of reception cells is no more than $O(n^{d+1})$ and provide examples with $\Omega(n)$ reception cells for a single station.

Do d-dimensional zones enjoy some form of weaker convexity? Although this remains an open question, we make a major advance in this context.

We consider the $(d+1)$-dimensional SINR diagram of a wireless network whose stations are embedded in d-dimensional space, and establish a much stronger property. It turns out, that while in the d-dimensional space, the network's SINR diagram might be highly fractured, going one dimension higher miraculously "heals" the reception zones, which become connected (in fact, hyperbolically connected or hyperbolically convex). This may have practical ramifications. For instance, considering stations located in the two-dimensional plane, one realizes that their reception zones in three-dimensional space are connected, which aids in answering point location queries in this realistic setting.

Finally, we consider the point location task, defined as follows. Given a set of broadcasting stations S and a point p, we are interested in knowing whether the transmission of station s is correctly received at p. We present a construction scheme of a data structure (per station) that maintains a partition of the plane into three zones: a zone of all points that correctly receive the transmissions of s, that is, points p with $\mathrm{SINR}(s, p) \geq \beta$; a zone where the transmission of s cannot be correctly received, that is, points p with $\mathrm{SINR}(s, p) < \beta$; and a zone of uncertainty corresponding to points that might receive the transmission in a somewhat lower quality, that is, points p with $\mathrm{SINR}(s, p) \geq (1-\varepsilon)^{2\alpha} \cdot \beta$, where ε is predefined performance parameter. Using this data structure, a point location query can be answered in logarithmic time.

1.3. Related Work

In the engineering community, the physical interference (SINR) model has been scrutinized for almost four decades. Focusing on topological, geometric, and algorithmic aspects, our starting point is the work of Avin et al. [2009], where it is proven that in the uniform case, namely, when all transmitters use the same power, the reception zones are convex and fat. Several papers consider the nonuniform case and have shown that the capacity of wireless networks increases when transmitters can adapt their transmission power. In their seminal paper, Agarwal and Erickson [1999], analyze the capacity of wireless networks in the physical and protocol models. Moscibroda [2007] analyze the worst-case capacity of wireless networks, without any assumption on the deployment of nodes in the plane, as opposed to almost all previous works on this problem. Nonuniform power assignments can clearly outperform a uniform assignment [Moscibroda et al. 2006; Moscibroda and Wattenhofer 2006] and increase the capacity of a network. Therefore, the majority of the recent literature on capacity and scheduling addresses nonuniform power.

Since in the uniform setting, the reception zones are convex and fat [Avin et al. 2009], the singularity points of a zone can be easily handled. When transmission power is nonuniform, handling the singularity points becomes a major challenge. We remark that recently, Gabrielov et al. have shown that the number of singular points of functions similar to the interference function is finite [2007]. Maxwell conjectures that the number of singularity points in the interference function is bound by $(n-1)^2$, where n is the number of transmitters; [Maxwell 1954]. For illustration, see Figure 1(a).

Another challenge that one has to deal with in nonuniform networks is the possible existence of regions with a very small gradient in the SINR function, as exemplified in Figure 1(b), which reflects the fact that the area containing all points p such that $\mathrm{SINR}(s_i, p) \in [\beta, \beta+\varepsilon]$ cannot be bounded even for small $\varepsilon > 0$.

It is hoped that a better understanding of the topology of the SINR diagram will improve our understanding of the joint problem of scheduling and power control. The complexity of this problem in the physical model, taking into account the geometry of the problem, is unknown. Nevertheless, many algorithms and heuristics have been

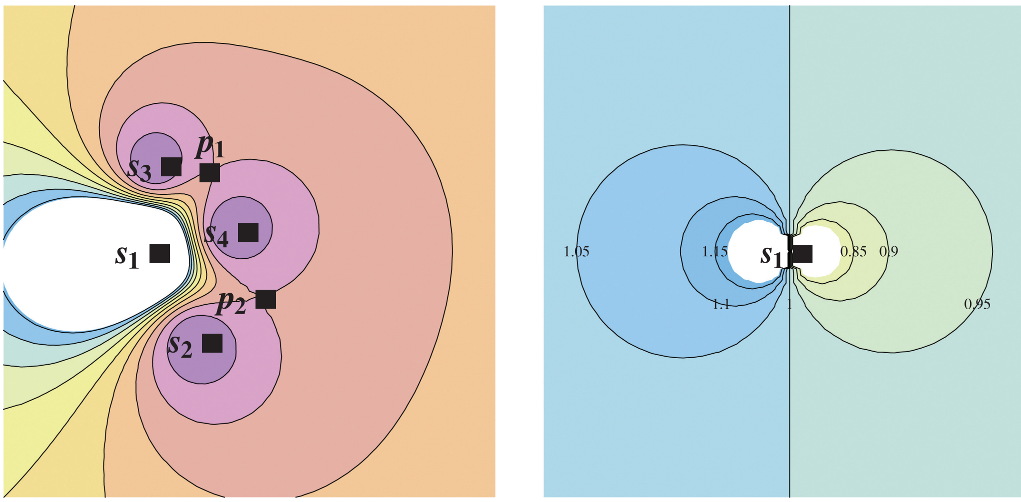

Fig. 1. Topographic view of a reception region (with no noise). Heights indicate SINR thresholds. (a) SINR diagram of 4-station nonuniform power network: singular points (p_1 and p_2) and contour lines of $\text{SINR}(s_1, (x, y))$. (b) Low gradient regions in SINR diagram in 2-station uniform power network. $\mathcal{H}_1$ is unbounded when $\beta \leq 1$ but finite for $\beta > 1$, illustrating the impossibility of getting uniformly bounds of the area between two SINR curves corresponding to two different threshold levels.

suggested (e.g., [Borbash and Ephremides 2006; Cruz and Santhanam 2003; Lee et al. 1995; Moscibroda and Wattenhofer 2006; Wang et al. 2005; Zander 1992]. See Moscibroda et al. [2007] and Goussevskaia et al. [2010] for a more detailed discussion of these approaches. Recently, Kesselheim [2011] has shown how to achieve a constant approximation for the capacity problem with power control, for doubling metric spaces. His algorithm yields $O(\log n)$ approximation for general metrics. Halldorsson and Mitra [2011] show tight characterizations of capacity maximization under power control, using oblivious power assignments in general metrics.

Finally, turning to the stochastic setting, the relation between stochastic SINR diagram (formed by modeling the SINR as a marked point process) and classical stochastic geometry models such as Poisson-Voronoi tessellations, has been studied extensively. See Baccelli and Blaszczyszyn [2009] for a detailed analysis, results, and applications of this approach.

2. PRELIMINARIES

2.1. Geometric Notions

Throughout, we consider the d-dimensional Euclidean space $\mathbb{R}^d$ (for $d \in \mathbb{Z}_{\geq 1}$). The *distance* between points p and point q is denoted by $\text{dist}(p, q) = \|q - p\|$. A *ball* of radius r centered at point $p \in \mathbb{R}^d$ is the set of all points at distance at most r from p, denoted by $B^d(p, r) = \{q \in \mathbb{R}^d \mid \text{dist}(p, q) \leq r\}$. Unless stated otherwise, we assume the two-dimensional Euclidean plane and omit d. The basic notions of open, closed, bounded, compact, and connected sets of points are defined in the standard manner. A point set P is said to be *open* if all points $p \in P$ are internal points, and *closed* if its complement $\bar{P}$ is open. If there exists some real r such that $\text{dist}(p, q) \leq r$ for every two points $p, q \in P$, then P is said to be *bounded*. A *compact* set is a set that is both closed and bounded. The *closure* of P, denoted $cl(P)$, is the smallest closed set containing P. The *boundary* of a point set P, denoted by $\Phi(P)$, is the intersection of the closure of P and the closure of its complement, that is, $\Phi(P) = cl(P) \cap cl(\bar{P})$. Let $L(\Phi(P))$ denote the length of $\Phi(P)$. A *connected set* is a point set P that cannot be partitioned to two nonempty subsets P_1, P_2 such that each of the subsets has no point in common with the closure of the other (i.e., P is connected if for every $P_1, P_2 \neq \emptyset$ such that $P_1 \cap P_2 = \emptyset$ and $P_1 \cup P_2 = P$, either $P_1 \cap cl(P_2) \neq \emptyset$ or $P_2 \cap cl(P_1) \neq \emptyset$.). A *maximal connected subset* $P_1 \subseteq P$ is a connected point set such that $P_1 \cup \{p\}$ is no longer connected for every $p \in P \setminus P_1$.

Unless stated otherwise, a zone refers to the union of an open connected set and some subset of its boundary. It may also refer to a single point or to the finite union of zones.

Let $F : \mathbb{R}^d \to \mathbb{R}^d$ and let $p \in \mathbb{R}^d$. Then F is the *characteristic polynomial* of a zone Z if $p \in Z \Leftrightarrow F(p) \leq 0$.

Denote the *area* of a bounded zone Z (assuming that it is well-defined) by area(Z). For a nonempty bounded zone $Z \neq \emptyset$ and an internal $p \in Z$, denote the maximal and minimal radii of Z with respect to p by

$$\delta(p, Z) = \sup\{r > 0 \mid Z \supseteq B(p, r)\}, \qquad \Delta(p, Z) = \inf\{r > 0 \mid Z \subseteq B(p, r)\},$$

and define the *fatness parameter* of Z *with respect to* p to be $\varphi(p, Z) = \Delta(p, Z)/\delta(p, Z)$. The zone Z is said to be *fat* with respect to p if $\varphi(p, Z)$ is bounded by some constant.

2.2. Wireless Networks

We consider a wireless network $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha\rangle$, where $d \in \mathbb{Z}_{\geq 1}$ is the dimension, $S = \{s_1, s_2, \ldots, s_n\}$ is a set of transmitting *radio stations* embedded in the d-dimensional space, ψ is an assignment of a positive real *transmitting power* ψ_i to each station s_i, $N \geq 0$ is the *background noise*, $\beta \geq 1$ is a constant that serves as the *reception threshold* (to be explained soon), and $\alpha > 0$ is the *path-loss parameter*. We sometimes wish to consider a network obtained from $\mathcal{A}$ by modifying one of the parameters while keeping all other parameters unchanged. To this end, we employ the following notation. Let $\mathcal{A}_{d'}$ be a network identical to $\mathcal{A}$ except its dimension is $d' \neq d$. $\mathcal{A}_{\beta'}$ and $\mathcal{A}_{\alpha'}$ are defined in the same manner. For notational simplicity, s_i also refers to the point $(x^i_1, \ldots, x^i_d)$ in the d-dimensional space $\mathbb{R}^d$ where the station s_i resides, and moreover, when $d = 2$, the point s_i in the Euclidean plane is denoted (x_i, y_i). The network is assumed to contain at least two stations, that is, $n \geq 2$. The *signal energy* of station s_i at point $p \neq s_i$ is defined to be $\mathrm{E}_{\mathcal{A}}(s_i, p) = \psi_i \cdot \mathrm{dist}^{-\alpha}(s_i, p)$. The *signal energy* of a set of stations $T \subseteq S$ at a point $p \notin T$ is defined to be $\mathrm{E}_{\mathcal{A}}(T, p) = \sum_{s_i \in T} \mathrm{E}_{\mathcal{A}}(s_i, p)$. Consider some point $p \notin S$ and a target station s_i that should be received at p. We define the *interference* of s_j to be the signal energy of s_j at p, $j \neq i$ denoted $\mathrm{I}_{\mathcal{A}}(s_j, p) = \mathrm{E}_{\mathcal{A}}(s_j, p)$. The *interference* of a set of stations $T \subseteq S \setminus \{s_i\}$ at a point $p \notin S$ is defined to be $\mathrm{I}_{\mathcal{A}}(T, p) = \mathrm{E}_{\mathcal{A}}(T, p)$. The *signal to interference and noise ratio* (SINR) of s_i at point p is defined as

$$\mathrm{SINR}_{\mathcal{A}}(s_i, p) = \frac{\mathrm{E}_{\mathcal{A}}(s_i, p)}{\mathrm{I}_{\mathcal{A}}(S - \{s_i\}, p) + N} = \frac{\psi_i \cdot \mathrm{dist}^{-\alpha}(s_i, p)}{\sum_{j \neq i} \psi_j \cdot \mathrm{dist}^{-\alpha}(s_j, p) + N}. \tag{1}$$

Observe that $\mathrm{SINR}_{\mathcal{A}}(s_i, p)$ is always positive, since the transmitting powers and the distances of the stations from p are always positive and the background noise is nonnegative.

In certain contexts, it is convenient to consider the reciprocal of the SINR function, namely, SINR^{-1} defined as

$$\mathrm{SINR}^{-1}_{\mathcal{A}}(s_i, p) = \frac{\mathrm{I}_{\mathcal{A}}(S - \{s_i\}, p) + N}{\mathrm{E}_{\mathcal{A}}(s_i, p)}. \tag{2}$$

When the network $\mathcal{A}$ is clear from the context, we may omit it and write simply $\mathrm{E}(s_i, p)$, $\mathrm{I}(s_j, p)$, $\mathrm{SINR}(s_i, p)$, and $\mathrm{SINR}^{-1}(s_i, p)$.

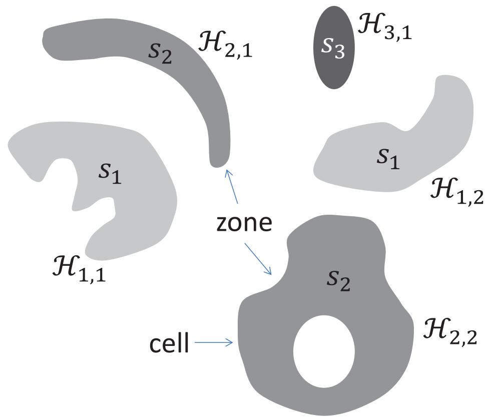

Fig. 2. Schematic representation of reception map. A map consists of reception zones: Each zone is composed of connected component(s), denoted by *cell(s)*.

The fundamental rule of the SINR model is that the transmission of station s_i is received correctly at point $p \notin S$ if and only if its SINR at p is not smaller than the reception threshold of the network, that is, $\mathrm{SINR}(s_i, p) \geq \beta$. If this is the case, then we say that s_i is *heard* at p. We refer to the set of points that hear station s_i as the *reception zone* of s_i, defined as

$$\mathcal{H}_i(\mathcal{A}) \ = \ \{p \in \mathbb{R}^d - S \mid \mathrm{SINR}_{\mathcal{A}}(s_i, p) \geq \beta\} \cup \{s_i\}.$$

This definition is necessary, since $\mathrm{SINR}(s_i, p)$ is undefined at points p in S and in particular at $p = s_i$ itself. In the same manner, we refer to the set of points where no station $s_i \in S$ can be heard (due to the background noise and interference) by

$$\mathcal{H}_\emptyset(\mathcal{A}) \ = \ \{p \in \mathbb{R}^d - S \mid \mathrm{SINR}(s_i, p) < \beta, \ \ \forall s_i \in S\}.$$

An SINR diagram $\mathcal{H}(\mathcal{A}) = \{\mathcal{H}_i(\mathcal{A}), \ \ 1 \leq i \leq n\} \cup \{\mathcal{H}_\emptyset(\mathcal{A})\}$ is a "reception map" characterizing the reception zones of the stations. This map partitions the plane into $n+1$ zones: a zone for each station $\mathcal{H}_i(\mathcal{A})$, $1 \leq i \leq n$, and a zone $\mathcal{H}_\emptyset(\mathcal{A})$ where no successful reception exists to any of the stations.

It is important to note that a reception zone, $\mathcal{H}_i(\mathcal{A})$, is not necessarily connected.

Hereafter, the set of points where the transmissions of a given station are successfully received is referred to as its *reception zone*, and a *cell* is a maximal connected set or component in a given reception zone. Hence the reception zone is a set of cells, given by $\mathcal{H}_i(\mathcal{A}) = \{\mathcal{H}_{i,1}(\mathcal{A}), \ldots \mathcal{H}_{i,\tau_i(\mathcal{A})}(\mathcal{A})\}$, where $\tau_i(\mathcal{A})$ is the number of cells in $\mathcal{H}_i(\mathcal{A})$. Analogously, $\mathcal{H}_\emptyset(\mathcal{A})$ is composed of $\tau_\emptyset(\mathcal{A})$ connected cells, $\mathcal{H}_{\emptyset,j}(\mathcal{A})$. Overall, the topology of a wireless network $\mathcal{A}$ is arranged in three levels: the *reception map* is at the top of the hierarchy. It is composed of $n+1$ reception zones, $\mathcal{H}_i(\mathcal{A}), i \in \{1, \ldots n, \emptyset\}$. Each zone $\mathcal{H}_i(\mathcal{A})$ is composed of $\tau_i(\mathcal{A})$ reception cells.[1] For a pictorial description, see Figure 2.

