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Capacity for the doubly dirty multiple access channel with partial side information at the transmitters

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Abstract: In this study, the authors establish achievable rate and capacity regions for three kinds of Gaussian multiple access channels (MACs) with side information (SI) partially (estimated or sensed version) and non-causally known at the transmitters. Actually, the authors show that the lattice strategy is optimal for Gaussian MACs with partial SI, the same as it is for Gaussian MACs with full SI studied by Philosof–Zamir. The rate and capacity regions, while showing that partiality in SI reduces the achievable rates, subsumes directly the Philosof–Zamir rate regions and also the Gueguen–Sayrac work and the Costa theorem indirectly as special cases.

1 Introduction

The study of communication channels with interference known at the transmitters as side information (SI) is of particular interest. In these channels, transmitters can make use of such knowledge to encode their information in order to mitigate negative effect of the interference, and hence, to achieve reliable communication with high data rates, often with rates equal to the rates of interference free channels, although the interference is not known to the receiver. SI is not necessarily an interference; it might be channel state or other information available in a way for the transmitter and or receiver. The SI known at the transmitter (SIT) generally helps the transmitter optimise its transmission.

Shannon [1] studied encoding for a single-user with causal SIT. Capacity of a discrete (i.e. finite input and output alphabets) and memoryless channel (DMC) with non-causal SIT was characterised by Gel'fand and Pinsker [2]. Costa [3] studied the Gaussian channel with interference non-causally known only at the transmitter (Fig. 1). He derived the capacity by extending the discrete alphabet Gel'fand–Pinsker (GP) capacity theorem to continuous alphabet Gaussian channel with input average power constraint $\mathbb{E}[X^2] \le P_X$, additive interference S_1 with average power P_s and additive white Gaussian noise (AWGN) Z with zero mean and average power N and independent of both S_1 and X. Costa named his coding strategy as dirty paper coding (DPC) and found the capacity $C = (1/2)\log(1 + ((P_X)/N))$. Surprisingly, his result shows that the capacity is independent of the interference S_1. Cover and Chiang [4] unified the results for discrete alphabet and memoryless channels and generalised the GP theorem to two-sided state information channels.

Gueguen–Sayrac [5] derived capacity of the Costa channel with partial SI knowledge ($\tilde{S}$) (Fig. 2) as $C = (1/2)\log(1 + ((P_X)/(D + N)))$, where $\mathbb{E}[(S - \tilde{S})^2] = \mathrm{D}$ and $\mathbb{E}[\tilde{S}^2] = \mathbb{E}[S\tilde{S}] = P_S$. Compared with the DPC with exact SI [3], it is observed that partiality in SI reduces the capacity.

In the multi-user setting, state dependent discrete (i.e. finite input, output and state alphabets) and memoryless channels have been studied. Das and Narayan [6] determined capacity region of time-varying multiple access channels (MACs) with various situations of SIT and SIR (SI at receiver). Jafar [7], presented a general framework for the capacity region of DM-MACs with causal and non-causal independent SI. Philosof–Zamir [8, 9] extended Jafar's work and provided achievable rate regions for the DM-MAC with correlated SI known non-causally at the encoders by using a random binning technique.

They also considered the Gaussian MAC with SI at both transmitters (doubly dirty MAC) in the high-SNR and strong interference regime [8–13]. They proved that positive rates are not achieved by using the Costa Gaussian binning for the doubly dirty MAC suffering from strong interferences [13]. In contrast, Philosof–Zamir [13] investigated the doubly dirty MAC by extended exploiting of Willems interference concentration idea [14] and showed that lattice-strategies can achieve positive rates independent of the interference power. Furthermore, Philosof–Zamir [13] showed that in some cases, depending on the noise variance and power constraints, high dimensional lattice strategies are in fact optimal. The DM-MAC with partial SI (e.g. [15, 16]) and the GP channel (e.g. [17]) have been studied more.

In this paper, the effect of partiality in SI at the transmitters in achieving the rate regions of (i) Gaussian doubly dirty MAC, (ii) Gaussian dirty MAC with a single dirty user and (iii) Gaussian dirty MAC with common interference are studied. It is shown that the achievable rate and capacity regions are reduced because of the partiality in SI just the same as in the Costa theorem for point to point SI channel [5]. It is readily seen that our rate regions include the regions of the MACs with full SI as special cases.

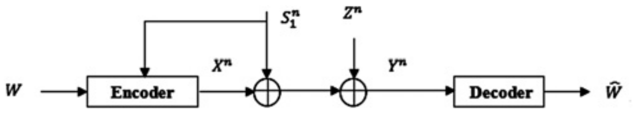

Fig. 1 *Gaussian channel with additive interference known at the transmitter*

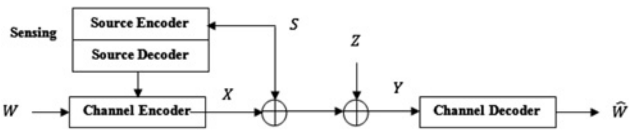

Fig. 2 *Costa channel with partial SIT [5]*

The rest of the paper is organised as follows. At the end of this section, notations are stated and some basic terminologies for lattices are reviewed. Section 2 includes the system models and our results based on lattice strategies. In particular, Section 2.1 is devoted to the Gaussian doubly dirty MAC with partial SIT. The Gaussian dirty MAC with a single dirty user and the helper problem with partial SIT and the Gaussian dirty MAC with common partial SIT are investigated in Sections 2.2 and 2.3, respectively. We have conclusion in Section 3.

1.1 Notations and lattice coding

Throughout this paper, boldface letters and capital letters denote vectors, for example, $\boldsymbol{X}$, $\boldsymbol{Y}$ and discrete random variables, for example, X, Y, respectively. A random variable X takes values in a set $\mathcal{X}$. $\|\cdot\|$ denotes the Euclidean norm of vectors, and $\mathbb{E}[.]$ is used to denote the expectation. The Gaussian distribution with mean μ and square deviation σ^2 is denoted by $N(\mu;\ \sigma^2)$. u.c.e$\{.\}$ and $[x]^+$ stands for the upper convex envelope and $\max(0, x)$, respectively.

Now, we review briefly some basic terminology from lattice theory that we will use in subsequent sections. An n-dimensional lattice Λ which is specified by the generator matrix $G \in \mathbb{R}^{n\times n}$, is the set of all integer linear combinations of the basis (columns) vectors of the $\boldsymbol{G}$. A point $1 \in \mathbb{R}^n$ belongs to the lattice if and only if it can be written as $1 = G.i$, where $i \in \mathbb{Z}^n$ and $\mathbb{Z} = \{0,\ \pm 1,\ \pm 2,\ \ldots\}$. The nearest neighbour quantiser of a lattice Λ is defined by $Q_\Lambda(x) \triangleq \arg\min_{l\in\Lambda} x - l$. The set of all points in $\mathbb{R}^n$ which are closer to the lattice point 1 than to any other lattice point is called the Voronoi region of 1. The fundamental Voronoi region $\mathcal{V}$ is the Voronoi region of $1 = 0$. The volumes of all Voronoi regions of lattice Λ are the same as the volume of the fundamental Voronoi region, that is, $V = \text{Vol}\left(\mathcal{V}(\Lambda)\right) = \int_{\mathcal{V}} \text{d}x$. The modulo-$\Lambda$ operation is defined as the quantisation error of x with respect to lattice Λ, that is

$$x \bmod \Lambda = x - Q_\Lambda(x) \tag{1}$$

The modulo-code operation satisfies

$$\left[x \bmod \Lambda + y\right] \bmod \Lambda = \left[x + y\right] \bmod \Lambda \tag{2}$$