The following definition is useful in our later arguments. Let $F^i_{\mathcal{A}}(p)$, $p \in \mathbb{R}^d$ be the *characteristic polynomial* of $\mathcal{H}_i(\mathcal{A})$ given by

$$F^i_{\mathcal{A}}(p) \ = \ \beta \left(\sum_{k \neq i} \psi_k \prod_{l \neq k} \mathrm{dist}^\alpha(s_l, p) + N \cdot \prod_{k} \mathrm{dist}^\alpha(s_k, p) \right) - \psi_i \prod_{k \neq i} \mathrm{dist}^\alpha(s_k, p). \qquad (3)$$

Then $p \in \mathcal{H}_i(\mathcal{A})$, if and only if $F^i_{\mathcal{A}}(p) \leq 0$.

[1]When $\mathcal{A}$ is clear from context, we may omit it and simply write $\mathcal{H}_i$, τ_i, and $F_i(p)$. When refering to reception zones $\mathcal{H}_i(\mathcal{A}_{d'})$ or $\mathcal{H}_i(\mathcal{A}_{\beta'})$, we may omit $\mathcal{A}$ and simply write $\mathcal{H}_i(d')$ and $\mathcal{H}_i(\beta)$.

Avin et al. [2009] discuss the relationships between an SINR diagram on a set of stations S with *uniform* powers and the corresponding *Voronoi diagram* on S. Specifically, it is shown that the n reception zones $\mathcal{H}_i(\mathcal{A})$ are strictly contained in the corresponding Voronoi cells Vor_i. SINR diagrams with nonuniform powers are related to the *weighted Voronoi diagram* of the stations instead of to the Voronoi diagram.

In the weighted version of the Voronoi diagram [Aurenhammer and Edelsbrunner 1984], we consider a weighted system $V = \langle S, w\rangle$, where $S = \{s_1, \ldots, s_n\}$ represents a set of n points in d-dimensional Euclidean space and $w = \{w_1, \ldots, w_n\}$ is an assignment of weights $w_i \in \mathbb{R}_{>0}$ to each point $s_i \in S$. The *weighted Voronoi diagram* of $V = \langle S, w\rangle$ partitions the planes into n zones, where

$$\text{WVor}_i(V) = \left\{ p \in \mathbb{R}^d \;\middle|\; \frac{w_i}{\text{dist}(s_i, p)} > \frac{w_j}{\text{dist}(s_j, p)},\ \text{for any } j \neq i \right\},$$

denotes the *zones* (of influence) of a point s_i in S, for every $i \in \{1, \ldots, n\}$. The weighted Voronoi map denoted by $\text{WVor}(V)$ is composed of *cells, edges*, and *vertices*. A cell corresponds to a *maximal connected component* in $\text{WVor}_i(V)$, $i \in \{1, \ldots, n\}$. An edge is the relative interior of the intersection of two closed cells. Finally, a vertex is an endpoint of an edge. In the unweighted Voronoi diagram, each zone $\text{WVor}_i(V)$ corresponds to one connected cell. On the contrary, a weighted Voronoi map is composed of $O(n^2)$ cells, as was shown by Aurenhammer and Edelsbrunner [1984]. For a given wireless network $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha\rangle$, we define the corresponding weighted Voronoi system $V_{\mathcal{A}} = \langle S^{\mathcal{A}}, w^{\mathcal{A}}\rangle$ in the following manner. The set of points $S^{\mathcal{A}}$ corresponds to S positions and $w_i^{\mathcal{A}} = \psi_i^{1/\alpha}$, for every $1 \leq i \leq n$. In what follows, we formally express the relation between $\mathcal{H}(\mathcal{A})$ and $\text{WVor}(V_{\mathcal{A}})$.

LEMMA 2.1. *$\mathcal{H}_i(\mathcal{A}) \subseteq \text{WVor}_i(V_{\mathcal{A}})$, for every $i \in \{1, \ldots, n\}$ and $\beta \geq 1$.*

PROOF. Consider a point $p \in \mathbb{R}^d$ such that $p \in \mathcal{H}_i(\mathcal{A})$. Let $d_i = \text{dist}(s_i, p)$. We prove that $p \in \text{WVor}_i(V_{\mathcal{A}})$. Since $p \in \mathcal{H}_i(\mathcal{A})$, by (1),

$$\frac{\psi_i}{d_i^\alpha} \geq \beta \cdot \left(\sum_{j \neq i} \frac{\psi_j}{d_j^\alpha} + N \right) \geq \frac{\psi_k}{d_k^\alpha} \left(1 + \sum_{j \neq k,i} \frac{\psi_j/\psi_k}{(d_j/d_k)^\alpha} \right),$$

where $\psi_k/d_k^\alpha = \max_{j \neq i}(\psi_j/d_j^\alpha)$, and hence

$$\frac{\psi_i}{d_i^\alpha} > \frac{\psi_k}{d_k^\alpha} \quad \text{and} \quad \frac{\psi_i^{1/\alpha}}{d_i} > \frac{\psi_k^{1/\alpha}}{d_k}.$$

The choice of w_i implies that $p \in \text{WVor}_i(V_{\mathcal{A}})$, and the claim holds. □

Consider the way the reception map $\mathcal{H}(\mathcal{A}_\alpha)$ of a given network $\mathcal{A}_\alpha$ changes as α goes to infinity while the other parameters (e.g., the set of stations, β the noise etc.) are fixed. The map $\mathcal{H}(\mathcal{A}_\alpha)$ converges to is denoted by

$$\mathcal{H}(\mathcal{A}_\infty) = \lim_{\alpha \to \infty} \mathcal{H}(\mathcal{A}_\alpha).$$

LEMMA 2.2. *$\mathcal{H}_i(\mathcal{A}_\infty) \subseteq \text{Vor}_i$, for every $i \in \{1, \ldots, n\}$.*

PROOF. By Lemma 2.1, $\mathcal{H}_i(\mathcal{A}) \subseteq \text{WVor}_i(V_{\mathcal{A}})$. It follows that $\mathcal{H}_i(\mathcal{A}_\infty) \subseteq \text{WVor}_i(V_{\mathcal{A}_\infty})$ for $w_i = \lim_{\alpha \to \infty} \psi_i^{1/\alpha} = 1$. But $\text{WVor}_i(V_{\mathcal{A}_\infty})$ is simply Vor_i. This can also be seen by considering the SINR function: as α gets larger, the power of the station becomes negligible compared to distance between the station and the point p. In other words, it gets closer to the uniform Voronoi diagram. □

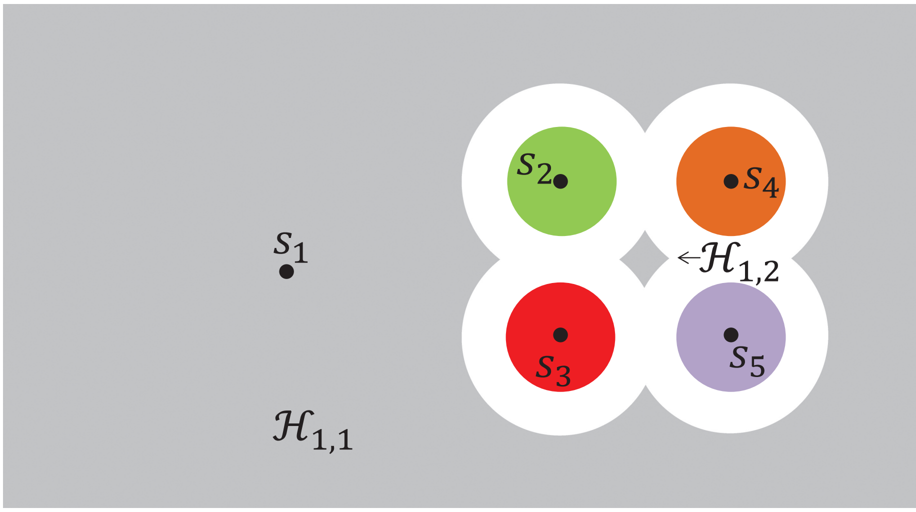

Fig. 3. An instance of a 5-station system with two connected cells of $\mathcal{H}_1$.

We conclude this section by stating an important technical lemma from Avin et al. [2009] that will be useful in our later arguments.

LEMMA 2.3 ([AVIN ET AL. 2009]). *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a mapping consisting of rotation, translation, and scaling by a factor of $\sigma > 0$. Consider some network $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha \rangle$ and let $f(\mathcal{A}) = \langle d, f(S), \psi, N/\sigma^2, \beta, \alpha \rangle$, where $f(S) = \{f(s_i) \mid s_i \in S\}$. Then for every station s_i and for all points $p \notin S$, we have $\mathrm{SINR}_{\mathcal{A}}(s_i, p) = \mathrm{SINR}_{f(\mathcal{A})}(f(s_i), f(p))$.*

3. DISCONNECTIVITY OF NONUNIFORM POWER SINR DIAGRAMS

The SINR diagram $\mathcal{H}(\mathcal{A})$ is a central concept of this article. We are interested in gaining some basic understanding of its topology. Specifically, we aim toward finding some "niceness" properties of reception zones and studying their usability in algorithmic applications. In previous work, Avin et al. [2009] consider the simpler case where all stations transmit with the same power. For a uniform power network, the reception zone of each station is known to be connected and to exhibit some desirable properties such as fatness and convexity. In the current work, we study the general (and common) case of nonuniform transmission powers.

By considering a 2-station network with nonuniform power, it is apparent that the reception zones of nonuniform power networks are not convex, however, connectivity is maintained. Unfortunately, although this is true for 2-stations systems, it does not hold in general. Connectivity might be broken even in networks with a small number of participants, as illustrated by the 5-station system of Figure 3, where the reception zone of s_1 is composed of two connected cells. This raises the immediate question of bounding the maximal number of cells a given SINR diagram might have.

A seemingly promising approach to studying this question is considering the corresponding weighted Voronoi diagrams. Recall that by Lemma 2.1, $\mathcal{H}_i(\mathcal{A}) \subseteq \mathrm{WVOR}_i(V_{\mathcal{A}})$. It therefore seems plausible that the number of weighted Voronoi cells (bounded by $O(n^2)$ [Aurenhammer and Edelsbrunner 1984]) might upper bound the number of connected cells in the corresponding SINR diagram. Unfortunately, this does not hold in general, since it might be the case that a single weighted Voronoi cell corresponds to several connected SINR cells. This phenomenon is formally stated in the following lemma.

LEMMA 3.1. *There exists a wireless network $\mathcal{A}^*$ such that a given cell of the corresponding weighted Voronoi diagram* $\text{WVOR}(V_{\mathcal{A}^*})$ *contains more than one cell of* $\mathcal{H}(\mathcal{A}^*)$.

PROOF. Let $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha \rangle$ be a wireless network, where $S = \{s_1, \ldots, s_n\}$ and $\mathcal{H}_1(\mathcal{A})$ is not connected, that is, $\mathcal{H}_1(\mathcal{A})$ is composed of more than one cell. Let $\mathcal{A}_m = \langle d, \mathcal{S}_m, \Theta_m, N, \beta, \alpha \rangle$, where $\mathcal{S}_m = \{s_1\} \cup \{s_2^1, \ldots, s_2^m\} \cup \ldots \cup \{s_n^1, \ldots, s_n^m\}$, $\Theta_m = \{\theta_1\} \cup \{\theta_2^1, \ldots, \theta_2^m\} \cup \ldots \cup \{\theta_n^1, \ldots, \theta_n^m\}$, where $\theta_1 = \psi_1$ and $\theta_i^l = \psi_i/m$, for every $i = 2, \ldots, n$ and $l = 1, \ldots, m$. To avoid cumbersome notation, let $V_m = V_{\mathcal{A}_m}$ and let $\text{WVOR}(V_m)$ be the corresponding weighted Voronoi diagram of $\mathcal{A}_m$. In what follows, we show that for sufficiently large m^*, the network $\mathcal{A}_{m^*}$ satisfies the conditions of the desired network $\mathcal{A}^*$. It is easy to verify that for large enough m^*, the weighted zone $\text{WVOR}_1(V_{m^*})$ is connected.

On the other hand, $\text{WVOR}_1(V_{m^*})$ contains more than one connected cell of $\mathcal{H}_1(\mathcal{A}_{m^*})$. First, observe that $\mathcal{H}_1(\mathcal{A}) = \mathcal{H}_1(\mathcal{A}_{m^*})$, and therefore $\mathcal{H}_1(\mathcal{A}_{m^*})$ is not connected as well. This follows by noting that $\text{E}_{\mathcal{A}}(s_1, p) = \text{E}_{\mathcal{A}_m}(s_1, p)$ and $\text{I}_{\mathcal{A}}(S \setminus \{s_1\}, p) = \text{I}_{\mathcal{A}_m}(\mathcal{S}_m \setminus \{s_1\}, p)$. Next, by the connectivity of $\text{WVOR}_1(V_{m^*})$ and Lemma 2.1, it follows that $\mathcal{H}_1(\mathcal{A}_{m^*}) \subseteq \text{WVOR}_1(V_{m^*})$. Since $\mathcal{H}_1(\mathcal{A}_{m^*})$ is not connected, the lemma follows. □

This lemma illustrates that the structural complexity of the SINR diagram cannot be fully captured by the weighted Voronoi diagram. Specifically, it implies that the number of connected cells in a nonuniform SINR diagram cannot be bounded by the number of weighted Voronoi cells, hence a different approach is needed. This challenge is extensively discussed in this article, where we obtain bounds and provide extreme constructions with respect to the the number of connected cells for a given station. We conjecture that the obtained upper bounds are not tight, and our constructions are close to the limit. Yet so far, no formal proof is available.

4. NUMBER OF CELLS IN NONUNIFORM SINR DIAGRAMS

In this section, we aim to achieve bounds for the number of connected cells in nonuniform diagrams. Section 4.1 provides upper bounds for the number of cells. Section 4.2 presents an extreme construction that, we believe, maximizes the number of cells of a single transmitter.

4.1. Upper Bound

We now consider the general case of a network of the form $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha = 2 \rangle$, and establish upper and lower bounds on the number of connected cells. To obtain an upper bound on the number of cells, we apply the following theorem of Milnor [1964] and Thom [1965].

THEOREM 4.1 (THEOREM 5.4 OF [WALLACH 1996]). *Let $f_1 \ldots, f_m$ be polynomials in $\mathbb{R}^d$ with $\mathtt{deg}(f_i) < K$. Then, $V = \{x = (x_1, \ldots, x_d) \mid f_i(x) = 0$ for every $i \in \{1, \ldots m\}\}$ has at most $K(2K-1)^{d-1}$ cells.*

The following is a direct consequence of Theorem 4.1.

LEMMA 4.2. $\sum_{i=1}^{n} \tau_i = O(n^{d+1})$.

PROOF. Consider $F_{\mathcal{A}}^i(p)$, the *characteristic polynomial* of $\mathcal{H}_i(\mathcal{A})$ given in Eq. (3). As $\mathtt{deg}(F_{\mathcal{A}}^i(p)) \leq 2 \cdot n$, Theorem 4.1 implies that $\tau_i(\mathcal{A}) = O(n^d)$. Summing over all n stations yields the lemma. □

In the same manner, we can also bound the number of connected cells in $\mathcal{H}_\emptyset(\mathcal{A})$, where no station is received correctly.

COROLLARY 4.3. $\tau_\emptyset(\mathcal{A}) = O(n^{2d})$.