The second moment per dimension of a uniform distribution over $\mathcal{V}$ is given by $\sigma_\Lambda^2 = (((1/n)\int_{\mathcal{V}} x^2\ \text{d}x)/V)$. The normalised second moment of the lattice Λ is defined as $G(\Lambda) = \left((\sigma_\Lambda^2)/\left(V^{(2/n)}\right)\right)$, which is always greater than $(1/(2\pi e))$. It is known [18] that when dimension n goes to infinity, there exist good lattice quantisers Λ_n with (approximately) sphere Voronoi regions such that $G(\Lambda_n) \to (1/(2\pi e))$. This means that we can model the quantisation noise of a good lattice with a white Gaussian noise. The Crypto lemma [19] states that for any $\boldsymbol{x}$ distributed over $\mathcal{V}$ and independent of U, which is uniformly distributed over $\mathcal{V}$, $(\boldsymbol{x} + U)\text{mod}\Lambda$ is independent of $\boldsymbol{x}$ and uniformly distributed over $\mathcal{V}$. Also, assume that $\boldsymbol{D}\sim\text{Unif}(\mathcal{V})$, that is, $\boldsymbol{D}$ is an n-dimensional random vector distributed uniformly over $\mathcal{V}$. The differential entropy of $\boldsymbol{D}$ is as follows [13]

$$h(\boldsymbol{D}) = \log_2(V) = \log_2\left(\frac{\sigma_\Lambda^2}{G(\Lambda)}\right)^{(n/2)} = \frac{n}{2}\log_2\left(\frac{\sigma_\Lambda^2}{G(\Lambda)}\right) \tag{3}$$

Consequently, for good lattice quantisers, we have $h(\boldsymbol{D}) \simeq (n/2)\log_2\left(2\pi e\sigma_\Lambda^2\right)$. An extensive study of lattices and lattice quantisation can be found in [20].

2 Main results

In this section, Gaussian doubly dirty MAC with partial SIT (Section 2.1), Gaussian MAC with a single dirty user and the helper problem with partial SIT (Section 2.2) and the Gaussian dirty MAC with common partial SIT (Section 2.3) are investigated. Actually, the effect of partial SI known non-causally at the transmitters is examined. We show that the achievable rate and capacity regions for cases with partial SIT are reduced. It is readily seen that our results include the regions of the MACs with full SI [13] as special cases.

2.1 Doubly dirty MAC with partial SI

In this part, the Gaussian doubly dirty MAC with partial SIT (Fig. 3) is studied. X_i, $i = 1, 2$, is the channel input transmitted by user i which is subjected to the power constraint P_i and the channel output is given by

$$\begin{aligned} Y &= X_1 + X_2 + S_1 + S_2 + Z \\ &= X_1 + X_2 + \tilde{S}_1 + \tilde{S}_2 + A_1 + A_2 + Z \end{aligned} \tag{4}$$

where Z is an AWGN with zero mean and variance N $\left(Z \sim \mathcal{N}(0,\ N)\right)$. The interference signal S_i, $i = 1, 2$, is i.i.d. Gaussian with variance $Q_i(S_i \sim \mathcal{N}(0,\ Q_i))$. $\tilde{S}_i$ is the best compressed representation of interference (S_i) which is non-causally known at the ith transmitter and satisfies (i) $\mathbb{E}\left[\left(S_i - \tilde{S}_i\right)^2\right] = \mathbb{E}\left[\left(A_i\right)^2\right] = E_i$, (ii) $A_i \sim \mathcal{N}\left(0,\ E_i\right)$ and (iii) $\mathbb{E}\left[\tilde{S}_i^2\right] = \mathbb{E}\left[S_i\tilde{S}_i\right] = P_{\tilde{s}_i}$, $i = 1,\ 2$. In the following theorems, after obtaining an outer bound on the capacity region of the Gaussian doubly dirty MAC with partial SI,

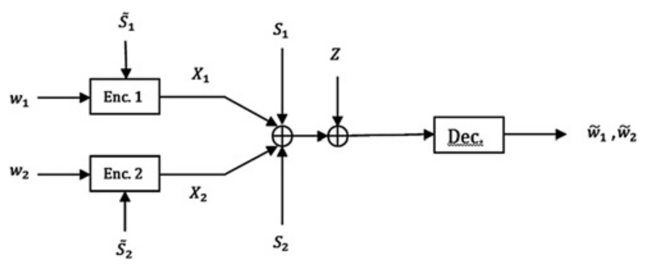

Fig. 3 *Doubly dirty MAC with partial SIT*

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achievable rate regions based on the lattice strategies are derived.

2.1.1 Outer bound for the capacity of the doubly dirty MAC with partial SIT

Theorem 1: Capacity region of the doubly dirty MAC with two independent estimated interference sequences $\tilde{\boldsymbol{S}}_1$ and $\tilde{\boldsymbol{S}}_2$ non-causally known at encoder 1 and encoder 2, respectively, is contained in the following region as long as $\boldsymbol{S}_i$ and $\tilde{\boldsymbol{S}}_i (i = 1,\ 2)$ are strong (i.e. $Q_i,\ P_{\tilde{s}_i} \to +\infty$)

$$R_1 + R_2 \le \frac{1}{2}\log\left(1 + \frac{\min(P_1,\ P_2)}{E_1 + E_2 + N}\right) \tag{5}$$

Corollary 1: Theorem 1 is reduced to [13, Corollary 2] by $E_1 = E_2 = 0$.

Proof: The proof is given in Appendix 1.

2.1.2 Inner bounds for the capacity of the doubly dirty MAC with partial SIT:

In the Gaussian doubly dirty MAC with partial SI based on the lattice transmission scheme (Fig. 4), encoder i, $i = 1, 2$, uses the lattice $\Lambda_i = k_i\Lambda$ (where k_i is a real number) with second moment $\sigma_i^2 = P_i$ and fundamental Voronoi region $\mathcal{V}_i$. The information of user i is carried by $\boldsymbol{V}_i$, where $\boldsymbol{V}_1 \in \text{Unif}(\mathcal{V}_1)$ and $\boldsymbol{V}_2 \in \text{Unif}(\mathcal{V}_2)$ are independent. Let $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ be two independent dither signals which are uniformly distributed over $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. Decoder knows both $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, whereas encoder i only knows dither signal $\boldsymbol{D}_i$. The transmitted signals by encoder 1 and encoder 2 are generated, respectively, as follows

$$\boldsymbol{X}_1 = [\boldsymbol{V}_1 - \alpha_1\tilde{\boldsymbol{S}}_1 + \boldsymbol{D}_1]\text{mod}\Lambda_1$$
$$\boldsymbol{X}_1 = [\boldsymbol{V}_2 - \alpha_2\tilde{\boldsymbol{S}}_2 + \boldsymbol{D}_2]\text{mod}\Lambda_2$$

where $\alpha_1,\ \alpha_2 \in [0,\ 1]$. The power constraints are satisfied because for any $\boldsymbol{V}_i = \boldsymbol{v}_i$, $\boldsymbol{X}_i \sim \text{Unif}(\mathcal{V}_i)$, where $\boldsymbol{X}_i$ is independent of $\boldsymbol{V}_i$. The decoder, upon receiving $\boldsymbol{Y}$, using lattice $\Lambda_r = k_r\Lambda$, computes $\boldsymbol{Y}'$ as follows

$$\boldsymbol{Y}' = [\alpha_r\boldsymbol{Y} - \gamma\boldsymbol{D}_1 - \beta\boldsymbol{D}_2]\text{mod}\Lambda_r \tag{6}$$

Note that, to achieve the desired results in each situation, we need to determine the basic lattice Λ and the scalars α_1, α_2, α_r, k_1, k_2, k_r, γ, β for the situation properly.

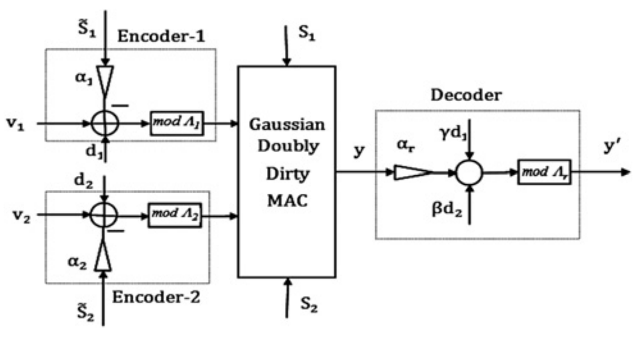

Fig. 4 *Gaussian doubly dirty MAC with partial SIT*

In the following theorems, it is provided conditions under which lattice-strategies are optimal. By considering two cases, imbalanced doubly dirty MAC with partial SIT $(E_1 + E_2 + N \le \sqrt{P_1P_2} - \min(P_1,\ P_2))$ and nearly balanced doubly dirty MAC with partial SIT $(E_1 + E_2 + N \ge \sqrt{P_1P_2} - \min(P_1,\ P_2))$, the capacity and achievable rate regions are given.

Theorem 2: In imbalanced case for $P_1 \ne P_2$, capacity region of the doubly dirty MAC with partial SI in transmitters, in the limit of strong interference meets the outer bound of Theorem 1 and is given by the set of all rate pairs $(R_1,\ R_2)$ satisfying

$$R_1 + R_2 \le \frac{1}{2}\log_2\left(1 + \frac{\min(P_1,\ P_2)}{E_1 + E_2 + N}\right) \tag{7}$$

Corollary 2: Theorem 2 is reduced to [13, Theorem 2] by $E_1 = E_2 = 0$.

Proof: The proof is given in Appendix 2.

In the following, an inner bound for the nearly balanced case is derived. First, the symmetric (exactly balanced) case, that is, $P_1 = P_2 = P$, and then, the general nearly balanced case is considered.

Theorem 3: In exactly balanced case, achievable rate region of the doubly dirty MAC with partial SIT is given by the set of all rate pairs $(R_1,\ R_2)$ satisfying

$$R_1 + R_2 \le \text{u.c.e}\left\{\left[\frac{1}{2}\log\left(\frac{1}{2} + \frac{P}{E_1 + E_2 + N}\right)\right]^+\right\} \tag{8}$$

where u.c.e is the upper convex envelope with respect to $(P/(E_1 + E_2 + N))$.

Corollary 3: Equation (8) in Theorem 3 is reduced to [13, equation (75)] by $E_1 = E_2 = 0$.

Proof: The proof is given in Appendix 3.

Theorem 4: In general nearly balanced case, achievable rate region of the doubly dirty MAC with partial SIT is given for any interferences by the set of all rate pairs $(R_1,\ R_2)$ satisfying as follows

$$R_1 + R_2 \le \text{u.c.e}\left\{\left[\frac{1}{2}\log_2\left(\frac{P_1 + P_2 + E_1 + E_2 + N}{2(E_1 + E_2 + N) + (\sqrt{P_1} - \sqrt{P_2})^2}\right)\right]^+\right\} \tag{9}$$

where the upper convex envelope is with respect to P_1, P_2.

Corollary 4: Theorem 4 is reduced to [13, Theorem 3] by $E_1 = E_2 = 0$.

Corollary 5: Theorem 4 is reduced to Theorem 3 by considering $P_1 = P_2 = P$.

Proof: The proof is given in Appendix 4.

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2.2 MAC with a single dirty user with partial SI

Fig. 5 shows the extended model of Gaussian dirty MAC with a single dirty user with partial SI and the helper problem, where only user 2 has a message to send and the user 1 helps user 2 to transmit at the highest possible rate. The input of decoder is $Y = X_1 + X_2 + S_1 + Z = X_1 + X_2 + \tilde{S}_1 + A_1 + Z$, where partial SI $(\tilde{S}_1)$ is only known for user 1 (informed user) which satisfies (i) $\mathbb{E}[(S_1 - \tilde{S}_1)^2] = E_1$, (ii) $(S_1 - \tilde{S}_1) = A_1 \sim \mathcal{N}(0,\ E_1)$ and (iii) $\mathbb{E}[S_1\tilde{S}_1] = \mathbb{E}[\tilde{S}_1^2] = P_{\tilde{s}_1}$. In the following Theorems, after deriving an outer bound for the capacity of this model, inner bounds are obtained.

2.2.1 Outer bound for the capacity of Gaussian MAC with a single dirty user with partial SIT

Theorem 5: Capacity region of the Gaussian MAC with a single dirty user with partial SIT and helper problem $(R_1 = 0)$ is contained in the following region as long as $\tilde{\boldsymbol{S}}_1$ and $\boldsymbol{S}_1$ are strong (i.e. $Q_1,\ P_{\tilde{s}_1} \to +\infty$)

$$R_2 \le \frac{1}{2}\log\left(1 + \frac{\min(P_1,\ P_2)}{E_1 + N}\right) \tag{10}$$

Corollary 6: Equation (10) is reduced to [13, Corollary 1] by considering $E_1 = 0$.

Proof: We can use the proof of Theorem 1. Note that here we have $X_1 = f(\tilde{S}_1)$ and $X_2 = g(W_2)$. □

2.2.2 Inner bounds for the capacity of Gaussian MAC with a single dirty user and with partial SIT:

A lattice-based transmission scheme is used for the Gaussian MAC with a single dirty user and with partial SIT. Any good code for quantisation and channel coding is used by the informed user, whereas any good code for channel coding is used by the uninformed user. Same as the previous section, the results depend on how balanced the SNRs are. Hence, it is defined imbalanced case, where $N + E_1 \le |P_1 - P_2|$, and nearly balanced case, where $N + E_1 \ge |P_1 - P_2|$. Using lattice-alignment transmission schemes, achievable rate regions are derived for these two cases.

Theorem 6: Considering a MAC with a single dirty user and partial SI where $N + E_1 \le |P_1 - P_2|$ (imbalanced case), capacity of the helper problem $(R_1 = 0)$ in the limit of strong interference is derived as follows

$$C_{\text{helper}}(P_1,\ P_2) = \frac{1}{2}\log\left(1 + \frac{\min(P_1,\ P_2)}{E_1 + N}\right) \tag{11}$$

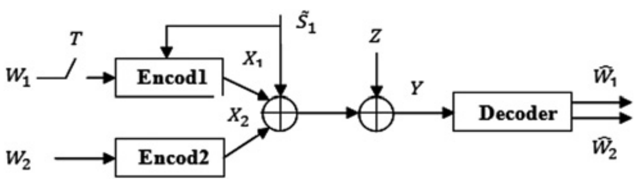

Fig. 5 *MAC with helper problem and a single dirty user with partial SIT*

Corollary 7: Theorem 6 is reduced to [13, Theorem 4] by $E_1 = 0$.

Proof: The proof is given in Appendix 5.

Theorem 7: Capacity region of the helper problem in the nearly balanced case satisfies

$$C_{\text{helper}}(P_1,\ P_2,\ N) \ge \text{u.c.e}\left\{\frac{1}{2}\log\left(1 + \frac{4P_1P_2}{(P_2 - P_1 + E_1 + N)^2 + 4P_1(E_1 + N)}\right)\right\} \tag{12}$$

where the upper convex envelope is with respect to P_1 and P_2.