PROOF. We first show that for $\beta \geq 1$, the characteristic polynomial of $\mathcal{H}_\emptyset(\mathcal{A})$ (also known as the *noise polynomial*) is

$$F^{\emptyset}_{\mathcal{A}}(p) = -\prod_{i=1}^{n} F^{i}_{\mathcal{A}}(p). \tag{4}$$

It is required to show that $p \in \mathcal{H}_\emptyset(\mathcal{A})$ if and only if $F^{\emptyset}_{\mathcal{A}}(p) < 0$. The first direction is trivial, as if $p \in \mathcal{H}_\emptyset(\mathcal{A})$, then $F^{i}_{\mathcal{A}}(p) > 0$ for every $i \in \{1, \ldots, n\}$, and hence $F^{\emptyset}_{\mathcal{A}}(p) < 0$. For the opposite direction, observe that if $p \notin \mathcal{H}_\emptyset(\mathcal{A})$, then there exists exactly one station s_j such that $p \in \mathcal{H}_j(\mathcal{A})$ and $F^j(\mathcal{A}, p) \leq 0$. This is due to the fact that when $\beta \geq 1$, reception zones for different stations do not overlap. Hence $F^{i}_{\mathcal{A}}(p) > 0$ for any $i \neq j$, and therefore $F^{\emptyset}_{\mathcal{A}}(p) \geq 0$ as required.

Consequently, the degree of the noise polynomial $F^{\emptyset}_{\mathcal{A}}(p)$ is bounded by $\mathtt{deg}(F^{\emptyset}_{\mathcal{A}}(p)) \leq 2 \cdot n^2$. By Theorem 4.1, it then follows that $\tau_\emptyset(d) < O(n^{2d})$. □

4.2. Construction of $\Omega(n)$ Connected Cells for a Single Station

We focus on the station s_1 with transmission power ψ_1 and devise a construction scheme that aims to maximize the number of connected cells τ_1. This construction achieves $\tau_1 = \Omega(n)$. We believe that this construction is close to the maximum possible, that is, we suspect that $\sum_{i=1}^{n} \tau_i = \Theta(n)$, yet no proof is currently available. For completeness, we refer the interested reader to an alternative construction with $\tau_1 = \Omega(\log \psi_1)$ cells, using the *wires* technique [Kantor et al. 2011].

We begin by providing a high-level intuition for such a construction and then describe it in more detail for the case where $d = 2$ and $\alpha = 2$. The same bounds can be obtained for any given fixed d and $\alpha \geq 1$. First, we describe the recipe for generating a *single* reception cell of s_1 (later we repeat it to create $\Omega(n)$ cells). The key idea is to take some compact domain D (e.g., a unit ball) that is sufficiently far away from s_1. This would guarantee that the received signal strength of s_1 at every point in D is roughly the same, and if s_1 is sufficiently strong (compared to the other interfering stations), then it would still be possible for some points on D to receive the transmission of s_1 successfully. Next, place a constant number $c = f(d, \alpha)$ of weak interfering stations on the boundary of D. The precise locations and the powers of these stations are set so that the interference experienced at any point on the boundary of D is strictly stronger than the interference experienced by some internal point p inside D (e.g., the ball center). Hence, the point p would have a larger SINR value (with respect to s_1) than any point on D's boundary (i.e., p would be part of the cell of s_1). To create $\Omega(n)$ cells, select many such domains D_i at sufficiently large distances from each other on the boundary of a very large circle centered at s_1. Since the domains are distant from each other, the interference at any point inside each domain D_i would be massively dominated by the interfering stations located on the boundary of D_i (i.e., the interference from stations on the other domains is negligible). As a result, for each of the $\Omega(n)$ domains D_i, the $\mathrm{SINR}(s_1, p)$ value evaluated in some internal point inside D_i is higher than that of any point on D_i's boundary, which allows us to set the desired SINR threshold. This completes the high-level description.

The Construction. For the sake of simplicity, throughout the remainder of this section, we consider the case where $\alpha = 2$ and the two-dimensional Euclidean plane, that is, $d = 2$. The goal of the construction is as follows. Given $n \geq 1$, find a placement of $4n+1$ stations $S = \{s_0, \ldots, s_{4n}\}$ and a power assignment ψ such that $\tau_0 = n + 1$, that is, s_0 is correctly received in $n + 1$ different connected cells.

Let us partition $S \setminus \{s_0\}$ into n quadruples $S_i = \{s_{4i+1}, \ldots, s_{4i+4}\}$, $0 \leq i \leq n - 1$, each corresponding to the vertices of an axis-aligned square. We assume the SINR

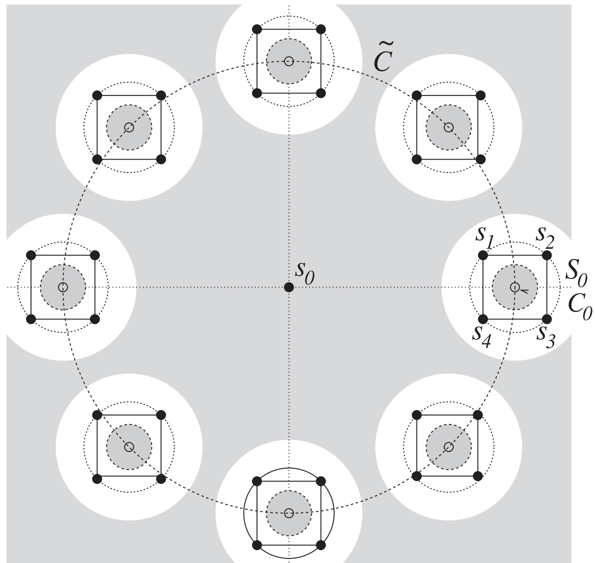

Fig. 4. Geometric view of the construction.

parameters $\alpha = 2$, $N = 1$, $\beta = 1$, and $\psi_i = 1$ for $i > 0$. The value of ψ_0 and the positions of S will be determined later on. The resulting network is $\mathcal{A} = \langle d = 2, S, \psi, N = 1, \beta = 1, \alpha = 2\rangle$. We next present the construction and then analyze the resulting structure.

Locate station s_0 at the origin $(0, 0)$ and draw a circle $\tilde{C}$ of radius R around it. Place n points $C_0, \ldots, C_{n-1}$ at equidistant locations on $\tilde{C}$, with $C_i = \langle R\cos(\frac{2\pi}{n}i),\ R\sin(\frac{2\pi}{n}i)\rangle$ for $0 \le i \le n-1$. Around each point C_i draw a unit circle. Locate the stations of S_i on the vertices of the axis-aligned $\sqrt{2} \times \sqrt{2}$ square enclosed by the ith unit-circle. Let $\hat{S}_i$ be the square defined by its four vertices S_i (see Figure 4).

We make use of the following equalities.

FACT 4.4. *(a)* $\mathrm{dist}(C_0, C_i) = 2R\sin\left(\frac{\pi}{n}i\right)$. *(b)* $\displaystyle\sum_{i=1}^{n-1} \frac{1}{\sin^2\left(\frac{\pi}{n}i\right)} = \frac{n^2-1}{3}$.

PROOF. By definition, $C_0 = (R, 0)$, then

$$\begin{aligned}
\mathrm{dist}(C_0, C_i)^2 &= \left(R - R\cos\left(\frac{2\pi}{n}i\right)\right)^2 + R^2 \cdot \sin^2\left(\frac{2\pi}{n}i\right) \\
&= R^2\left(1 - 2\cos\left(\frac{2\pi}{n}i\right) + \cos^2\left(\frac{2\pi}{n}i\right)\right) + R^2 \cdot \sin^2\left(\frac{2\pi}{n}i\right) \\
&= 2R^2\left(1 - \cos\left(\frac{2\pi}{n}i\right)\right) = 2R^2\left(1 - \left(\cos^2\left(\frac{\pi}{n}i\right) - \sin^2\left(\frac{\pi}{n}i\right)\right)\right) \\
&= 4R^2 \cdot \sin^2\left(\frac{\pi}{n}i\right),
\end{aligned}$$

where the penultimate equality follows by the trigonometry equality that $\cos 2\theta = \cos^2\theta - \sin^2\theta$. Part (a) follows. Part (b) is a general inequality verified using Wolfram [2010]. □

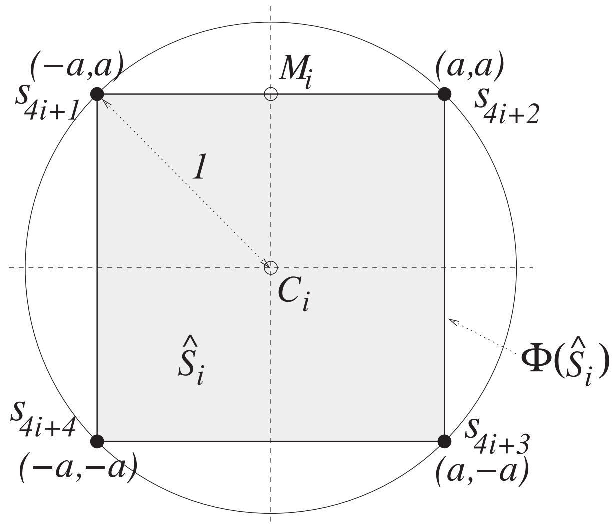

Fig. 5. Zoom-in on the unit circle of S_i and on the square $\hat{S}_i$.

COROLLARY 4.5. $\sum_{i=1}^{n-1} \frac{1}{\text{dist}(C_0, C_i)^2} = \frac{n^2-1}{12R^2}$.

PROOF. By Fact 4.4(a),

$$\sum_{i=1}^{n-1} \frac{1}{\text{dist}(C_0, C_i)^2} = \frac{1}{4R^2} \cdot \sum_{i=1}^{n-1} \frac{1}{\sin^2\left(\frac{\pi}{n} i\right)}, \frac{n^2-1}{12R^2},$$

where the last inequality follows by Fact 4.4(b). □

LEMMA 4.6. *For every* $0 \le i \le n-1$*, the following hold.*

(a) For the center point C_i*,* $\text{I}(S_i, C_i) = 4$.
(b) For any boundary point $p \in \Phi(\hat{S}_i)$*,* $\text{I}(S_i, p) \ge 4\frac{4}{5}$.

PROOF. For convenience, let us translate the square S_i to the origin, that is, map C_i to $(0,0)$. Let $S_i = \{(-a,a),(a,a),(a,-a),(-a,-a)\}$ be the vertices of the resulting $2a \times 2a$ square, where $a = 1/\sqrt{2}$. (see Figure 5). The interference of S_i on the center point $C_i = (0,0)$ is given by $\text{I}(S_i, C_i) = \text{I}(S_i,(0,0)) = 4 \cdot (1/(\sqrt{2}a)^2) = 4$, implying part (a) of the lemma.

We next prove part (b). Due to symmetry, we may restrict attention to a single square edge, say, the upper edge $e = \{p = (x,y) \mid -a \le x \le a \ , y = a\}$. The interference of the four stations of S_i on a point $p = (x,y) \in e$ is given by

$$\text{I}(S_i,(x,a)) = \frac{1}{(x-a)^2} + \frac{1}{(x+a)^2} + \frac{1}{(x-a)^2+4a^2} + \frac{1}{(x+a)^2+4a^2}. \tag{5}$$

Let $M_i = (0,a)$ be the middle point on edge e. Consider the first derivative of the interference function

$$\begin{aligned}\frac{\partial \mathrm{I}(S_i,(x,a))}{\partial x} &= -\frac{2}{(x-a)^3}-\frac{2}{(x+a)^3}-\frac{2(x-a)}{(4a^2+(x-a)^2)^2}-\frac{2(x+a)}{(4a^2+(x+a)^2)^2}\\ &= \frac{8x(935a^{10}+765a^8x^2+286a^6x^4+50a^4x^6+11a^2x^8+x^{10})}{(a-x)^3(a+x)^3(5a^2-2ax+x^2)^2(5a^2+2ax+x^2)^2}.\end{aligned}$$

The point M_i is the only local optimum of $\mathrm{I}(S_i,(x,a))$ in the range $x \in (-a,a)$, as $\frac{\partial \mathrm{I}(S_i,(x,a))}{\partial x}=0$ implies $x=0$. The second derivative is given by

$$\begin{aligned}\partial^2 \mathrm{I}(S_i,(x,a))/\partial x^2 \;=\; & 6/(x-a)^4+6/(a+x)^4+(8(x-a)^2)/(4a^2+(x-a)^2)^3\\ & -2/(4a^2+(x-a)^2)^2+(8(a+x)^2)/(4a^2+(a+x)^2)^3\\ & -2/(4a^2+(a+x)^2)^2.\end{aligned}$$

Since $\partial^2 \mathrm{I}(S_i,(x=0,a))/\partial x^2 = 1496/(125a^2) > 0$, we conclude that M_i is indeed a local minimum. In particular, we get that $\mathrm{I}(S_i,M_i)=\mathrm{I}(S_i,(0,a))=12/(5a^2)=24/5$, and $\mathrm{I}(S_i,M_i) \le \mathrm{I}(S_i,p)$ for any boundary point $p=(x,y)\in\Phi(\hat{S}_i)$, that is not an edge midpoint, establishing part (b) of the lemma. □

4.2.1. Construction Strategy. The desired construction should impose two requirements for each $0 \le i \le n-1$:

(1) (R1) $\mathrm{SINR}(s_0, C_i) \ge 1$,
(2) (R2) $\mathrm{SINR}(s_0, p) < 1$ for every boundary point $p \in \Phi(\hat{S}_i)$.

Requirement (R1) guarantees that s_0 is correctly received at n regions, namely, the immediate ε-neighborhoods of the points C_i for sufficiently small $\varepsilon > 0$, whereas requirement (R2) implies also that s_0 is not received on any point on the perimeters of the n squares, and hence guarantees the n reception regions to be disconnected cells.

Having fixed the station locations up to the choice of R and the transmission powers of all stations except s_0, it remains to select values for R and ψ_0 that will ensure (R1) and (R2). We employ the following strategy. For each boundary point $p \in \Phi(\hat{S}_i)$, we establish an *overestimate* for the signal energy received at M_i from s_0, and an *underestimate* for the interference caused by $S \setminus \{s_0\}$. For each C_i, we establish an underestimate for the signal energy received at C_i from s_0, and an overestimate for the interference caused by $S \setminus \{s_0\}$. We then select ψ_0 and R that satisfy requirements ($R1$) and ($R2$) under these stricter conditions. Intuitively, by choosing the radius R to be "large enough," we ensure that the signal energy generated by station s_0 and the interference generated on square i by all other stations $S \setminus \{s_0, s_{4i+1}, \ldots, s_{4i+4}\}$ is almost identical on the center point C_i and on the boundary points of the square surrounding it, $\Phi(\hat{S}_i)$. This implies that the difference in the *SINR* between the center point C_i and the points of $\Phi(\hat{S}_i)$ is influence mostly by the difference in the signal energy generates by four stations $s_{4i+1}, \ldots, s_{4i+4}$ on the square $\hat{S}_i$, which is at least $1/4$. Therefore, if we first choose a sufficiently large R, then it is possible to set the power ψ_0 so that it satisfies requirements ($R1$) and ($R2$) as well.

LEMMA 4.7. *If* $R \ge \sin^{-1}(\pi/n)$ *and* $\psi_0 \ge 5R^2 + 4(n^2-1)/3$, *then requirement (R1) holds, namely,* $\mathrm{SINR}(s_0, C_i) \ge 1$ *for every* $0 \le i \le n-1$.

PROOF. Let $\widehat{s_j} = s_{4j+k}$ for some $k \in \{1,\ldots,4\}$ be the closest station to C_i in S_j, that is, such that $\mathrm{dist}(C_i,\widehat{s_j}) = \min_{1\le l\le 4}\{\mathrm{dist}(C_i, s_{4j+l})\}$. To overestimate the interference of S_j on C_i, we eliminate the other three stations of S_j and assign $\widehat{s_j}$ transmission power

$\widehat{\psi}_j = 4$. By the triangle inequality, $\mathrm{dist}(\widehat{s}_j, C_i) > \mathrm{dist}(C_i, C_j) - 1$, and therefore,

$$\mathrm{I}(S \setminus (S_i \cup \{s_0\}), C_i) \;=\; \sum_{j \neq i} \mathrm{I}(S_j, C_i) \;<\; \sum_{j \neq i} \mathrm{I}(\widehat{s}_j, C_i) \;<\; \sum_{j \neq i} \frac{4}{(\mathrm{dist}(C_i, C_j) - 1)^2}.$$

By Fact 4.4(a),

$$\mathrm{I}(S \setminus (S_i \cup \{s_0\}), C_i) \;=\; \sum_{i \neq 0} \frac{4}{\left(2R \cdot \sin\left(\frac{\pi}{n} i\right) - 1\right)^2} \;\leq\; \sum_{i \neq 0} \frac{4}{(R \cdot \sin(\frac{\pi}{n} i))^2},$$

where the last inequality follows by the fact that $R \cdot \sin(\frac{\pi}{n} i) \geq 1$ for every i (by the first assumption of the lemma). By Corollary 4.5,

$$\mathrm{I}(S \setminus (S_i \cup \{s_0\}), C_i) \;<\; \frac{4(n^2 - 1)}{3R^2}.$$

By Lemma 4.6(a), it follows that $\mathrm{I}(S \setminus (S_i \cup \{s_0\}), C_i) < 4 + 4(n^2 - 1)/(3R^2)$. Finally, by plugging this into Eq. (1), recalling that $N = 1$, we get that

$$\mathrm{SINR}(s_0, C_i) \;\geq\; \frac{\psi_0 \cdot R^{-2}}{5 + 4(n^2 - 1)/(3R^2)} > 1,$$

where the last inequality follows by the second assumption of the lemma. Hence, requirement (R1) holds. □

We now turn to selecting R and ψ_0 ensuring requirement ($R2$) at every boundary point $p \in \Phi(\hat{S}_i)$. By construction, $R - 1$ is the minimal distance from the origin to any point on a unit circle centered at C_i. Hence we have the following.

OBSERVATION 4.8. $\mathrm{E}(s_0, p) < \psi_0/(R - 1)^2$, *for every* $p \in \Phi(\hat{S}_i)$.

LEMMA 4.9. *If* $R \geq \sin^{-1}(\pi/n)$ *and* $\psi_0 < (5\frac{4}{5} + \frac{n^2-1}{27R^2}) \cdot (R - 1)^2$, *then requirement (R2) holds, namely,* $\mathrm{SINR}(s_0, p) < 1$ *for every* $p \in \Phi(\hat{S}_i)$, $0 \leq i \leq n - 1$.

PROOF. We underestimate $\mathrm{I}(S_j, p)$, $p \in \Phi(\hat{S}_i)$ by considering only the station $\widehat{s}_j = s_{4j+k}$ (for some $k \in \{1, \ldots, 4\}$) closest to p in S_j. The distance $\mathrm{dist}(\widehat{s}_j, p)$ can be overestimated by the distance between p and center C_j. Formally, we have

$$\mathrm{I}(S_j, p) \;>\; \mathrm{I}(\widehat{s}_j, p) \;>\; \mathrm{I}(C_j, p) > \frac{1}{(\mathrm{dist}(C_j, C_i) + 1)^2} \;>\; \frac{4}{(3 \cdot \mathrm{dist}(C_j, C_i))^2}. \quad (6)$$

To see the last inequality, note that since $R \geq \sin^{-1}(\pi/n)$ by the first assumption of the lemma, Fact 4.4(a) guarantees that $\mathrm{dist}(C_0, C_1) \geq 2$. As $\mathrm{dist}(C_0, C_1) = \min_{i \neq j} \mathrm{dist}(C_i, C_j)$, it follows that also $\mathrm{dist}(C_i, C_j) \geq 2$ for every i and j. We therefore have, by Inequality (6) and Fact 4.4(b), that

$$\mathrm{I}(S \setminus (S_i \cup \{s_0\}), p) \;=\; \sum_{j \neq i} \mathrm{I}(S_j, p) \;>\; \frac{4}{9} \cdot \sum_{j \neq i} \mathrm{dist}(C_j, C_i)^{-2} \;=\; \frac{n^2 - 1}{27R^2}.$$

Next, by combining Observation 4.8 and Eq. (1), we have that

$$\mathrm{SINR}(s_0, p) \;\leq\; \frac{\psi_0 \cdot (R - 1)^{-2}}{5\frac{4}{5} + (n^2 - 1)/(27R^2)} < 1,$$

where the last inequality follows by the second assumption of the lemma. The lemma follows. □

Finally, we combine the conditions developed in the previous sections for requirements (R1) and (R2) (Lemmas 4.7 and 4.9) and show that there exists a feasible solution, namely, a choice of R and ψ_0 such that both requirements hold.

Clearly, R should be greater than $\sin^{-1}(\pi/n)$. Let $U = (5\frac{4}{5} + \frac{n^2-1}{27R^2}) \cdot (R-1)^2$ and $L = (5 + \frac{4(n^2-1)}{3R^2}) \cdot R^2$. Then, by Lemmas 4.7 and 4.9, ψ_0 should be chosen to satisfy $\psi_0 < U$ and $\psi_0 \geq L$. It is left to verify that for every n, there exists a choice of $R > \sin^{-1}(\pi/n)$ such that $U > L$. If this holds, then any choice of ψ_0 in the range $U > \psi_0 \geq L$ satisfies the requirements. Letting

$$\Delta = U - L = \left(5\frac{4}{5} + \frac{n^2-1}{27R^2}\right) \cdot (R-1)^2 - \left(5 + \frac{4(n^2-1)}{3R^2}\right) \cdot R^2, \tag{7}$$

it suffices to show that $\Delta > 0$ for sufficiently large R. This is done by developing Eq. (7) taking into account leading factors. For ease of analysis, let $n^* = n^2 - 1$. Then, by Eq. (7), we need R to satisfy $R^2 \cdot (4n^*/(3R^2) + 5) < (R-1)^2 \cdot (n^*/(27R^2) + 29/5)$. Multiplying by R^2 and rearranging, the requirement becomes

$$\frac{4}{5}R^4 - \frac{58}{5}R^3 + \frac{29}{5}R^2 > \left(\frac{35}{27}R^2 - \frac{2}{27}R + \frac{1}{27}\right) n^*.$$

For sufficiently large R, the left-hand side expression is greater than $\frac{3}{5}R^4$, and the right-hand side expression is smaller than $\frac{12}{5}R^2 \cdot n^*$, so it suffices to require that $\frac{3}{5}R^4 > \frac{12}{5}R^2 \cdot n^*$, or after simplification, that $R > 2n$. We therefore established the following.

THEOREM 4.10. *There exists a network $\mathcal{A}$ such that $\tau_1 = \Theta(n)$.*

5. CONNECTIVITY OF RECEPTION ZONES IN $\mathbb{R}^{d+1}$

In this section, we consider the case where the stations are embedded in $\mathbb{R}^d$ but study the topological properties of their reception zones in $\mathbb{R}^{d+1}$, where niceness properties emerge. We show that zones in $\mathbb{R}^{d+1}$ obey a stronger property, namely, *hyperbolic convexity*.

Let $S = \{s_1, \ldots, s_n\}$ be a set of stations embedded in the d-dimensional space $\mathbb{R}^d$. We consider on the network $\mathcal{A} = \langle d, S, \psi, N, \beta, 2\alpha \geq 2\rangle$ in $\mathbb{R}^d$ and the *reception map* $\mathcal{H}(\mathcal{A}_{d+1})$ created for it in $\mathbb{R}^{d+1}$. We assume without loss of generality that the stations are embedded in the hyperplane $x_{d+1} = 0$ in $\mathbb{R}^{d+1}$, with positions $(x_1^{s_i}, \ldots, x_d^{s_i}, 0)$. Throughout this section, we slightly abuse notation by occasionally considering a point $p = (x_1^p, \ldots, x_d^p)$ in $\mathbb{R}^d$ as a point in $\mathbb{R}^{d+1}$, namely, $(x_1^p, \ldots, x_d^p, 0)$. This section concerns what happens when we go one dimension higher and consider the SINR diagram in dimension $d+1$ for S. Recall that

$$\mathcal{H}_i(\mathcal{A}_{d+1}) = \{p \in \mathbb{R}^{d+1} \setminus \{S\} \mid \mathrm{SINR}(s_i, p) \geq \beta\} \cup \{s_i\}.$$

The following theorem shows that the situation improves dramatically in this setting.

THEOREM 5.1. *Given a network $\mathcal{A} = \langle d, S, \psi, N, \beta, 2\alpha \geq 2\rangle$, $\mathcal{H}_i(\mathcal{A}_{d+1})$ is connected for every $i \in \{1, \ldots, n\}$.*

In what follows, we concentrate on s_1 and show that $\mathcal{H}_1(\mathcal{A}_{d+1})$ is connected. Let $p = (x_1^p, \ldots, x_d^p, x_{d+1}^p) \in \mathbb{R}^{d+1}$ be any point that correctly receives the transmission of station $s_1 = (x_1^{s_1}, \ldots, x_d^{s_1}, 0)$. To prove that $\mathcal{H}_1(\mathcal{A}_{d+1})$ is connected, we show that there exists a continuous curve connecting s_1 and $p \in \mathbb{R}^{d+1}$ such that s_1 is correctly received at any point along this curve. In fact, we establish a stronger property, namely, that for any two points $p_1 = (x_1^{p_1}, \ldots, x_d^{p_1}, x_{d+1}^{p_1})$ and $p_2 = (x_1^{p_2}, \ldots, x_d^{p_2}, x_{d+1}^{p_2})$ in $\mathcal{H}_1(\mathcal{A}_{d+1})$,

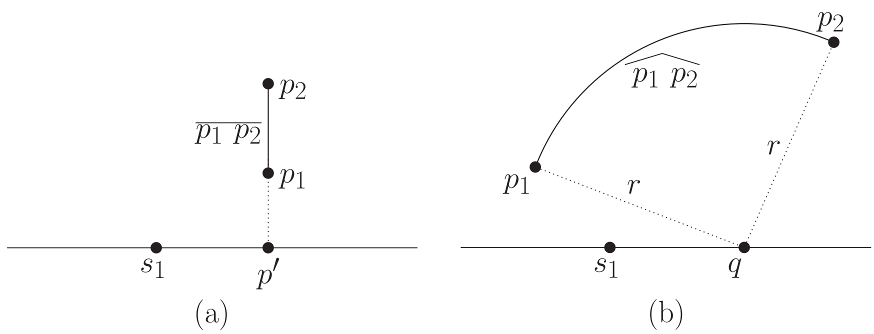

Fig. 6. The hyperbolic geodesic of points p_1 and p_2 corresponds to either (a) a vertical line (case HC1), or (b) a hyperbolic arc (case HC2).

residing on the same side of the hyperplane $x_{d+1} = 0$, that is, satisfying

$$sign\big(x_{d+1}^{p_1}\big) \cdot sign\big(x_{d+1}^{p_2}\big) \;\geq\; 0, \tag{8}$$

there exists a continuous curve connecting p_1 and p_2 in $\mathbb{R}^{d+1}$ such that s_1 is correctly received at any point along this curve. In particular, this curve corresponds to the *hyperbolic geodesic* of p_1 and p_2 denoted by $\mathrm{h}(p_1, p_2)$. Note that this indeed guarantees the connectivity of $\mathcal{H}_1(\mathcal{A}_{d+1})$ by taking $p_1 = s_1$.

We begin by recalling some facts about hyperbolic geometry, see Thurston [1997] for details. Specifically, we consider a standard model of hyperbolic planes, namely, the upper half-plane model. Under this model, the geodesic of two points $p_1, p_2 \in \mathbb{R}^d$ is either a vertical line or an arc, as will be formulated later. A point set P is *hyperbolic star-shaped* with respect to point $p_1 \in P$ if the hyperbolic geodesic of p_1 and every point $p_2 \in P$, is contained in the point-set P as well, for example, $\mathrm{h}(p_1, p_2) \subseteq P$ (where p_1 and p_2 satisfy Eq. (8)). A point set P is *hyperbolic convex* if it is star-shaped with respect to any point $p_1 \in P$. In other words, for any two points $p_1, p_2 \in P$ obeying Eq. (8), $\mathrm{h}(p_1, p_2) \subseteq P$ as well. In this section, we show that the reception zone $\mathcal{H}_1(\mathcal{A}_{d+1})$ is hyperbolic convex and therefore connected.

We proceed by considering two cases, one for each type of hyperbolic geodesics.

Case HC1. $x_i^{p_1} = x_i^{p_2}$ for $i \in \{1, \ldots, d\}$; $\mathrm{h}(p_1, p_2)$ corresponds to a vertical line denoted by $\overline{p_1\, p_2}$, see points p_1 and p_2 of Figure 6(a).

Case HC2. There exists some $i \in \{1, \ldots, d\}$ such that $x_i^{p_1} \neq x_i^{p_2}$; $\mathrm{h}(p_1, p_2)$ corresponds to an arc, denoted by $\widehat{p_1\, p_2}$, see points p_2 and p_3 of Figure 6(b).

We next consider Case HC1 and show that if p_1 and p_2 are in $\mathcal{H}_1(\mathcal{A}_{d+1})$, then so is any point on the segment $\overline{p_1\, p_2}$. Then, we turn to Case HC2 and show that if p_1 and p_2 are in $\mathcal{H}_1(\mathcal{A}_{d+1})$, then there exists an arc $\widehat{p_1\, p_2}$ fully contained in $\mathcal{H}_1(\mathcal{A}_{d+1})$. In particular, for $p_1 = s_1$, there exists an arc $\widehat{s_1\, p_2}$, for every reception point $p_2 \in \mathcal{H}_1(\mathcal{A}_{d+1})$, such that $\widehat{s_1\, p_2} \subseteq \mathcal{H}_1(\mathcal{A}_{d+1})$, that is, the zone is hyperbolic star-shaped with respect to s_1, hence it is connected.

5.1. Analysis of Case HC1

For Case HC1, we state the following lemma.

LEMMA 5.2. *Let $p_1, p_2 \in \mathcal{H}_1(\mathcal{A}_{d+1})$ be points satisfying Eq. (8) such that $x_j^{p_1} = x_j^{p_2}$ for $j \in \{1, \ldots, d\}$. Then, $p \in \mathcal{H}_1(\mathcal{A}_{d+1})$, for every internal point $p \in \overline{p_1\, p_2}$. (For a pictorial description, see Figure 7(a).)*

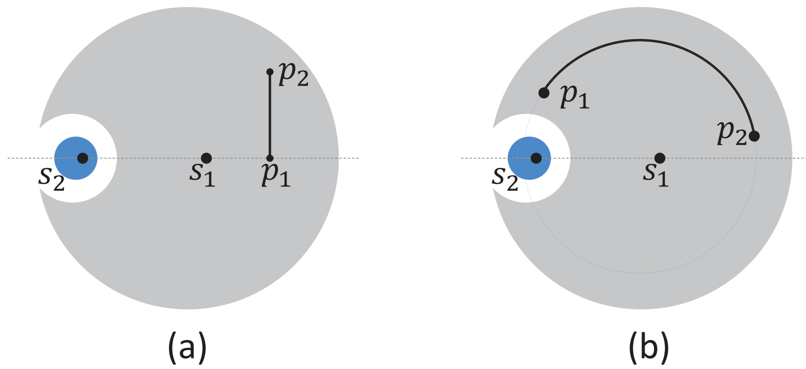

Fig. 7. Hyperbolic convexity in $\mathbb{R}^2$. The stations s_1 and s_2 are embedded in $\mathbb{R}^1$. (a) Convexity on a straight vertical line in $\mathbb{R}^2$, $\overline{p_1\, p_2}$, (b) Hyperbolic convexity on a circular arc in $\mathbb{R}^2$, $\widehat{p_1\, p_2}$.