If we consider exactly balanced case (nearly balanced case with equal power), then we will have $C_{\text{helper}}(P_1, P_2, N) \ge$ u.c.e$\{(1/2)\log(1 + ((4(\text{SNRP})^2)/(1 + 4\text{SNRP})))\}$, where signal to noise ratio-partial case $(\text{SNRP}) = (P/(E_1 + N))$ and the upper convex envelope is with respect to SNRP.

Corollary 7: Theorem 7 is reduced to [13, Lemma 5] by considering $E_1 = 0$.

Proof: The proof is given in Appendix 6.

2.3 MAC with partial common SIT

Fig. 6 shows the extended model of Gaussian dirty MAC with partial common SIT. Partial SI $(\tilde{\boldsymbol{S}}_c)$ instead of exact SI $(\boldsymbol{S}_c)$ is known non-causally at both transmitters which satisfies $\mathbb{E}[(\boldsymbol{S}_c - \tilde{\boldsymbol{S}}_c)^2] = E_1$ and $\mathbb{E}[\boldsymbol{S}_c\tilde{\boldsymbol{S}}_c] = \mathbb{E}[\tilde{\boldsymbol{S}}_c^2] = P_{\tilde{s}_c}$. The channel model is as $\boldsymbol{Y} = \boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{S}_c + \boldsymbol{Z}$. The capacity region of Gaussian MAC with partial common SIT is achieved using random binning and lattice strategies.

First, random binning is used. Using DPC with partial SI (like [5]) twice for each user, $(R_1,\ R_2)$ is achieved. The auxiliary random variables are considered as follows (as in the point to point case [5]), $U_1 = X_1 + \alpha_1\tilde{S}_c$ where X_1 and $\tilde{S}_c$ are independent and $\tilde{S}_c$ satisfies $\mathbb{E}[(S_c - \tilde{S}_c)^2] = E_1$, $\mathbb{E}[\tilde{S}_c^2] = \mathbb{E}[S_c\tilde{S}_c]$ and $U_2 = X_2 + \alpha_2\hat{S}_c$ where $\hat{S}_c = S_c - \alpha_1\tilde{S}_c$, X_2 and $\tilde{\hat{S}}_c$ are independent and $\tilde{\hat{S}}_c$ satisfies $\mathbb{E}[(\hat{S}_c - \tilde{\hat{S}}_c)^2] = E_2$, $\mathbb{E}[\tilde{\hat{S}}_c^2] = \mathbb{E}[\hat{S}_c\tilde{\hat{S}}_c]$. First, the model of writing on dirty paper for user 1 is considered. The channel model is supposed to be $Y = X_1 + S_c + Z_{\text{eq}}$ where $Z_{\text{eq}} = X_2 + Z$. Then, similar to [5], $\alpha_1 = ((P_1)/(P_1 + P_2 + N + E_1))$ and $R_1 = (1/2)\log(1 + ((P_1)/(P_2 + N + E_1)))$ can be obtained.

Then, the model of writing on dirty paper for user 2 is investigated. Considering $\hat{S}_c = S_c - \alpha_1\tilde{S}_c$, the equivalent channel is as $Y' = Y - U_1 = X_2 + \hat{S}_c + Z$. Hence, user 2 can achieve R_2 by using $\alpha_2 = ((P_2)/(P_2 + E_2 + N))$ as $R_2 = (1/2)\log(1 + ((P_2)/(E_2 + N)))$. Then, the corner point (13) of

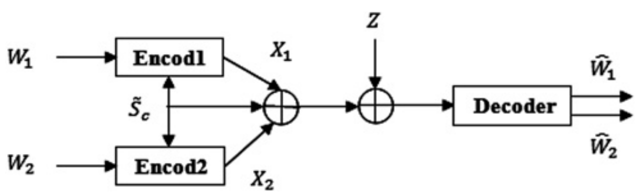

Fig. 6 *MAC with common partial SI*

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the pentagon is achieved

$$(R_1, R_2) = \left(\frac{1}{2}\log\left(1+\frac{P_1}{P_2+N+E_1}\right),\ \frac{1}{2}\log\left(1+\frac{P_2}{E_2+N}\right)\right) \tag{13}$$

In the following, lattice strategy is used for deriving the capacity region of Gaussian MAC with common SIT. User 1 and user 2 use the lattices Λ_1 and Λ_2 with second moments P_1 and P_2, respectively. The transmitted signals by encoders are as follows

$$\begin{aligned} \boldsymbol{X}_1 &= \left[\boldsymbol{V}_1 - \alpha_1\tilde{\boldsymbol{S}}_c + \boldsymbol{D}_1\right]\bmod \Lambda_1; \\ \boldsymbol{X}_2 &= \left[\boldsymbol{V}_2 - \alpha_2\tilde{\hat{\boldsymbol{S}}}_c + \boldsymbol{D}_2\right]\bmod \Lambda_2 \end{aligned} \tag{14}$$

where $\tilde{\boldsymbol{S}}_c$ is the estimated $\boldsymbol{S}_c$, $\hat{\boldsymbol{S}}_c = \boldsymbol{S}_c - \alpha_1\tilde{\boldsymbol{S}}_c$, $\tilde{\hat{\boldsymbol{S}}}_c$ is the estimated

$$\mathbb{E}\left[\left(\boldsymbol{S}_c - \tilde{\boldsymbol{S}}_c\right)^2\right] = E_1, \quad \mathbb{E}\left[\tilde{\boldsymbol{S}}_c^2\right] = \mathbb{E}\left[\boldsymbol{S}_c\tilde{\boldsymbol{S}}_c\right]$$

and

$$\mathbb{E}\left[\left(\hat{\boldsymbol{S}}_c - \tilde{\hat{\boldsymbol{S}}}_c\right)^2\right] = E_2, \quad \mathbb{E}\left[\tilde{\hat{\boldsymbol{S}}}_c^2\right] = \mathbb{E}\left[\hat{\mathbf{S}}_c\tilde{\hat{\boldsymbol{S}}}_c\right]$$

Fig. 7 shows the decoder of Gaussian MAC with common SIT. In the first stage, the decoder calculates $\boldsymbol{Y}'$ as follows

$$\begin{aligned} \boldsymbol{Y}' &= \left[\alpha_1\boldsymbol{Y} - \boldsymbol{D}_1\right]\bmod \Lambda_1 \\ &= \left[\alpha_1\left(\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{S}_c + \boldsymbol{Z}\right) - \boldsymbol{D}_1\right]\bmod \Lambda_1 \end{aligned} \tag{15}$$

$$\begin{aligned} &= \left[\boldsymbol{V}_1 - (1-\alpha_1)\boldsymbol{X}_1 + \alpha_1\left(\boldsymbol{S}_c - \tilde{\boldsymbol{S}}_c\right)\right. \\ &\quad \left. + \alpha_1\boldsymbol{X}_2 + \alpha_1\boldsymbol{Z}\right]\bmod \Lambda_1 \end{aligned} \tag{16}$$

The rate achieved by user 1 is given as follows

$$R_1 = \frac{1}{n}I\left(\boldsymbol{V}_1; \boldsymbol{Y}'\right) = \frac{1}{n}\left\{h\left(\boldsymbol{Y}'\right) - h\left(\boldsymbol{Y}'|\boldsymbol{V}_1\right)\right\} \tag{17}$$

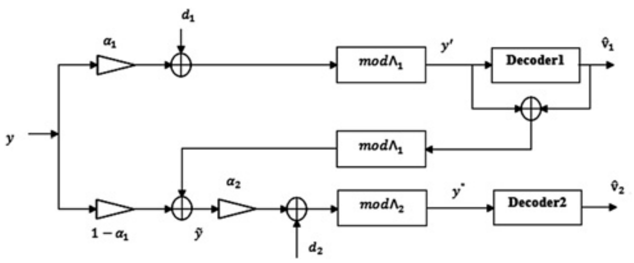

Fig. 7 *Decoder of MAC with common interference [13]*

$$\begin{aligned} &\geq \frac{1}{2}\log\left(\frac{P_1}{G(\Lambda_1)}\right) \\ &\quad - \frac{1}{2}\log\left(2\pi e\left((1-\alpha_1)^2P_1 + \alpha_1^2(P_2 + E_1 + N)\right)\right) \end{aligned} \tag{18}$$

$$\begin{aligned} &= \frac{1}{2}\log\left(\frac{P_1}{(1-\alpha_1)^2P_1 + \alpha_1^2(P_2 + E_1 + N)}\right) \\ &\quad - \frac{1}{2}\log\left(2\pi eG(\Lambda_2)\right) \end{aligned} \tag{19}$$

Using the optimal minimum mean-square error (MMSE) factor for user 1, $\alpha_1^* = \left((P_1)/(P_1 + P_2 + E_1 + N)\right)$ is obtained. For lattice that is good for quantisation, the achievable rate is given by $R_1 \leq (1/2)\log(1 + ((P_1)/(P_2 + E_1 + N)))$.

In the second stage, the effective noise is considered as follows

$$\begin{aligned} \hat{\boldsymbol{Z}}_{\text{eq}} &= \left[\boldsymbol{Y}' - \hat{\boldsymbol{V}}_1\right]\bmod \Lambda_1 \\ &= \left[-(1-\alpha_1)\boldsymbol{X}_1 + \alpha_1\left(\boldsymbol{S}_c - \tilde{\boldsymbol{S}}_c\right) + \alpha_1\boldsymbol{X}_2 + \alpha_1\boldsymbol{Z}\right]\bmod \Lambda_1 \end{aligned} \tag{20}$$

Since

$$\begin{aligned} &\frac{1}{n}E\left\{\left\|-(1-\alpha_1)\boldsymbol{X}_1 + \alpha_1\left(\boldsymbol{S}_c - \tilde{\boldsymbol{S}}_c\right) + \alpha_1\left(\boldsymbol{X}_2 + \boldsymbol{Z}\right)\right\|^2\right\} \\ &= \frac{P_1(P_2 + N + E_1)}{P_1 + P_2 + N + E_1} \leq P_1 \end{aligned}$$

we have

$$\hat{\boldsymbol{Z}}_{\text{eq}} = -(1-\alpha_1)\boldsymbol{X}_1 + \alpha_1\left(\boldsymbol{S}_c - \tilde{\boldsymbol{S}}_c\right) + \alpha_1\left(\boldsymbol{X}_2 + \boldsymbol{Z}\right)$$

with high probability. Hence, the decoder calculates

$$\tilde{\boldsymbol{Y}} = (1-\alpha_1)\boldsymbol{Y} + \hat{\boldsymbol{Z}}_{\text{eq}} \tag{21}$$

$$\begin{aligned} &= (1-\alpha_1)\left(\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{S}_c + \boldsymbol{Z}\right) - (1-\alpha_1)\boldsymbol{X}_1 \\ &\quad + \alpha_1\left(\boldsymbol{S}_c - \tilde{\boldsymbol{S}}_c\right) + \alpha_1\left(\boldsymbol{X}_2 + \boldsymbol{Z}\right) \end{aligned} \tag{22}$$

$$= \boldsymbol{X}_2 + \boldsymbol{S}_c - \alpha_1\tilde{\boldsymbol{S}}_c + \boldsymbol{Z} = \boldsymbol{X}_2 + \hat{\boldsymbol{S}}_c + \boldsymbol{Z} \tag{23}$$

where $\hat{\boldsymbol{S}}_c = \boldsymbol{S}_c - \alpha_1\tilde{\boldsymbol{S}}_c$. In the third stage, the decoder calculates

$$\begin{aligned} \boldsymbol{Y}'' &= \left[\alpha_2\tilde{\boldsymbol{Y}} - \boldsymbol{D}_2\right]\bmod \Lambda_2 \\ &= \left[\alpha_2\left(\boldsymbol{X}_2 + \hat{\boldsymbol{S}}_c + \boldsymbol{Z}\right) - \boldsymbol{D}_2\right]\bmod \Lambda_2 \end{aligned} \tag{24}$$

$$= \left[\mathbf{V}_2 - (1-\alpha_2)\boldsymbol{X}_2 + \alpha_2\left(\hat{\boldsymbol{S}}_c - \tilde{\hat{\boldsymbol{S}}}_c\right) + \alpha_2\boldsymbol{Z}\right]\bmod \Lambda_2 \tag{25}$$

The rate achieved by user 2 is given by

$$R_2 = \frac{1}{n} I(V_2; Y'') = \frac{1}{n}\{h(Y'') - h(Y''|V_2)\} \quad (26)$$

$$\geq \frac{1}{2}\log\left(\frac{P_2}{(1-\alpha_2)^2 P_2 + \alpha_2^2(E_2+N)}\right) - \frac{1}{2}\log(2\pi e G(\Lambda_2)) \quad (27)$$

Using the optimal MMSE factor for user 2, we have $\alpha_2^* = ((P_2)/(P_2+E_2+N))$. For lattice that is good for quantisation, the achievable rate is given by $R_2 \leq (1/2)\log(1 + ((P_2)/(E_2+N)))$.

Similarly, we can obtain the achievability of the second corner point $((1/2)\log(1+((P_1)/(E_1+N))), (1/2)\log(1+((P_2)/(P_1+E_2+N))))$ by first decoding user 2 and then decoding user 1. Using time sharing for the corner points, the capacity region follows. Supposing $E_1 = E_2 = 0$, we can see the results of [13, section VIII].

3 Conclusion

In this paper, Gaussian doubly dirty MAC, Gaussian MAC with common interference and the helper problem were studied. Since at high SNR with strong interference regime, random binning strategy is not able to give positive rates, achievable rate regions and outer bounds of these models were obtained by using lattice strategy. It is shown that in some cases (based on how the SNRs are balanced), capacity is achieved. It was shown that partiality in SI at the transmitters reduces the achievable rate or capacity regions the same as that occurs for the Costa Gaussian channel.

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5 Appendix 1

Proof of Theorem 1: We bound the sum rate $R_1 + R_2$ as follows

$$n(R_1+R_2) \leq h(W_1, W_2) = h(W_1, W_2|\boldsymbol{Y}) + I(W_1, W_2; \boldsymbol{Y}) \leq n\varepsilon_n + I(W_1, W_2; \boldsymbol{Y}) \quad (28)$$

$$= n\varepsilon_n + I(W_1, W_2, \tilde{\boldsymbol{S}}_1, \tilde{\boldsymbol{S}}_2; \boldsymbol{Y}) - I(\tilde{\boldsymbol{S}}_1, \tilde{\boldsymbol{S}}_2; \boldsymbol{Y}|W_1, W_2)$$
$$= n\varepsilon_n + h(\boldsymbol{Y}) - h(\boldsymbol{Y}|W_1, W_2, \tilde{\boldsymbol{S}}_1, \tilde{\boldsymbol{S}}_2) - h(\tilde{\boldsymbol{S}}_1, \tilde{\boldsymbol{S}}_2|W_1, W_2) + h(\tilde{\boldsymbol{S}}_1, \tilde{\boldsymbol{S}}_2|W_1, W_2, \boldsymbol{Y})$$
$$= n\varepsilon_n + h(\boldsymbol{Y}) - h(\boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{Z}) - h(\tilde{\boldsymbol{S}}_1, \tilde{\boldsymbol{S}}_2|W_1, W_2) + h(\tilde{\boldsymbol{S}}_1, \tilde{\boldsymbol{S}}_2|W_1, W_2, \boldsymbol{Y}) \quad (29)$$