PROOF. Assume without loss of generality that $x_{d+1}^{p_1} < x_{d+1}^{p_2}$. Consider an internal point $p = (x_1^{p_1}, \ldots, x_d^{p_1}, x_{d+1}^{p}) \in \overline{p_1\, p_2}$, that is, $x_{d+1}^{p_1} < x_{d+1}^{p} < x_{d+1}^{p_2}$. Due to symmetry, we may restrict attention to $x_{d+1}^{p_1} \geq 0$. (To simplify notations, when it is clear from the context, we may omit p from x_{d+1}^{p} and write x_{d+1}.) Let $p' = (x_1^{p_1}, \ldots, x_d^{p_1}, 0)$. For ease of notation, let $a_i = \text{dist}^2(s_i, p')$ and let $b_i(x_{d+1}) = a_i + x_{d+1}^2$. Note that $\text{dist}^{2\alpha}(s_i, p) = b_i^{\alpha}(x_{d+1})$, for every $i \in \{1, \ldots, n\}$. Thus, the SINR function of s_1 restricted to such point p is given by

$$\text{SINR}(s_1, p) = \frac{\frac{\psi_1}{b_1(x_{d+1})^{\alpha}}}{\sum_{i=2}^{n} \frac{\psi_i}{b_i(x_{d+1})^{\alpha}} + N}.$$

Let $l_i(x_{d+1}) = b_1(x_{d+1})/b_i(x_{d+1})$ and $m_i(x_{d+1}) = (a_i - a_1)/b_i(x_{d+1})^2$. In this context, it may be convenient to consider the reciprocal of the SINR function (Eq. (2)),

$$\text{SINR}^{-1}(s_1, p) = \sum_{i=2}^{n} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha}(x_{d+1}) + \frac{N \cdot b_1^{\alpha}(x_{d+1})}{\psi_1}. \tag{9}$$

We first show that this function is twice differentiable in x_{d+1} on $\overline{p_1\, p_2}$. In particular, it is sufficient to show that it is continuous. Assume the contrary. Since the function is undefined only at stations positions, discontinuity implies that there might exist some station $s_i \in \overline{p_1\, p_2}$, where $2 \leq i \leq n$. Since $x_{d+1}^{s_i} = 0$, only p_1 might correspond to such s_i. But p_1 is a reception point of s_1, contradiction. To characterize the optimum points, we next consider the first and second derivatives of the function SINR^{-1} on $\overline{p_1\, p_2}$.

Note that $\frac{\partial l_i(x_{d+1})}{\partial x_{d+1}} = 2x_{d+1} \cdot m_i(x_{d+1})$, $\frac{\partial m_i(x_{d+1})}{\partial x_{d+1}} = -4x_{d+1} m_i(x_{d+1})/b_i(x_{d+1})$, and $\frac{\partial b_i(x_{d+1})}{\partial x_{d+1}} = 2x_{d+1}$. Thus,

$$\frac{\partial \text{SINR}^{-1}(s_1, p)}{\partial x_{d+1}} = 2\alpha \cdot x_{d+1} \left(\sum_{i=2}^{n} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-1}(x_{d+1}) \cdot m_i(x_{d+1}) + \frac{N \cdot b_1^{\alpha-1}(x_{d+1})}{\psi_1} \right), \tag{10}$$

and

$$\frac{\partial^2 \mathrm{SINR}^{-1}(s_1, p)}{\partial x_{d+1}^2} = 2\alpha \left(\sum_{i=2}^{n} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-1}(x_{d+1}) \cdot m_i(x_{d+1}) + \frac{N \cdot b_1^{\alpha-1}(x_{d+1})}{\psi_1} \right) + 4\alpha \cdot (\alpha - 1) \cdot x_{d+1}^2 \cdot \left(\sum_{i=2}^{n} \frac{\psi_i}{\psi_1} l_i^{\alpha-2}(x_{d+1}) \cdot m_i^2(x_{d+1}) + \frac{N \cdot b_1^{\alpha-2}(x_{d+1})}{\psi_1} \right) - 8\alpha x_{d+1}^2 \sum_{i=2}^{n} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-1}(x_{d+1}) \cdot \frac{m_i(x_{d+1})}{b_i(x_{d+1})}. \quad (11)$$

Let $\mathcal{J}_{pos} = \{i \in \{2, \ldots, n\} \mid a_i \geq a_1\}$ and $\mathcal{J}_{neg} = \{i \in \{2, \ldots, n\} \mid a_i < a_1\}$.

We distinguish between two cases.

Case 1. $\mathcal{J}_{neg} = \emptyset$. In this case, $m_i(x_{d+1}) \geq 0$, therefore, by Eq. (10), we get that $\frac{\partial \mathrm{SINR}^{-1}(s_1,p)}{\partial x_{d+1}} \geq 0$, for every $p \in \overline{p_1\, p_2}$. This implies that $\mathrm{SINR}^{-1}(s_1, p) \leq \mathrm{SINR}^{-1}(s_1, p_2)$, thus $\mathrm{SINR}(s_1, p) \geq \mathrm{SINR}(s_1, p_2) \geq \beta$ as required.

Case 2. $\mathcal{J}_{neg} \neq \emptyset$. There exists some $2 \leq i \leq n$ such that $a_i < a_1$. This implies the possible existence of other optimum points. Consider an optimum point of the form $p_{opt} = (x_1^{p_1}, \ldots, x_d^{p_1}, x_{d+1}^{opt})$, where $x_{d+1}^{opt} \neq 0$. Thus by Eq. (10), we have that

$$\frac{\partial \mathrm{SINR}^{-1}(s_1, p_{opt})}{\partial x_{d+1}^{opt}} = 2\alpha \cdot x_{d+1}^{opt} \left(\sum_{i=2}^{n} \left(\frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-1}\left(x_{d+1}^{opt}\right) \cdot m_i\left(x_{d+1}^{opt}\right) \right) + \frac{N \cdot b_1^{\alpha-1}\left(x_{d+1}^{opt}\right)}{\psi_1} \right) = 0. \quad (12)$$

In turn, this implies that

$$\sum_{i=2}^{n} \frac{\psi_i}{\psi_1} \left(l_i^{\alpha-1}\left(x_{d+1}^{opt}\right) \cdot m_i\left(x_{d+1}^{opt}\right) \right) + \frac{N \cdot b_1^{\alpha-1}\left(x_{d+1}^{opt}\right)}{\psi_1} = 0. \quad (13)$$

Plugging this equality into Eq. (12), the second derivative of SINR^{-1} at p_{opt} becomes

$$\frac{\partial^2 \mathrm{SINR}^{-1}(s_1, p_{opt})}{\partial \left(x_{d+1}^{opt}\right)^2} = 4\alpha(\alpha - 1)\left(x_{d+1}^{opt}\right)^2 \left(\sum_{i=2}^{n} \frac{\psi_i}{\psi_1} l_i^{\alpha-2}\left(x_{d+1}^{opt}\right) \cdot m_i^2\left(x_{d+1}^{opt}\right) + \frac{N \cdot b_1^{\alpha-1}\left(x_{d+1}^{opt}\right)}{\psi_1} \right) - 8\alpha\left(x_{d+1}^{opt}\right)^2 \sum_{i=2}^{n} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-1}\left(x_{d+1}^{opt}\right) \cdot \frac{m_i\left(x_{d+1}^{opt}\right)}{b_i\left(x_{d+1}^{opt}\right)}. \quad (14)$$

To prove the lemma, we wish to show that the SINR function has no-local minimum on the vertical line segment, $\overline{p_1\, p_2}$, or that the second derivative of SINR^{-1} restricted to extreme internal points in the segment is nonnegative. Define

$$\wp\left(x_{d+1}^{opt}\right) = -8\alpha\left(x_{d+1}^{opt}\right)^2 \sum_{i=2}^{n} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-1}\left(x_{d+1}^{opt}\right) \cdot \frac{m_i(x_{d+1})}{b_i(x_{d+1})}. \quad (15)$$

Since $\alpha \geq 1$, $l_i(x_{d+1}^{opt}) \geq 0$, and $b_1(x_{d+1}^{opt}) \geq 0$, thus by Eq. (14), it is sufficient to show that $\wp(x_{d+1}^{opt}) \geq 0$. Note that $\mathcal{J}_{pos}$ and $\mathcal{J}_{neg}$ separate Eq. (15) into its positive and negative terms. Let

$$S_{pos} = \sum_{i \in \mathcal{J}_{pos}} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-1}\left(x_{d+1}^{opt}\right) \cdot \frac{m_i\left(x_{d+1}^{opt}\right)}{b_i\left(x_{d+1}^{opt}\right)} \quad \text{and} \quad S_{neg} = \sum_{i \in \mathcal{J}_{neg}} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-1}\left(x_{d+1}^{opt}\right) \cdot \frac{m_i\left(x_{d+1}^{opt}\right)}{b_i\left(x_{d+1}^{opt}\right)}.$$

Then, $\wp(x^{opt}_{d+1}) = -8(x^{opt}_{d+1})^2(S_{pos}+S_{neg})$. Recall that by the definition of $b_1(x^{opt}_{d+1})$, it follows that

$$b_1\big(x^{opt}_{d+1}\big) \; < \; b_i\big(x^{opt}_{d+1}\big) \quad \text{iff} \quad a_i \; > \; a_1,$$

for any $1 \le i \le n$. Since $b_i(x^{opt}_{d+1}) > 0$ for every i, we have that

$$S_{pos} \; \le \; \sum_{i \in \mathcal{J}_{pos}} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-1}\big(x^{opt}_{d+1}\big) \cdot \frac{m_i\big(x^{opt}_{d+1}\big)}{b_1\big(x^{opt}_{d+1}\big)} \quad \text{and} \quad S_{neg} \; < \; \sum_{i \in \mathcal{J}_{neg}} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-1}\big(x^{opt}_{d+1}\big) \cdot \frac{m_i\big(x^{opt}_{d+1}\big)}{b_1\big(x^{opt}_{d+1}\big)},$$

This implies that

$$S_{pos}+S_{neg} \; < \; \frac{1}{b_1\big(x^{opt}_{d+1}\big)} \cdot \left(\sum_{i=2}^{n} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-1}\big(x^{opt}_{d+1}\big) \cdot m_i\big(x^{opt}_{d+1}\big) \right) = \frac{1}{b_1\big(x^{opt}_{d+1}\big)} \cdot \frac{-N \cdot b_1^{\alpha-1}\big(x^{opt}_{d+1}\big)}{\psi_1} \le 0,$$

where the equality holds by Eq. (13). Thus, $\wp(x^{opt}_{d+1}) \ge 0$ and $\partial^2 \mathrm{SINR}^{-1}(s_1, p_{opt})/\partial(x^{opt}_{d+1})^2 \ge 0$. That is, any local optimum point other than $p' = (x^{p_1}_1, \ldots, x^{p_1}_d, 0)$ is a local minimum. Since Eq. (9) is continuous and twice differentiable in $\overline{p_1\,p_2}$, this case corresponds to three local optimum points: two local minima, namely, $(x^{p_1}_1, \ldots, x^{p_1}_d, x^{opt}_{d+1})$ and $(x^{p_1}_1, \ldots, x^{p_1}_d, -x^{opt}_{d+1})$, and one local maximum point, $p' = (x^{p_1}_1, \ldots, x^{p_1}_d, 0)$ in between. In sum, there is no local maximum inside $\overline{p_1\,p_2}$, which implies that

$$\mathrm{SINR}^{-1}(s_1, p) \le \max\{(\mathrm{SINR}^{-1}(s_1, p_1)), \mathrm{SINR}^{-1}(s_1, p_2)\},$$

and thus

$$\mathrm{SINR}(s_1, p) \ge \min\{\mathrm{SINR}(s_1, p_1), \mathrm{SINR}(s_1, p_2)\} \ge \beta.$$

The lemma follows. □

A direct consequences of this claim is the following.

Corollary 5.3. *Let $p_1, p_2 \in \mathcal{H}_1(\mathcal{A}_{d+1})$ be points satisfying Eq. (8) s.t $x^{p_1}_j = x^{p_2}_j$ for $j \in \{1, \ldots, d\}$. Then the line L extrapolated by the segment $\overline{p_1\,p_2}$ intersects $\mathcal{H}_1(\mathcal{A}_{d+1})$ at most four times.*

Proof. The proof follows immediately by the fact that L has at most three extremum points. Note that in general, the number of intersections is bounded by $O(n)$, due to the degree of the SINR function. □

5.2. Analysis of Case HC2

Let $p_1, p_2 \in \mathbb{R}^{d+1}$ be two points of interest such that $x^{p_1}_j \ne x^{p_2}_j$ for some $j \in \{1, \ldots, d\}$ (recall that p_1 and p_2 obey Inequality (8)). The hyperbolic geodesic of p_1 and p_2, $\widehat{p_1\,p_2}$, is defined as follows. Let $p^d_1 = (x^{p_1}_1, \ldots, x^{p_1}_d, 0)$ and $p^d_2 = (x^{p_2}_1, \ldots, x^{p_2}_d, 0)$ be the projection of the points p_1 and p_2 to the hyperplane $x_{d+1} = 0$, respectively. Consider a point $q \in \mathbb{R}^d \times \{0\}$ equidistant from p_1 and p_2 and positioned on the line defined by the points p^d_1 and p^d_2. Let $r = \mathrm{dist}(p_1, q) = \mathrm{dist}(p_2, q)$. The hyperbolic geodesic, $\widehat{p_1\,p_2}$, corresponds to the shorter arc connecting p_1 and p_2 on the circumference $\Phi(B^{d+1}(q, r))$.

Lemma 5.4. *Let $p_1, p_2 \in \mathcal{H}_1(\mathcal{A}_{d+1})$ obeying (8). Then $\widehat{p_1\,p_2} \subseteq \mathcal{H}_1(\mathcal{A}_{d+1})$. (For a pictorial description, see Figure 7(b)).*

Proof. By Lemma 2.3, we may assume without loss of generality that $x^{p_1}_j = x^{p_2}_j$ for $j \in \{2, \ldots, d\}$, and by q definition, it follows that $x^q_j = x^{p_1}_j$ for $j \in \{2, \ldots, d\}$. Due to symmetry, we may restrict attention to the case where $x^{p_1}_{d+1} \ge 0$ and $x^{p_2}_{d+1} \ge 0$. We begin by showing that the SINR function has no local minimum on $\widehat{p_1\, p_2}$. Recall that $r = \text{dist}(q, p_1)$. The circumference $\Phi(B^{d+1}(q,r))$ is defined by the following equation:

$$\sum_{j=1}^{d} \left(x_j - x_j^q\right)^2 + x_{d+1}^2 \;=\; r^2. \tag{16}$$

Equivalently, the x_{d+1} coordinate of points on the circumference can be expressed as

$$x_{d+1} = \pm \sqrt{r^2 - \sum_{j=1}^{d} \left(x_j - x_j^q\right)^2}.$$

Let $g(x_1, \ldots, x_d) = \sqrt{r^2 - \sum_{j=1}^{d}(x_j - x_j^q)^2}$, for every $(x_1, \ldots, x_d) \in \mathbb{R}^d$. We consider the function $\text{SINR}^{-1}(s_1, p)$ of Eq. (2), restricted to a point $p = (x_1, \ldots, x_d, g(x_1, \ldots, x_d))$ on $\Phi(B^{d+1}(q,r))$. For ease of notation, let $a_i = (x_1^{s_i} - x_1^q)$ and $b_i = \sum_{j=1}^{d}((x_j^{s_i})^2 - (x_j^q)^2) - 2\sum_{j=2}^{d}(x_j^{s_i} - x_j^q)x_j + r^2$. We then have that

$$\text{dist}^2(s_i, p) \;=\; \sum_{j=1}^{d+1} \left(x_j^{s_i} - x_j\right)^2 \;=\; b_i - 2a_i x_1. \tag{17}$$

Let $l_i(x_1) = \text{dist}^2(s_1, p)/\text{dist}^2(s_i, p)$. By plugging Eq. (17) into the SINR^{-1} function (Eq. (2)), we get

$$\text{SINR}^{-1}(s_1, p) \;=\; \sum_{i=2}^{n} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha}(x_1) + \frac{N(b_1 - 2a_1x_1)^{\alpha}}{\psi_1}. \tag{18}$$

Note that since $x_j^{p_1} = x_j^{p_2}$ for $j \in \{2, \ldots, d\}$, it follows by the definition of $\widehat{p_1\, p_2}$, that $x_j^{p} = x_j^{p_1}$ for $j \in \{2, \ldots, d\}$ for every $p \in \widehat{p_1\, p_2}$. To characterize the optimum points, it is sufficient, therefore, to consider the derivatives of Eq. (18) with respect to x_1 only (i.e., treating x_j for $j \in \{2, \ldots, d\}$ as constants). It is important to note that Eq. (18) is twice differentiable on $\widehat{p_1\, p_2}$. To see this, it in enough to argue that it is continuous or that no station s_i other than s_1 belongs to $\widehat{p_1\, p_2}$ (this is indeed a sufficient condition for continuity). Assume, to the contrary, that there might be some station $s_i \in \widehat{p_1\, p_2}$ for $2 \le i \le n$. Since the arc endpoints, p_1 and p_2, are in $\mathcal{H}_1(\mathcal{A}_{d+1})$, neither of them correspond to s_i, for $i > 1$. It follows that s_i occurs at some internal point on the arc. Since p_1 and p_2 satisfy Inequality (8), it follows that $x_{d+1}^{p} > 0$ for every $p \in \widehat{p_1\, p_2} \setminus \{p_1, p_2\}$. Yet, $x_{d+1}^{s_i} = 0$, for every $s_i \in S$, yielding a contradiction. Define $m_i(x_1) = 2\,(a_i b_1 - a_1 b_i)\,/\,(b_1 - 2a_1x_1)^2$, for examples, $m_i(x_1) = \partial l_i(x_1)/\partial x_1$. Note that $\partial m_i(x_1)/\partial x_1 = \frac{4a_i \cdot m_i(x_1)}{b_1 - 2a_1x_1} = \frac{-4a_i \cdot m_i(x_1)}{\text{dist}^2(s_i, p)}$.