$$= n\varepsilon_n + h(\boldsymbol{Y}) - h(\boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{Z}) - h(\tilde{\boldsymbol{S}}_1, \tilde{\boldsymbol{S}}_2) + h(\tilde{\boldsymbol{S}}_1|W_1, W_2, \boldsymbol{Y}) + h(\tilde{\boldsymbol{S}}_2|W_1, W_2, \boldsymbol{Y}, \tilde{\boldsymbol{S}}_1) \quad (30)$$

$$\leq n\varepsilon_n + h(\boldsymbol{Y}) + h(\tilde{\boldsymbol{S}}_1|\boldsymbol{Y}) + h(\tilde{\boldsymbol{S}}_2|W_1, W_2, \boldsymbol{Y}, \tilde{\boldsymbol{S}}_1) - h(\boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{Z}) - h(\tilde{\boldsymbol{S}}_1, \tilde{\boldsymbol{S}}_2) \quad (31)$$

$$\leq n\varepsilon_n + h(\boldsymbol{Y}|\tilde{\boldsymbol{S}}_1) + h(\tilde{\boldsymbol{S}}_1) + h(\boldsymbol{X}_2 + \boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{Z}) - h(\boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{Z}) - h(\tilde{\boldsymbol{S}}_1) - h(\tilde{\boldsymbol{S}}_2) \quad (32)$$

$$\leq n\varepsilon_n + h(\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{A}_1 + \boldsymbol{A}_2 + \tilde{\boldsymbol{S}}_2 + \boldsymbol{Z}) + h(\boldsymbol{X}_2 + \boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{Z}) - h(\boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{Z}) - h(\tilde{\boldsymbol{S}}_2)$$
$$\leq n\varepsilon_n + \frac{n}{2}\log\left(\frac{\left(\sqrt{P_1} + \sqrt{P_2} + \sqrt{E_1} + \sqrt{E_2} + \sqrt{P_{\tilde{s}_2}}\right)^2 + N}{P_{\tilde{s}_2}} \cdot \times \frac{P_2 + E_1 + E_2 + N}{E_1 + E_2 + N}\right) \quad (33)$$

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where $\varepsilon_n \to 0$ as $n \to \infty$. Here, (28) is because of Fano's inequality, (29) follows from (4) and the fact that $\mathbb{E}\left[\tilde{S}_i^2\right] = \mathbb{E}\left[S_i\tilde{S}_i\right] = P_{\tilde{s}_i}$, $i = 1, 2$ and X_i, is a function of $\left(\tilde{\boldsymbol{S}}_i, W_i\right)$, (30) follows from mutually independence of $\left\{\tilde{\boldsymbol{S}}_1, \tilde{\boldsymbol{S}}_2, W_1, W_2\right\}$, (31) comes from the fact that conditioning reduces entropy, (32) follows from (4) and the facts that $\boldsymbol{X}_1$ is a function of $\left(\tilde{\boldsymbol{S}}_1, W_1\right)$ and the conditioning does not increase entropy, and (33) follows from the fact that Gaussian distribution maximises differential entropy for a fixed second moment and Cauchy–Schwarz inequality. On the other hand, we can also rewrite (30)–(33) as follows

$$
\begin{aligned}
& n\left(R_1 + R_2\right) \\
& \leq n\varepsilon_n + h(\boldsymbol{Y}) - h\left(\boldsymbol{A}_1 + \boldsymbol{A}_1 + \boldsymbol{Z}\right) - h\left(\tilde{\boldsymbol{S}}_1, \tilde{\boldsymbol{S}}_2\right) \\
& \quad + h\left(\tilde{\boldsymbol{S}}_2 | W_1, W_2, \boldsymbol{Y}\right) + h\left(\tilde{\boldsymbol{S}}_1 | W_1, W_2, \boldsymbol{Y}_1, \tilde{\boldsymbol{S}}_2\right) \\
& \leq n\varepsilon_n + h(\boldsymbol{Y}) + h\left(\tilde{\boldsymbol{S}}_2 | \boldsymbol{Y}\right) + h\left(\tilde{\boldsymbol{S}}_1 | W_1, W_2, \boldsymbol{Y}, \tilde{\boldsymbol{S}}_2\right) \\
& \quad - h\left(\boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{Z}\right) - h\left(\tilde{\boldsymbol{S}}_1, \tilde{\boldsymbol{S}}_2\right) \\
& \leq n\varepsilon_n + h\left(\boldsymbol{Y} | \tilde{\boldsymbol{S}}_2\right) + h\left(\tilde{\boldsymbol{S}}_2\right) + h\left(\boldsymbol{X}_1 + \boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{Z}\right) \\
& \quad - h\left(\boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{Z}\right) - h\left(\tilde{\boldsymbol{S}}_1\right) - h\left(\tilde{\boldsymbol{S}}_2\right) \\
& \leq n\varepsilon_n + h\left(\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{A}_1 + \boldsymbol{A}_2 + \tilde{\boldsymbol{S}}_1 + \boldsymbol{Z}\right) \\
& \quad + h\left(\boldsymbol{X}_1 + \boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{Z}\right) - h\left(\boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{Z}\right) - h\left(\tilde{\boldsymbol{S}}_1\right) \\
& \leq n\varepsilon_n + \frac{n}{2}\log\left(\frac{\left(\sqrt{P_1} + \sqrt{P_2} + \sqrt{E_1} + \sqrt{E_2} + \sqrt{P_{\tilde{s}_1}}\right)^2 + N}{P_{\tilde{s}_1}}\right. \cdot \\
& \quad \left. \times \frac{P_1 + E_1 + E_2 + N}{E_1 + E_2 + N}\right)
\end{aligned} \tag{34}
$$

Then for asymptotic case of strong interference ($P_{\tilde{s}_1}$, $P_{\tilde{s}_2} \to +\infty$), we obtain $R_1 + R_2 \leq (1/2)\log(1 + ((\min(P_1, P_2))/(E_1 + E_2 + N)))$.

6 Appendix 2

Proof of Theorem 2: The converse part has been proved in Theorem 1. In the following, achievability regions are studied. Four cases are considered, the first case user 1 is a helper for user 2 where $P_1 \geq P_2((P_2 + N + E_1 + E_2)/(P_2))^2$, the second case user 1 is a helper for user 2 where $P_2 \geq P_1((P_1 + N + E_1 + E_2)/(P_1))^2$, the third case user 2 is a helper for user 1 where $P_1 \geq P_2((P_2 + N + E_1 + E_2)/(P_2))^2$, the fourth case user 2 is a helper for user 1 where $P_2 \geq P_1((P_1 + N + E_1 + E_2)/(P_1))^2$. Now, the achievability for the first case is investigated.