Consider an optimum point $p_{opt} \in \widehat{p_1\, p_2} \setminus \{p_1, p_2\}$. This optimum point satisfies

$$\frac{\partial \text{SINR}^{-1}(s_1, p_{opt})}{\partial x_1^{opt}} \;=\; \alpha \left(\sum_{i=2}^{n} \frac{\psi_i}{\psi_1} \cdot l_i\left(x_1^{opt}\right)^{\alpha-1} \cdot m_i\left(x_1^{opt}\right) - \frac{2a_1 \cdot N \cdot \left(b_1 - 2a_1 x_1^{opt}\right)^{\alpha-1}}{\psi_1} \right) = 0. \tag{19}$$

The second derivative with respect to x_1^{opt} is given by

$$\frac{\partial^2 \mathrm{SINR}^{-1}(s_1, p_{opt})}{\partial \left(x_1^{opt}\right)^2} = \alpha(\alpha-1)\left(\sum_{i=2}^{n} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-2}\left(x_1^{opt}\right) \cdot m_i^2\left(x_1^{opt}\right) + 4a_1^2 \cdot \frac{N\left(b_1 - 2a_1 x_1^{opt}\right)^{\alpha-2}}{\psi_1}\right) + 4\alpha \sum_{i=2}^{n} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-1}\left(x_1^{opt}\right) \cdot \frac{a_i \cdot m_i\left(x_1^{opt}\right)}{\mathrm{dist}^2(s_i, p_{opt})}.$$

Define

$$\wp(p_{opt}) = 4\alpha \sum_{i=2}^{n} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-1}\left(x_1^{opt}\right) \cdot \frac{a_i \cdot m_i\left(x_1^{opt}\right)}{\mathrm{dist}^2(s_i, p_{opt})}.$$

Since $\alpha \geq 1$, it is sufficient to show that $\wp(p_{opt}) \geq 0$. We separate the summation of Eq. (20) into two parts: S_{pos} and S_{neg}, the summation of elements that correspond to positive (respectively, negative) elements in the left term of Eq. (19). Formally, letting $\mathcal{J}_{pos} = \{i \in \{2, \ldots, n\} \mid a_i b_1 \geq a_1 b_i\}$ and $\mathcal{J}_{neg} = \{i \in \{2, \ldots, n\} \mid a_i b_1 < a_1 b_i\}$, we have $\wp(p_{opt}) = 4\alpha(S_{pos} + S_{neg})$, where

$$S_{pos} = \sum_{i \in \mathcal{J}_{pos}} \frac{\psi_i}{\psi_1} \cdot l_i\left(x_1^{opt}\right)^{\alpha-1} \cdot \frac{a_i \cdot m_i\left(x_1^{opt}\right)}{\mathrm{dist}^2(s_i, p_{opt})} \quad \text{and} \quad S_{neg} = \sum_{i \in \mathcal{J}_{neg}} \frac{\psi_i}{\psi_1} \cdot l_i\left(x_1^{opt}\right)^{\alpha-1} \cdot \frac{a_i \cdot m_i\left(x_1^{opt}\right)}{\mathrm{dist}^2(s_i, p_{opt})}.$$

Let $c_i(x) = a_i/\mathrm{dist}^2(s_i, p_{opt})$. Then, $sign(c_i(x_1^p)) = sign(a_i)$ for any $p \in \Phi(B^{d+1}(q,r)) \setminus \{s_1\}$. Therefore it follows that $c_1(x_1^{opt}) \leq c_i(x_1^{opt})$ if $a_i b_1 \geq a_1 b_i$ (i.e., $i \in \mathcal{J}_{pos}$), and that $c_1(x_1^{opt}) > c_i(x_1^{opt})$ if $a_i b_1 < a_1 b_i$ (i.e., $i \in \mathcal{J}_{neg}$), implying that

$$S_{pos} \geq \sum_{i \in \mathcal{J}_{pos}} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-1}\left(x_1^{opt}\right) \cdot c_1\left(x_1^{opt}\right) \cdot m_i\left(x_1^{opt}\right) \quad \text{and} \quad S_{neg} > \sum_{i \in \mathcal{J}_{neg}} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-1}\left(x_1^{opt}\right) \cdot c_1\left(x_1^{opt}\right) \cdot m_i\left(x_1^{opt}\right).$$

Therefore,

$$\begin{aligned}
\wp(p_{opt}) &\geq 4\alpha \cdot c_1\left(x_1^{opt}\right) \sum_{i=2}^{n} \frac{\psi_i}{\psi_1} \cdot l_i^{\alpha-1}\left(x_1^{opt}\right) \cdot m_i\left(x_1^{opt}\right) \\
&= 8\alpha \cdot c_1\left(x_1^{opt}\right) \cdot \frac{a_1 \cdot N \cdot (b_1 - 2a_1 x_1)^{\alpha-1}}{\psi_1} \\
&= 8\alpha \cdot \frac{a_1^2 \cdot N}{\psi_1 \cdot \mathrm{dist}^{4-2\alpha}(s_1, p_{opt})} \geq 0,
\end{aligned}$$

where the second equality follows by Eq. (19). It therefore holds that $\partial^2 \mathrm{SINR}^{-1}(s_1, p_{opt})/\partial(x_1^{opt})^2 \geq 0$ as required. We showed that there is no local maximum point of $\mathrm{SINR}^{-1}(s_1, p)$ on $\widehat{p_1\, p_2}$. Thus, there is no local minimum point of $\mathrm{SINR}(s_1, p)$ on $\widehat{p_1\, p_2}$. Hence,

$$\mathrm{SINR}(s_1, p) \geq \min(\mathrm{SINR}(s_1, p_1), \mathrm{SINR}(s_1, p_2))) \geq \beta$$

for every point $p \in \widehat{p_1\, p_2}$, as required. Lemma 5.4 follows. □

Finally, we turn to complete the proof for Thm. 5.1.

PROOF OF THEOREM 5.1. By Lemma 5.4, $\mathcal{H}_1(\mathcal{A}_{d+1})$ is hyperbolic convex. It follows that $\mathcal{H}_1(\mathcal{A}_{d+1})$ is hyperbolic star-shaped with respect to s_1 and is therefore connected. □

5.3. Application for Testing Reception Conditions

We now describe a direct implication of the hyperbolic convexity property of $\mathcal{H}_i(\mathcal{A}_{d+1})$. Let $C \in \mathbb{R}^{d+1}$ be a closed shape (not necessarily convex) that does not contain any station, $C \cap S = \emptyset$, contained in the positive (or negative) half-plane $x_{d+1} > 0$ (resp. $x_{d+1} < 0$), that is, Inequality (8) is satisfied for every two points $p_1, p_2 \in C$. The following corollary uses the hyperbolic convexity of $\mathcal{H}_i(\mathcal{A}_{d+1})$ to show that if $\Phi(C)$ receives the transmission by s_i successfully, so does any internal point $p \in C$. In addition, if no point on the boundary, $\Phi(C)$, is able to receive the transmission by s_i successfully, then $\mathrm{SINR}(s_i, p) < \beta$ for any internal point $p \in C$. In other words, for any closed shape C such that $\Phi(C) \cap \Phi(\mathcal{H}_i(\mathcal{A}_{d+1})) = \emptyset$, by testing merely the boundary $\Phi(C)$ for reception of s_i, one can deduce about the reception of an internal point $p \in C$.

COROLLARY 5.5. *(a) if $\Phi(C) \subseteq \mathcal{H}_i(\mathcal{A}_{d+1})$, then $C \subseteq \mathcal{H}_i(\mathcal{A}_{d+1})$. (b) if $\Phi(C) \cap \mathcal{H}_i(\mathcal{A}_{d+1}) = \emptyset$, then $C \cap \mathcal{H}_i(\mathcal{A}_{d+1}) = \emptyset$.*

PROOF. Property (a) follows by Lemma 5.2. To prove property (b), assume, by way of contradiction, that there exists a point $p \in C$ such that $\mathrm{SINR}_{\mathcal{A}}(s_i, p) \geq \beta$. By Theorem 5.1, $\mathcal{H}_i(\mathcal{A}_{d+1})$ is connected and is hyperbolic star-shaped with respect to s_i. This implies that there exists an arc $\widehat{p\, s_i}$ such that $\widehat{p\, s_i} \subseteq \mathcal{H}_i(\mathcal{A}_{d+1})$. Since p is an internal point and $s_i \notin C$, the arc $\widehat{p\, s_i}$ must intersect $\Phi(C)$, implying that there exists some point $q \in \Phi(C)$ such that $\mathrm{SINR}_{\mathcal{A}}(s_i, q) \geq \beta$, contradiction. □

6. THE FATNESS OF THE RECEPTION ZONES

In Section 5, we show that the reception zone $\mathcal{H}_i(\mathcal{A}_{d+1})$ of each station s_i in a nonuniform power network is hyperbolic-convex. In this section, we develop a deeper understanding of the shape of the reception zones $\mathcal{H}_i$ and $\mathcal{H}_i(\mathcal{A}_{d+1})$ by analyzing their fatness. Consider a nonuniform power network $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha \rangle$, where $S = \{s_1, \ldots, s_n\}$ and $\alpha > 0$ and $\beta > 1$ are constants. We focus on s_1 and assume that its location is not shared by any other station (otherwise, its reception zone is $\mathcal{H}_1 = \{s_1\}$). In addition, without loss of generality, we let the minimal transmission energy be 1 and denote the maximal transmission energy by Ψ. We next establish explicit bounds on the maximal and minimal radii $\Delta(s_1, \mathcal{H}_1)$ and $\delta(s_1, \mathcal{H}_1)$ of the zone $\mathcal{H}_1$, and provide a bound on the perimeter of $\mathcal{H}_i$ by bounding the length of the curve $\Phi(\mathcal{H}_i)$.

6.1. Explicit Bounds

We first establish an explicit lower bound on $\delta(s_1, \mathcal{H}_1(\mathcal{A}))$ and an explicit upper bound on $\Delta(s_1, \mathcal{H}_1(\mathcal{A}))$. To avoid cumbersome notation, we assume a two-dimensional space ($d = 2$) throughout this section; the proof trivially generalizes to arbitrary dimensions d.

Fix $\kappa = \min\{\mathrm{dist}(s_1, s_i) \mid i > 1\}$. For establishing a lower bound on $\delta(s_1, \mathcal{H}_1)$, an extreme scenario (making δ as small as possible) would be to place s_1 at $(0, 0)$ with $\psi_1 = 1$ and all other $n - 1$ stations at $(\kappa, 0)$ with $\psi_i = \Psi$ for $i \in \{2, \ldots, n\}$. For the sake of analysis, let us replace the noise N by a new imaginary station s_{n+1} located at $(\kappa, 0)$ whose power is $N \cdot \kappa^2$. This introduces the nonuniform power network $\mathcal{A}^\delta = \langle 2, \{(0, 0), (\kappa, 0), \ldots, (\kappa, 0)\}, \{1, \Psi, \ldots, \Psi, N \cdot \kappa^2\}, 0, \beta, \alpha \rangle$. Note that the signal energy of the new station s_{n+1} at point $(x, 0)$ satisfies (1) $\mathrm{E}(s, (x, 0)) > N$ for all $0 < x < \kappa$; (2) $\mathrm{E}(s_{n+1}, (x, 0)) = N$ for $x = 0$; and (3) $\mathrm{E}(s_{n+1}, (x, 0)) < N$ for all $x < 0$. Therefore, the value of $\delta(s_1, \mathcal{H}_1)$ can only get smaller by this replacement, that is, $\delta(s_1, \mathcal{H}_1(\mathcal{A}^\delta)) < \delta(s_1, \mathcal{H}_1(\mathcal{A}))$. The point q_δ whose distance to s_1 realizes $\delta(s_1, \mathcal{H}_1)$ is thus located at $(\hat{d}, 0)$

for some $0 < \hat{d} < \kappa$ that satisfies the equation $\mathrm{SINR}_{\mathcal{A}^\delta}(s_1, q_\delta) = \beta$, or $\hat{d}^{-2}/((\Psi(n-1) + N \cdot \kappa^2)(\kappa - \hat{d})^{-2}) = \beta$. Solving for $\hat{d}$ yields

$$\hat{d} = \frac{\kappa}{\sqrt{\beta(\Psi(n-1) + N \cdot \kappa^2)} + 1} \geq \frac{\kappa}{2\sqrt{2\beta \cdot \Psi \cdot n)}},$$

where the inequality follows by assuming that $N \cdot \kappa^2 \leq \Psi \cdot n$. Hence we have the following.

LEMMA 6.1. $\delta(s_1, \mathcal{H}_1(\mathcal{A})) \geq \Omega(\kappa/\sqrt{\Psi \cdot n})$.

To establish an upper bound on $\Delta(s_1, \mathcal{H}_1(\mathcal{A}))$, consider the case where s_1 transmits with power Ψ while the other stations remain silent ($\psi_i = 0$, for $i > 1$). The point q_Δ whose distance to s_1 realizes $\Delta(s_1, \mathcal{H}_1)$ is thus located at $(\pm\hat{d}, 0)$ for $\hat{d} \leq \sqrt{\Psi/(\beta \cdot N)}$, hence we get the following.

LEMMA 6.2. $\Delta(s_1, \mathcal{H}_1) \leq \sqrt{\Psi/(\beta \cdot N)}$.

COROLLARY 6.3. *The fatness parameter of $\mathcal{H}_1(\mathcal{A})$ with respect to s_1 satisfies*

$$\varphi(\mathcal{H}_1(\mathcal{A})) \leq O\left(\frac{\Psi}{\kappa} \cdot \sqrt{\frac{n}{N}}\right).$$

6.2. Bounding the Perimeter of $\mathcal{H}_1(\mathcal{A})$

We next provide an upper bound on the perimeter length $\mathrm{per}(\mathcal{H}_1(\mathcal{A}))$. The perimeter of a cell $\mathcal{H}_{1,i}(\mathcal{A})$ is the length of the closed curve given by $\Phi(\mathcal{H}_{1,i}(\mathcal{A}))$. The perimeter length of a zone is the sum of the perimeters of the cells it contains. Again we assume $d = 2$ for clarity of presentation, yet the bound can be naturally extended to any dimension d. In the case of an uniform power network, a bound on the perimeter of $\mathcal{H}_1(\mathcal{A})$ is simply given by the perimeter of the large disk of radius $\Delta(s_1, \mathcal{H}_1(\mathcal{A}))$. In the case of a nonuniform power network, $\mathcal{H}_1(\mathcal{A})$ is nonconvex, and therefore, the trivial bound of $2\pi \cdot \Delta(s_1, \mathcal{H}_1(\mathcal{A})) = O(\sqrt{\Psi/N})$ does not hold. We begin by providing the following useful fact in this context.

FACT 6.4 ([SANTALÓ 2004]). *Let C_{out} be a closed curve of length l_{out}. Let C_{in} be a curve of length l_{in} enclosed by C_{out}. Let $\mathbb{Y}(L, C)$ be the number of intersection points between the straight line L and the curve C. Then there exists a straight line L such that $\mathbb{Y}(L, C_{in}) \geq 2l_{in}/l_{out}$.*

COROLLARY 6.5. $\mathrm{per}(\mathcal{H}_1(\mathcal{A})) \leq O(\Delta(s_1, \mathcal{H}_1(\mathcal{A})) \cdot n^3)$.

PROOF. We first bound from above the perimeter of a cell $\mathcal{H}_{1,i}(\mathcal{A}) \subseteq \mathcal{H}_1$. Let $f(L)$ be the projection of $F^1_{\mathcal{A}}(p = (x, y))$ on the line $L = ax + b$. Then, $\mathtt{deg}(f) \leq 2n$ (when $N \neq 0$) and $\mathbb{Y}(L, \Phi(\mathcal{H}_{1,i}(\mathcal{A}))) \leq 2n$. Recall that any connected cell $\mathcal{H}_{1,i}(\mathcal{A})$ is enclosed by a disk of radius $\Delta(s_1, \mathcal{H}_1(\mathcal{A}))$. Combining this with Fact 6.4, we have that there exists a line L such that

$$\frac{2 \cdot \mathrm{per}(\mathcal{H}_{1,i}(\mathcal{A}))}{2\pi \cdot \Delta(s_1, \mathcal{H}_1(\mathcal{A}))} \leq \mathbb{Y}(L, \Phi(\mathcal{H}_{1,i}(\mathcal{A}))) \leq 2n.$$

Hence, $\mathrm{per}(\mathcal{H}_{1,i}(\mathcal{A})) \leq 2\pi \cdot \Delta(s_1, \mathcal{H}_1(\mathcal{A})) \cdot n$ for every $i \in \{1, \ldots, O(n^2)\}$. Summing over the connected cells of s_1, whose number is at most $O(n^2)$ by Theorem 4.1, the claim follows. $\square$

In summary, we get the following.

THEOREM 6.6. *In a nonuniform network* $\mathcal{A} = \langle d = 2, S, \psi, N, \beta, \alpha = 2\rangle$, *where* $S = \{s_1, \ldots, s_{n-1}\}$ *and* $\alpha > 0$ *and* $\beta > 1$ *are constants, if* $\kappa = \min\{\text{dist}(s_1, s_i) \mid i > 1\} > 0$, *then*

$$\Omega\left(\frac{\kappa^2}{\Psi \cdot n}\right) \le \text{area}(\mathcal{H}_1(\mathcal{A})) \le O\left(\frac{\Psi}{N}\right),$$

$$\Omega\left(\frac{\kappa}{\sqrt{\Psi \cdot n}}\right) \le \text{per}(\mathcal{H}_1(\mathcal{A})) \le O\left(n^3 \cdot \sqrt{\frac{\Psi}{N}}\right).$$

7. APPROXIMATE POINT LOCATION

Our goal in this section is to show how a data structure supporting approximate point location queries is constructed. The performance of our constructions (e.g., memory requirements, query time) would be a function of the fatness bound established in the previous section.

7.1. The Setting

Consider a nonuniform power network $\mathcal{A} = \langle d = 2, \{s_1, \ldots, s_n\}, \Psi, N, \beta, \alpha = 2\rangle$. Given some point $p \in \mathbb{R}^2$, we are interested in the question: is s_1 heard at p under the interference of $S \setminus \{s_1\}$ and background noise N? One can directly compute $\text{SINR}_{\mathcal{A}}(s_1, p)$ in time $\Theta(n)$ and answer this question. However, typically, this question is asked for many different points p, thus linear time computations may be too expensive. Our goal in this section is to provide mechanisms that answer some approximated variants of this question much faster. In Section 7.2, we present a point location scheme for the case where all stations are aligned on a line. In Section 7.3, we provide several schemes for point location for the general case where the stations are embedded in $\mathbb{R}^d$. Generally speaking, the mechanisms we present construct an efficient data structure that maintains a partition of the Euclidean plane. We consider two types of data structures. The first partitions the plane, for every station s_i with reception zone $\mathcal{H}_i$, into three disjoint zones, $\mathbb{R}^2 = \mathcal{H}_i^+ \cup \mathcal{H}_i^- \cup \mathcal{H}_i^?$, such that (1) $\mathcal{H}_i^+ \subseteq \mathcal{H}_i$; (2) $\mathcal{H}_i^- \cap \mathcal{H}_i = \emptyset$; and (3) $\mathcal{H}_i^?$ is a bounded set determined by the requested accuracy level of the algorithm. The second type partitions the plane, again for a given station s_i, into two disjoint zones $\mathbb{R}^2 = \mathcal{H}_i^+ \cup \mathcal{H}_i^-$, such that the set of misclassified points is bounded. We construct a separate data structure $\texttt{QDS}_i$ for every $1 \le i \le n$. The final data structure $\texttt{DS}$ is the union of the n data structures $\texttt{QDS}_1, \ldots, \texttt{QDS}_n$.

Given a query point $p \in \mathbb{R}^2$, $\texttt{QDS}_i$ answers in logarithmic time (with respect to the fatness parameter and number of stations) whether p is in $\mathcal{H}_i^+$, $\mathcal{H}_i^-$, or $\mathcal{H}_i^?$ (this last case being possible only for $\texttt{QDS}_i$ of the first type).

Recall that by Lemma 2.1, a point p cannot be in $\mathcal{H}_i$ unless it belongs to $\text{WV}\textsc{or}_i(V_{\mathcal{A}})$, where $\text{WV}\textsc{or}_i(V_{\mathcal{A}})$ is the weighted Voronoi cell of s_i with weight $w_i = \psi_i^{1/\alpha}$. Thus for such a point p there is no need to query the data structure $\texttt{QDS}_j$ for any $j \neq i$.

Due to Aurenhammer and Edelsbrunner [1984], a weighted Voronoi diagram of quadratic size for the n stations is constructed in $O(n^2)$ preprocessing time. Then given a query point $p \in \mathbb{R}^2$, the station s_i such that $p \in \text{WV}\textsc{or}(s_i)$ can be identified in time $O(\log n)$. We then invoke the appropriate data structure $\texttt{QDS}_i$.

We first provide some notation and then present the common framework for all point-location schemes discussed next. For ease of notation, we focus hereafter on station s_1 and its data structure $\texttt{QDS}_1$, and let $F_\beta(p)$ denote the characteristic polynomial of $\mathcal{H}_1(\mathcal{A})$, namely, $F^1_{\mathcal{A}}(p)$, see Eq. (3). In the same manner, the characteristic polynomial of $\mathcal{H}_1(\mathcal{A}_{\beta'})$, for $\beta' \neq \beta$, is given by $F_{\beta'}(p)$. Let $\texttt{QDS} = \texttt{QDS}_1$. In addition, for station $s_i \in S$, define Δ_i as the upper bound on $\Delta(s_i, \mathcal{H}_i)$, δ_i as the lower bound on $\delta(s_i, \mathcal{H}_i)$ and φ_i as the upper bound on $\varphi(\mathcal{H}_i)$ (i.e., Δ_i/δ_i).

Procedure SegTest $(\sigma, F_\beta(p))$

(1) Employ Sturm condition on σ for $F_\beta(p)$,
let t be the number of distinct intersection points of σ and $F_\beta(p)$.
(2) Evaluate $F_\beta(p)$ on σ endpoints, p_1 and p_2.
(a) If $t = 0$
— If $F_\beta(p_1) > 0$ *and* $F_\beta(p_2) > 0$ return $-$;
— Else, return $+$;
(b) Else, return $?$;

Fig. 8. Pseudocode of algorithm SegTest.

QDS is based upon imposing a $\gamma \in \mathbb{R}_{>0}$-spaced *grid*, denoted by G_γ, on the Euclidean plane, γ is determined later on. The notions of grid *columns*, *rows*, *vertices*, *edges*, and *cells* are defined in the natural manner. We assume that G_γ is aligned so that the point s_1 is a grid vertex.

The parameter γ is set to be sufficiently small so that the cell containing point s_1 is internal to the ball inscribed in $\mathcal{H}_1$, namely, $B(s_1, \delta_1)$.

In fact, we take $\gamma \le \min\{\delta_1/(2\sqrt{2})\}$ so that the ball of radius δ_1 centered at s_1 is guaranteed to contain $\Omega((\Delta_1/\delta_1)^2)$ cells (all of them are internal by definition). The main ingredient of our algorithm is a *segment testing* procedure [Avin et al. 2009], named hereafter procedure SegTest. Given a segment σ, this procedure returns the number of distinct intersection points of σ and $\Phi(\mathcal{H}_1(\mathcal{A}))$. The segment test is implemented to run in time $O(n^2)$ by employing the Sturm condition [Basu et al. 2003] of the projection of the polynomial $F_\beta(p)$ on σ and by direct calculation of the SINR function[2] in the endpoints of σ. In particular, the segment testing allows one to decide whether $\sigma \cap \mathcal{H}_1(\mathcal{A}) = \emptyset$ or not. Procedure SegTest, presented formally next, is common to the two point-location schemes presented later on.

Given a grid G_γ, procedure SegTest (Figure 8) is invoked for each of the four edges for every cell $c_i \in G_\gamma$ at distance at most Δ_1 from s_1. The overall number of invocations is thus bounded by $O(\pi \cdot \Delta_1^2/\gamma^2)$. The difference between the schemes we present is in the definition of the performance parameter ε. We conclude this section by evaluating the memory costs $M_{\mathcal{A}}(\texttt{QDS})$, preprocessing time $T_{\mathcal{A}}(\texttt{QDS})$, and query time $T_{\mathcal{A}}^{query}(\texttt{QDS})$ of the procedure and the schemes that use it, in terms of γ. Each of the schemes chooses γ so that the error is controlled (where the precise notion of error is scheme-dependent). We begin with bounding the size of the data structure QDS. Let C_γ denote the number of cells in G_γ. Then, due to area consideration, $C_\gamma = O((\Delta_1/\gamma)^2)$. It is required to keep the tag of each cell tag, therefore QDS is of size

$$M_{\mathcal{A}}(\texttt{QDS}) = O((\Delta_1/\gamma)^2). \tag{20}$$

Note that it is sufficient to keep in QDS only cells in $\mathcal{H}_1^+ \cup \mathcal{H}_1^?$. Next, we bound the preprocessing time complexity, $T_{\mathcal{A}}(\texttt{QDS})$. The dominating steps are the invocations of procedure SegTest (Figure 8). As the cost of a single SegTest invocation is $O(n^2)$ and there are $O(C_\gamma)$ invocations, we get

$$T_{\mathcal{A}}(\texttt{QDS}) = O(((n \cdot \Delta_1)/\gamma)^2). \tag{21}$$

[2]The segment test procedure can be implemented in time $O(n \log n)$ by applying advanced numerical techniques.

Algorithm SturmCellCollinear $(c_i, F_\beta(p))$

(1) For any edge e_j of c_i
— $t_j \leftarrow$ SegTest$(e_j, F_\beta(p))$;
(2) If $t_j = \ -\ $ for any $j \in \{1, ..., 4\}$ return $-$;
(3) If $t_j = \ +\ $ for any $j \in \{1, ..., 4\}$ return $+$;
(4) return ?;

Fig. 9. Pseudocode of algorithm SturmCellCollinear.

Finally, we analyze the time for a single point-location query, which corresponds to the time for finding the cell to which p belongs. This can be done by preforming binary search on C_γ cells. Recall that there is a prior step involving an access to the weighted Voronoi diagram data structure. As mentioned, that step is bounded by $O(\log n)$, which is dominated by $O(\log C_\gamma)$. Therefore,

$$T_{\mathcal{A}}^{query}(\texttt{QDS}) \ = \ O(\log C_\gamma) = O(\log(\Delta_1/\gamma)). \tag{22}$$

7.2. Collinear Networks

In this section, we focus on the Euclidean plane $\mathbb{R}^2$ and consider a special type of nonuniform power network. A network $\mathcal{A} = \langle d = 2, \{s_1, \ldots, s_{n-1}\}, \Psi, N, \beta, \alpha = 2\rangle$ is said to be *collinear* [Avin et al. 2009] if $s_1 = (0, 0)$ and $s_i = (a_i, 0)$ for $a_i \in \mathbb{R}$ for every $1 \le i \le n - 1$. The point-location task is simpler for collinear networks due to the following lemma.

Lemma 7.1. *Let $\mathcal{A}$ be a collinear nonuniform power network. Then $\mathcal{H}_1$ is hyperbolic-convex and therefore connected.*

Proof. The proof follows immediately by Theorem 5.1, setting $d = 1$. Specifically, the stations of collinear network are essentially embedded in $\mathbb{R}^1$, and therefore their two-dimensional reception zones $\mathcal{H}_i(\mathcal{A}_{d=2})$ are hyperbolic-convex. □

Note that by Lemma 7.1, the reception zones $\mathcal{H}_1$ of collinear network follow the property of Corollary 5.5. We now complete the description of the data structure QDS. Let $c_i \in G_\gamma$ be a grid cell. Procedure SturmCellCollinear (Figure 9) is a tagging mechanism invoked for every cell $c_i \in G_\gamma$ (in fact, due to symmetry, it is sufficient to restrict attention to the half space $y \ge 0$).

QDS maintains the collection of $\mathcal{H}_1^? \cup \mathcal{H}_1^+$ cells, where $c_i \in \mathcal{H}_1^?$ if SturmCellCollinear-$(c_i, \mathcal{H}_1)$ returns ? and $c_i \in \mathcal{H}_1^+$ if SturmCellCollinear$(c_+, \mathcal{H}_1)$ returns $+$. We begin by bounding the number of cells in $\mathcal{H}_1^?$. Let C_γ be the number of rows and columns in G_γ. Then $C_\gamma \le 4\pi\,\Delta_1/\gamma$. Since $\deg(F_\beta) \le 2n$, the number of intersection points of $F_\beta(p)$ with any grid row or column is at most $2n$ (see Eq. (3) for definition). Overall, we get that the total number of intersection points of $F_\beta(p)$ with any of the C_γ rows and columns in G_γ is at most[3] $2n \cdot C_\gamma$. Hence the total number of $\mathcal{H}_1^?$ cells is bounded by $2n \cdot C_\gamma$. Since the area of each cell is γ^2, it follows that

$$\text{area}\big(\mathcal{H}_1^?\big) \ \le \ 8\pi \cdot n \cdot \Delta_1 \cdot \gamma. \tag{23}$$

[3]Note that by the hyperbolic convexity property of $\mathcal{H}_1$ we have that the number of intersection points of $F_\beta(p)$ and any vertical line (grid column) is at most 4, see Corollary 5.3; To keep things simple we do not take it into account.

```
Algorithm SturmCell (c_i, F_β(p)))
(1) For any edge e_j of c_i
    — t_j ← SegTest(e_j, F_β(p));
(2) If t_j = − for any j ∈ {1,...,4} return −;
(3) Else return +;
```

Fig. 10. Pseudocode of algorithm SturmCell.

In order to guarantee that area($\mathcal{H}_1^?$) $\leq \varepsilon \cdot$ area($\mathcal{H}_1$), we demand that $8\pi \cdot n \cdot \Delta_1 \cdot \gamma \leq \varepsilon \cdot \pi \delta_1^2$ (this is sufficient as area($\mathcal{H}_1$) $\geq B(s_1, \delta_1)$). Therefore, it is sufficient to fix

$$\gamma = \frac{\varepsilon \delta_1}{8n \cdot \varphi}. \tag{24}$$

We are now ready to establish the correctness of procedure SturmCellCollinear (Figure 9).

LEMMA 7.2.
(a) If SturmCellCollinear($c_i, \mathcal{H}_1(\mathcal{A})$) *returns* $+$, *then* $c_i \subseteq \mathcal{H}_1(\mathcal{A})$.
(b) If SturmCellCollinear($c_i, \mathcal{H}_1(\mathcal{A})$) *returns* $-$, *then* $c_i \cap \mathcal{H}_1(\mathcal{A}) = \emptyset$.
(c) Let $c_i \subseteq B(s_1, \Delta)$ *be such that* SturmCellCollinear($c_i, H(s_1, \beta)$) *returns* ?. *Then the total area of such* c_i *cells is bounded from above by* $\varepsilon \cdot$ area($\mathcal{H}_1$).

PROOF. (a) and (b) follow by Corollary 5.5, where the grid cell c_i corresponds to a closed shape whose circumference is tested. Finally (c) is guaranteed by the way we set γ. □

Let $\varphi_{\max} = \max_{i=1}^{n}\{\varphi_i\}$ and $\varphi_{\text{sum}}^4 = \sum_{i=1}^{n} \varphi_i^4$. Throughout this section, we establish the following theorem, by Eqs. (21), (20), (22), and (24).