Applying the lattice transmission scheme and considering $\alpha_1 = k_1 = 1$, $k_2 = \alpha_2$, $\boldsymbol{V}_1 = 0$ and $\sigma_1^2 = P_1$, $\sigma_2^2 = \alpha_2^2 P_1$, the encoder 1 and encoder 2 send $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, respectively, that are generated as follows

$$
\begin{aligned}
\boldsymbol{X}_1 &= [-\tilde{\boldsymbol{S}}_1 + \boldsymbol{D}_1] \bmod \Lambda_1 \\
\boldsymbol{X}_2 &= [\boldsymbol{V}_2 - \alpha_2\tilde{\boldsymbol{S}}_2 + \boldsymbol{D}_2] \bmod \Lambda_2
\end{aligned} \tag{35}
$$

After receiving $\boldsymbol{Y}$, the decoder sets $k_r = \gamma = \alpha_r = \alpha_2$, $\beta = 1$ and computes $\boldsymbol{Y}' = [\alpha_2(\boldsymbol{Y} - \boldsymbol{D}_1) - \boldsymbol{D}_2] \bmod \Lambda_2$ as follows:

$$
\boldsymbol{Y}' = \left[\alpha_2\left(\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{Z} - \boldsymbol{D}_1\right) - \boldsymbol{D}_2\right] \bmod \Lambda_2 \tag{36}
$$

$$
\begin{aligned}
&= \left[\boldsymbol{V}_2 - \left(1 - \alpha_2\right)\boldsymbol{X}_2 + \alpha_2\left(\boldsymbol{S}_1 - \tilde{\boldsymbol{S}}_1\right) + \alpha_2\left(\boldsymbol{S}_2 - \tilde{\boldsymbol{S}}_2\right)\right. \\
&\quad \left. + \alpha_2\boldsymbol{Z} - \alpha_2 Q_{\Lambda_1}\left(-\tilde{\boldsymbol{S}}_1 + \boldsymbol{D}_1\right)\right] \bmod \Lambda_2
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
&= \left[\boldsymbol{V}_2 - \left(1 - \alpha_2\right)\boldsymbol{X}_2 + \alpha_2\left(\boldsymbol{S}_1 - \tilde{\boldsymbol{S}}_1\right) + \alpha_2\left(\boldsymbol{S}_2 - \tilde{\boldsymbol{S}}_2\right)\right. \\
&\quad \left. + \alpha_2\boldsymbol{Z}\right] \bmod \Lambda_2
\end{aligned} \tag{38}
$$

where (37) follows from (1), (2) and (35). Equation (38) follows from the fact that $\alpha_2 Q_{\Lambda_1}\left(-\tilde{\boldsymbol{S}}_1 + \boldsymbol{D}_1\right) \in \Lambda_2$, because $\Lambda_1 = \Lambda$ and $\Lambda_2 = \alpha_2\Lambda$. In fact, the interference signal with Λ_2 using the modulo-Λ_2 operation is aligned. The rate achieved by user 2 is given by

$$
R_2 = \frac{1}{n} I\left(\boldsymbol{V}_2; \boldsymbol{Y}'\right) = \frac{1}{n}\left\{h\left(\boldsymbol{Y}'\right) - h\left(\boldsymbol{Y}' | \boldsymbol{V}_2\right)\right\} \tag{39}
$$

$$
\begin{aligned}
&= \frac{1}{n}\left\{h\left(\boldsymbol{Y}'\right) - h\left(\left[-\left(1 - \alpha_2\right)\boldsymbol{X}_2 + \alpha_2\left(\boldsymbol{S}_1 - \tilde{\boldsymbol{S}}_1\right)\right.\right.\right. \\
&\quad \left.\left.\left. + \alpha_2\left(\boldsymbol{S}_2 - \tilde{\boldsymbol{S}}_2\right) + \alpha_2\boldsymbol{Z}\right] \bmod \Lambda_2\right)\right\}
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
&\geq \frac{1}{2}\log_2\left(\frac{P_2}{G\left(\Lambda_2\right)}\right) - \frac{1}{2}\log_2 \\
&\quad \times \left(2\pi e\left(\left(1 - \alpha_2\right)^2 P_2 + \alpha_2^2\left(E_1 + E_2 + N\right)\right)\right)
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
&= \frac{1}{2}\log_2\left(\frac{P_2}{\left(1 - \alpha_2\right)^2 P_2 + \alpha_2^2\left(E_1 + E_2 + N\right)}\right) \\
&\quad - \frac{1}{2}\log_2\left(2\pi e G\left(\Lambda_2\right)\right)
\end{aligned} \tag{42}
$$

where (41) is because of (i) $\boldsymbol{Y}'$ is uniformly distributed over $\mathcal{V}_2$, (ii) for fixed second moment, Gaussian distribution maximises the entropy and (iii) modulo operation reduces the second moment. For $P_1 = P_2((P_2 + N + E_1 + E_2/P_2))^2$, using the optimal MMSE factor for user 2, we have $\partial R_2 / \partial\alpha_2 = 0$, $\alpha_2^* = P_2/P_2 + E_1 + E_2 + N$. Hence, as long as Λ_2 is a good lattice for quantisation, we have

$$
R_2 \leq \frac{1}{2}\log_2\left(1 + \frac{P_2}{E_1 + E_2 + N}\right) \tag{43}
$$

Considering the second case, similarly, the following rate can be achieved.

$$
R_2 \leq \frac{1}{2}\log\left(1 + \frac{P_1}{E_1 + E_2 + N}\right) \tag{44}
$$

From (43) and (44), the achievable rate for the point (0, R_2) satisfies (see (45))

Similarly, for the third and fourth cases, the achievable rate region for the point (R_1, 0) can be given by (see (46))

Using time sharing between (45) and (46) for $E_1 + E_2 + N \leq \sqrt{P_1 P_2} - \min(P_1, P_2)$ and $P_1 \neq P_2$, any rate pair in the straight line $R_1 + R_2 \leq (1/2)\log(1 + ((\min(P_1, P_2))/(E_1 + E_2 + N)))$ is achievable and the proof is complete.

7 Appendix 3

Proof of Theorem 3: Using the lattice-alignment transmission scheme, we consider the case that $k_1 = k_2 = k_r = \beta = \gamma = 1$ and $\alpha_1 = \alpha_2 = \alpha_r = \alpha$. The encoder 1 and encoder 2 send $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, respectively, generated as follows

$$\begin{aligned} \boldsymbol{X}_1 &= [\boldsymbol{V}_1 - \alpha\tilde{\boldsymbol{S}}_1 + \boldsymbol{D}_1] \bmod \Lambda \\ \boldsymbol{X}_2 &= [\boldsymbol{V}_2 - \alpha\tilde{\boldsymbol{S}}_2 + \boldsymbol{D}_2] \bmod \Lambda \end{aligned} \tag{47}$$

The receiver calculates $\boldsymbol{Y}' = [\alpha\boldsymbol{Y} - \boldsymbol{D}_1 - \boldsymbol{D}_2] \bmod \Lambda$. Using (2) and (47), the equivalent channel is given by

$$\begin{aligned} \boldsymbol{Y}' = \big[&\boldsymbol{V}_1 + \boldsymbol{V}_2 - (1-\alpha)\boldsymbol{X}_1 - (1-\alpha)\boldsymbol{X}_2 \\ &+ \alpha(\boldsymbol{S}_1 - \tilde{\boldsymbol{S}}_1) + \alpha(\boldsymbol{S}_2 - \tilde{\boldsymbol{S}}_2) + \alpha\boldsymbol{Z}\big] \bmod \Lambda \end{aligned} \tag{48}$$

The sum of achievable rates is given by

$$R_1 + R_2 = \frac{1}{n} I(\boldsymbol{V}_1 \boldsymbol{V}_2; \boldsymbol{Y}') = \frac{1}{n}\{h(\boldsymbol{Y}') - h(\boldsymbol{Y}'|\boldsymbol{V}_1\boldsymbol{V}_2)\} \tag{49}$$

$$\geq \frac{1}{2}\log\left(\frac{P}{G(\Lambda)}\right) - \frac{1}{2}\log_2\left(2\pi e\left(2(1-\alpha)^2 P + \alpha^2(E_1 + E_2 + N)\right)\right) \tag{50}$$

$$= \frac{1}{2}\log_2\left(\frac{P}{2(1-\alpha)^2 P + \alpha^2(E_1 + E_2 + N)}\right) - \frac{1}{2}\log(2\pi e G(\Lambda)) \tag{51}$$

Using the optimal MMSE factor, we achieve $\alpha^* = ((2P)/(2P + E_1 + E_2 + N))$. Therefore any rate pair satisfying (52) is achievable

$$R_1 + R_2 \leq \left[\frac{1}{2}\log_2\left(\frac{1}{2} + \frac{P}{E_1 + E_2 + N}\right)\right]^+ \tag{52}$$

Clearly, using a time sharing argument, (8) can be achieved.