THEOREM 7.3. *Given a collinear nonuniform power network* $\mathcal{A}$, *it is possible to construct, in* $\widetilde{O}(n^4 \cdot \varphi_{\text{sum}}^4/\varepsilon^2)$ *preprocessing time, a data structure* DS *of size* $O(n^2 \cdot \varphi_{\text{sum}}^4/\varepsilon^2)$ *that imposes a* $(2n+1)$*-wise partition* $\bar{\mathcal{H}} = \langle \mathcal{H}_1^+, \ldots, \mathcal{H}_n^+, \mathcal{H}_1^?, \ldots, \mathcal{H}_n^?, \mathcal{H}^- \rangle$ *of the Euclidean plane* $\mathbb{R}^2$ *(i.e., the zones in* $\bar{\mathcal{H}}$ *are pairwise disjoint and* $\mathbb{R}^2 = \bigcup_{i=1}^{n} \mathcal{H}_i^+ \cup \mathcal{H}^- \cup \bigcup_{i=1}^{n} \mathcal{H}_i^?$*), such that for every* $1 \leq i \leq n$ *the following hold.*

(1) $\mathcal{H}_i^+ \subseteq \mathcal{H}_i$.
(2) $\mathcal{H}^- \cap \mathcal{H}_i = \emptyset$.
(3) $\mathcal{H}_i^?$ *is bounded and its area is at most an* ε*-fraction of the area of* $\mathcal{H}_i$. *Furthermore, given a query point* $p \in \mathbb{R}^2$, *it is possible to extract from* DS, *in time* $O(\log(n \cdot \varphi_{\max}/\varepsilon))$, *the zone in* $\bar{\mathcal{H}}$ *to which* p *belongs.*

7.3. General Networks

In this section, we assume the general setting where stations are embedded in $\mathbb{R}^d$ and therefore their reception zones $\mathcal{H}_i$ are not necessarily hyperbolic convex. Specifically, we cannot assume our zones to satisfy the property of Corollary 5.5. We begin by presenting the basic tagging procedure SturmCell (Figure 10) invoked on each cell $c_i \in B(s_1, \Delta_1)$.

Let $0 < \varepsilon < 1$ be a predetermined performance parameter. We construct in $O(\varphi^2 \cdot n^2/\varepsilon^2)$ preprocessing time a data structure QDS of size $O(\varphi^2/\varepsilon^2)$. QDS essentially partitions the Euclidean plane into three disjoint zones $\mathbb{R}^2 = \mathcal{H}_1^+ \cup \mathcal{H}_1^- \cup \mathcal{H}_1^?$, where (1) $\mathcal{H}_1^+ \subseteq \mathcal{H}_1$; (2) $\mathcal{H}_1^- \cap \mathcal{H}_1 = \emptyset$; and (3) $\mathcal{H}_1^? \subseteq \mathcal{H}_1(\hat{\beta})$, for $\hat{\beta} > (1-\varepsilon)^{2\alpha} \cdot \beta$. Procedure TagCell tests $\Phi(c_i)$ for high and low β, namely, $(1+\varepsilon)^\alpha \cdot \beta$ and $(1-\varepsilon)^\alpha \cdot \beta$ respectively. If there exists at least one point $p \in \Phi(c_i)$ such that SINR$(s_1, p) \geq (1+\varepsilon)^\alpha \cdot \beta$, Algorithm TagCell $(c_i, F(p))$

(1) Let $t_1 = \mathsf{SturmCell}(c_i, F_{(1+\varepsilon)^\alpha\beta}(p))$;
(2) If $t_1 \neq -$ return $+$;
(3) Let $t_2 = \mathsf{SturmCell}(c_i, F_{(1-\varepsilon)^\alpha\beta}(p))$;
(4) If $t_2 = -$ return $-$;
(5) Else, return $?$;

Fig. 11. Pseudocode of algorithm TagCell.

then the entire cell is declared to be in $\mathcal{H}_1$. In addition, if $\mathrm{SINR}(s_1, p) < (1-\varepsilon)^\alpha \cdot \beta$, for any point $p \in \Phi(c_i)$, then the cell is declared to be out of $\mathcal{H}_1$. Otherwise the cell c_i is "questionable." Essentially, the questionable cells correspond to the case where $(1-\varepsilon)^{2\alpha} \cdot \beta \leq \mathrm{SINR}(s_1, p) \leq (1+\varepsilon)^{2\alpha} \cdot \beta$ for any $p \in c_i$.

Let γ (grid resolution) be given by

$$\gamma = \frac{\varepsilon \delta_1}{3 \cdot \sqrt{2}} \tag{25}$$

The rest of this section is dedicated for establishing the correctness of procedure TagCell (Figure 11). We begin by showing that the SINR ratio of neighboring points within a grid cell c_i is similar. This is stated formally in the following lemma.

LEMMA 7.4. *Let* $\mathrm{SINR}(s_1, p) = \hat{\beta}$. *Then,* $\mathrm{SINR}(s_1, \widetilde{p}) \in [(\frac{1-\hat{\varepsilon}}{1+\hat{\varepsilon}})^\alpha \cdot \hat{\beta}, (\frac{1+\hat{\varepsilon}}{1-\hat{\varepsilon}})^\alpha \cdot \hat{\beta}]$ *for any* $\widetilde{p} \in B(p, \sqrt{2}\gamma)$, *where* $\hat{\varepsilon} = \varepsilon/3$.

PROOF. Let $\gamma' = \sqrt{2}\gamma$. Note that we are interested in the points p such that $p \notin B(s_i, \delta_1)$ for any $s_i \in S$ (since for other points p, the location is determined easily). It then follows that

$$\begin{aligned}\mathrm{E}(s_i, \widetilde{p}) &= \psi_i \cdot \mathrm{dist}^{-\alpha}(s_1, \widetilde{p}) \geq \psi_i \cdot (\mathrm{dist}(s_i, p) + \gamma')^{-\alpha} = \psi_i \cdot (\mathrm{dist}(s_i, p) + \hat{\varepsilon} \cdot \delta_1)^{-\alpha} \\ &\geq \psi_i \cdot ((\hat{\varepsilon} + 1) \cdot \mathrm{dist}(s_i, p))^{-\alpha} \geq \frac{1}{(\hat{\varepsilon}+1)^\alpha} \mathrm{E}(s_i, p),\end{aligned}$$

relying on the equality of $\gamma' = \hat{\varepsilon} \cdot \delta_1$, which follows by Eq. (25). In the same manner,

$$\begin{aligned}\mathrm{E}(s_i, \widetilde{p}) &= \psi_i \cdot \mathrm{dist}^{-\alpha}(s_i, \widetilde{p}) \leq \psi_i \cdot (\mathrm{dist}(s_i, p) - \gamma')^{-\alpha} \leq \psi_i \cdot (\mathrm{dist}(s_i, p) - \hat{\varepsilon}\delta_1)^{-\alpha} \\ &\leq \psi_i \cdot ((1 - \hat{\varepsilon}) \cdot \mathrm{dist}(s_i, p))^{-\alpha} \leq \frac{1}{(1-\hat{\varepsilon})^\alpha} \mathrm{E}(s_i, p),\end{aligned}$$

obtained by using Eq. (25) again. Overall, we get that

$$\begin{aligned}\mathrm{SINR}(s_1, \widetilde{p}) &= \frac{\mathrm{E}(s_1, \widetilde{p})}{\mathrm{I}(S \setminus \{s_1\}, \widetilde{p}) + N} \\ &\subseteq \left[\left(\frac{1-\hat{\varepsilon}}{1+\hat{\varepsilon}}\right)^\alpha \cdot \frac{\mathrm{E}(s_1, p)}{\mathrm{I}(S \setminus \{s_1\}, p) + N}, \left(\frac{1+\hat{\varepsilon}}{1-\hat{\varepsilon}}\right)^\alpha \cdot \frac{\mathrm{E}(s_1, p)}{\mathrm{I}(S \setminus \{s_1\}, p) + N}\right] \\ &\subseteq \left[\left(\frac{1-\hat{\varepsilon}}{1+\hat{\varepsilon}}\right)^\alpha \cdot \mathrm{SINR}(s_1, p), \left(\frac{1+\hat{\varepsilon}}{1-\hat{\varepsilon}}\right)^\alpha \cdot \mathrm{SINR}(s_1, p)\right],\end{aligned}$$

which yields our claim. □

We now turn to prove the correctness of procedure TagCell.

LEMMA 7.5.

(a) *If* TagCell$(c_i, F_\beta(p))$ *returns* $+$*, then* $c_i \subseteq \mathcal{H}_1$.
(b) *If* TagCell$(c_i, F_\beta(p))$ *returns* $-$*, then* $c_i \cap \mathcal{H}_1 = \emptyset$.
(c) *Let* $c_i \subseteq B(s_1, \Delta_1)$ *be such that* TagCell$(c_i, F_\beta(p))$ *returns ?. Then* $c_i \subseteq \mathcal{H}_1((1-\varepsilon)^{2\alpha} \cdot \beta)$*, or* $\mathrm{SINR}(s_1, p) \in [(1-\varepsilon)^{2\alpha}, (1+\varepsilon)^{2\alpha}]$*, for every* $p \in c_i$*;*

PROOF. Let $\hat{\varepsilon} = \varepsilon/3$. We begin with property (a). Let c_i be such that TagCell$(c_i, F_\beta(p))$ returns $+$. That implies that there exists a point $p \in \Phi(c_i)$ such that $\mathrm{SINR}(s_1, p) \geq (1+\varepsilon)^\alpha \cdot \beta$. By Lemma 7.4, it then follows that

$$\mathrm{SINR}(s_1, \widetilde{p}) \;\geq\; \left(\frac{1-\hat{\varepsilon}}{1+\hat{\varepsilon}}\right)^\alpha (1+\varepsilon)^\alpha \cdot \beta \;\geq\; \beta, \text{ for every } \widetilde{p} \in B(p, \sqrt{2}\gamma),$$

where the last inequality follows from the choice of $\hat{\varepsilon}$. In particular, this holds for any point $\widetilde{p}$ in c_i, and (a) is established.

Let c_i be such that TagCell$(c_i, F_\beta(p)) = -$. That implies that $\mathrm{SINR}(s_1, p) < (1-\varepsilon)^\alpha \cdot \beta$, for every $p \in \Phi(c_i)$. Assume, by the way of contradiction, that there exists some point $\widetilde{p} \in c_i$ such that $\mathrm{SINR}(s_1, \widetilde{p}) \geq \beta$. Then by Lemma 7.4, it must be the case that

$$\mathrm{SINR}(s_1, p) \;\geq\; \left(\frac{1-\hat{\varepsilon}}{1+\hat{\varepsilon}}\right)^\alpha \cdot \beta \geq (1-\varepsilon)^\alpha \cdot \beta, \text{ for every } p \in B(\widetilde{p}, \sqrt{2}\gamma).$$

Thus SturmCell$(c_i, F_{(1-\varepsilon)^\alpha \cdot \beta}(x, y))$ returns $+$, and we end with contradiction, which establishes (b).

Finally, it is left to prove (c). As SturmCell$(c_i, F_{(1+\varepsilon)^\alpha \cdot \beta}(x, y))$ does not return $+$, $\mathrm{SINR}(s_1, p) < (1+\varepsilon)^\alpha$, for every $p \in \Phi(c_i)$, and therefore,

$$\mathrm{SINR}(s_1, \widetilde{p}) \;\leq\; (1+\varepsilon)^\alpha \cdot \left(\frac{1+\hat{\varepsilon}}{1-\hat{\varepsilon}}\right)^\alpha \;\leq\; (1+\varepsilon)^{2\alpha} \cdot \beta \text{ for every } p \in c_i.$$

Next, as SturmCell$(c_i, F_{(1-\varepsilon)^\alpha \cdot \beta}(x, y))$ does not return $-$, there exists $p \in \Phi(c_i)$ such that $\mathrm{SINR}(s_1, p) \geq (1-\varepsilon)^\alpha$. Therefore,

$$\mathrm{SINR}(s_1, \widetilde{p}) \;\geq\; (1-\varepsilon)^\alpha \cdot \left(\frac{1-\hat{\varepsilon}}{1+\hat{\varepsilon}}\right)^\alpha \;\geq\; (1-\varepsilon)^{2\alpha} \cdot \beta \text{ for every } p \in c_i,$$

establishes the claim. □

Let $\varphi_{\max} = \max_{i=1}^n\{\varphi_i\}$ and $\varphi_{\mathrm{sum}} = \sum_{i=1}^n \varphi_i$. Combining Eqs. (20), (21), (22), and (25), we establish the following theorem.

THEOREM 7.6. *It is possible to construct, in* $O(n^2 \cdot \varphi_{\mathrm{sum}}/\varepsilon^2)$ *preprocessing time, a data structure* DS *(given by the union of* $\mathtt{QDS}_1, \ldots, \mathtt{QDS}_n$*) of size* $O(\varphi_{\mathrm{sum}}/\varepsilon^2)$ *that imposes a* $(2n+1)$*-wise partition* $\bar{\mathcal{H}} = \langle \mathcal{H}_1^+, \ldots, \mathcal{H}_n^+, \mathcal{H}_1^?, \ldots, \mathcal{H}_n^?, \mathcal{H}^- \rangle$ *of the Euclidean plane* $\mathbb{R}^2$ *(i.e., the zones in* $\bar{\mathcal{H}}$ *are pairwise disjoint and* $\mathbb{R}^2 = \bigcup_{i=1}^n \mathcal{H}_i^+ \cup \mathcal{H}^- \cup \bigcup_{i=1}^n \mathcal{H}_i^?$*) such that for every* $1 \leq i \leq n$*the following hold.*

(1) *(1)* $\mathcal{H}_i^+ \subseteq \mathcal{H}_i$.
(2) *(2)* $\mathcal{H}^- \cap \mathcal{H}_i = \emptyset$.
(3) *(3)* $\mathcal{H}_i^? \subseteq \mathcal{H}_i((1-\varepsilon)^{2\alpha} \cdot \beta)$.

Furthermore, given a query point $p \in \mathbb{R}^2$*, it is possible to extract from* DS*, in time* $O(\log(\varphi_{\max}/\varepsilon))$*, the zone in* $\bar{\mathcal{H}}$ *to which* p *belongs.*

Overall, in this section, we present several schemes for solving approximate point location tasks. We consider two approaches for interpreting the approximation parameter ϵ. The first approach (used in our two first schemes), following Avin et al. [2012], returns an uncertain answer for points coming from an ϵ fraction of the area of the reception zone. The second approach (used in our third scheme) interprets the approximation parameter ϵ as the amount of the allowed slack on the required SINR threshold β, that is, there is an uncertainty interval of $[(1-\epsilon)\beta, (1+\epsilon)\beta]$.

The quality of the first scheme depends heavily on the fatness parameter of the regions and on the number of connected components. Improving the bounds on these key parameters will have an immediate effect on the efficiency of this scheme. We then considered the restricted case of collinear networks where the queries are in $\mathbb{R}^2$. In this case, one can exploit the connectivity of the zones (in $\mathbb{R}^2$) to obtain a better point location scheme. For a recent work on batched point location schemes whose performance guarantees do not depend on the topological parameters of the reception regions, see Aronov and Katz [2014].

8. CONCLUSION

In this article, we introduce the study of SINR diagrams with nonuniform powers. A major long-term goal of the study of SINR diagrams is to develop the area of "wireless computational geometry" in which SINR diagrams play a role that is similar to that of Voronoi diagrams in computational geometry.

Towards achieving this goal, we focus on key topological properties of these maps: connectivity and convexity. Specifically, our main efforts went into bounding the number of connected components such diagrams may assume. We also aimed toward developing a weaker notion of convexity for these reception regions as well as their application in the fundamental point location task. Whereas our key result demonstrates the hyperbolic convexity of these regions in $\mathbb{R}^{d+1}$, the question of defining nice properties for these region in $\mathbb{R}^d$ remains open. An additional major open problem concerns the current gap on the number of cells in n station networks. Other geometrical parameters such as fatness and perimeter call for tighter bounds as well.

Finally, it would be of great interest to develop additional algorithmic applications for these networks, in particular one that can enjoy their hyperbolic convexity.

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