8 Appendix 4

Proof of Theorem 4: For proof of this theorem, considering user 2 as a helper, two cases $P_1 \leq P_2 \leq P_1((P_1 + N + E_1 + E_2)/(P_1))^2$ and $P_2 \leq P_1 \leq P_2((P_2 + N + E_1 + E_2)/(P_2))^2$ are investigated. Supposing the case one, achievable region for the rate pair (R_1, 0) is derived as follows

$$R_1 = \frac{1}{2}\log_2\left(\frac{P_1 + P_2 + E_1 + E_2 + N}{2(E_1 + E_2 + N) + \left(\sqrt{P_1} - \sqrt{P_2}\right)^2}\right) \tag{53}$$

Using the lattice transmission scheme, it is considered $k_1 = k_r = \beta = ((\alpha_1)/(\alpha_2))$, $k_2 = \gamma = 1$, $\alpha_r = \alpha_1$, $\boldsymbol{V}_2 = 0$ and $\sigma_2^2 = P_2$, $\sigma_1^2 = ((\alpha_1^2)/(\alpha_2^2))P_2$. The encoders send

$$\begin{aligned} \boldsymbol{X}_1 &= [\boldsymbol{V}_1 - \alpha_1\tilde{\boldsymbol{S}}_1 + \boldsymbol{D}_1] \bmod \Lambda_1 \\ \boldsymbol{X}_2 &= [-\alpha_2\tilde{\boldsymbol{S}}_2 + \boldsymbol{D}_2] \bmod \Lambda_2 \end{aligned} \tag{54}$$

The receiver calculates $\boldsymbol{Y}' = [\alpha_1\boldsymbol{Y} - \boldsymbol{D}_1 - \beta\boldsymbol{D}_2] \bmod \Lambda_1$. Using (54), (1), (2) and this point that $((\alpha_1)/(\alpha_2))Q_{\Lambda_2}(-\alpha_2\tilde{\boldsymbol{S}}_2 + \boldsymbol{D}_2) \in \Lambda_1$, we have

$$\begin{aligned} \boldsymbol{Y}' = \Big[&\boldsymbol{V}_1 - (1-\alpha_1)\boldsymbol{X}_1 - \frac{\alpha_1}{\alpha_2}(1-\alpha_2)\boldsymbol{X}_2 \\ &+ \alpha_1(\boldsymbol{S}_1 - \tilde{\boldsymbol{S}}_1) + \alpha_1(\boldsymbol{S}_2 - \tilde{\boldsymbol{S}}_2) + \alpha_1\boldsymbol{Z}\Big] \bmod \Lambda_1 \end{aligned} \tag{55}$$

Since $\boldsymbol{X}_1$ is independent of $\boldsymbol{V}_1$ and $\boldsymbol{X}_2$ is independent of $\boldsymbol{V}_1$ and $\boldsymbol{X}_1$, then R_1 can be obtained as follows

$$R_2 = \begin{cases} \frac{1}{2}\log_2\left(1 + \frac{P_1}{N + E_1 + E_2}\right), & P_2 \geq P_1\left(\frac{P_1 + N + E_1 + E_2}{P_1}\right)^2 \\ \frac{1}{2}\log_2\left(1 + \frac{P_2}{N + E_1 + E_2}\right), & P_1 \geq P_2\left(\frac{P_2 + N + E_1 + E_2}{P_2}\right)^2 \end{cases} \tag{45}$$

$$R_1 = \begin{cases} \frac{1}{2}\log_2\left(1 + \frac{P_1}{N + E_1 + E_2}\right), & P_2 \geq P_1\left(\frac{P_1 + N + E_1 + E_2}{P_1}\right)^2 \\ \frac{1}{2}\log_2\left(1 + \frac{P_2}{N + E_1 + E_2}\right), & P_1 \geq P_2\left(\frac{P_2 + N + E_1 + E_2}{P_2}\right)^2 \end{cases} \tag{46}$$

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$$R_1 = \frac{1}{n} I(\boldsymbol{V}_1; \boldsymbol{Y}') = \frac{1}{n}\{h(\boldsymbol{Y}') - h(\boldsymbol{Y}'|\boldsymbol{V}_1)\}$$
$$= \frac{1}{n}\Bigg\{h(\boldsymbol{Y}') - h\bigg(\Big[-(1-\alpha_1)\boldsymbol{X}_1 - \frac{\alpha_1}{\alpha_2}(1-\alpha_2)\boldsymbol{X}_2. + \alpha_1(\boldsymbol{S}_1 - \tilde{\boldsymbol{S}}_1) + \alpha_1(\boldsymbol{S}_2 - \tilde{\boldsymbol{S}}_2) + \alpha_1 \boldsymbol{Z}\Big]\bmod\Lambda_1\bigg)\Bigg\} \quad (56)$$

$$\geq \frac{1}{2}\log_2\left(\frac{P_1}{G(\Lambda_1)}\right) - \frac{1}{2}\log_2\left(2\pi e\left((1-\alpha_1)^2 P_1 + \left(\frac{\alpha_1}{\alpha_2}\right)^2 (1-\alpha_2)^2 P_2 + \alpha_1^2\left(E_1 + E_2 + N\right)\right)\right) \quad (57)$$

Considering $((\alpha_1)/(\alpha_2)) = \sqrt{((P_1)/(P_2))}$ and $G(\Lambda_1) \to (1/(2\pi e))$ as $n \to \infty$ (good lattice), $\alpha_1^* = ((\sqrt{P_1}(\sqrt{P_1} + \sqrt{P_2}))/(P_1 + P_2 + (E_1 + E_2 + N)))$ can be found by using the optimal MMSE factor. Therefore the rate region is achieved as (53).

Similarly, the rate pair $(R_1, 0)$ can be achieved for the second case as follows

$$(R_1, 0) = \left(\left[\frac{1}{2}\log_2\left(\frac{P_1 + P_2 + E_1 + E_2 + N}{2(E_1 + E_2 + N) + (\sqrt{P_1} - \sqrt{P_2})^2}\right)\right]^+, 0\right) \quad (58)$$

Owing to the symmetry, it can be shown that the achievable rate of the point $(0, R_2)$ for $E_1 + E_2 + N \geq \sqrt{P_1 P_2} - \min(P_1, P_2)$ is derived as follows

$$(0, R_2) = \left(0, \left[\frac{1}{2}\log_2\left(\frac{P_1 + P_2 + E_1 + E_2 + N}{2(E_1 + E_2 + N) + (\sqrt{P_1} - \sqrt{P_2})^2}\right)\right]^+\right) \quad (59)$$

Using a time sharing between the achievable rate pairs in (58) and (59), the proof is complete.

9 Appendix 5

Proof of Theorem 6: The converse part was proved in Theorem 5. In the followings, the achievability for two cases, $P_2 \geq P_1 + E_1 + N$ and $P_1 \geq P_2 + E_1 + N$ is investigated. In the first stage, the case $P_2 \geq P_1 + E_1 + N$ is studied. Using the lattice-alignment transmission scheme, we have $k_1 = k_r = \sqrt{((P_1)/(P_2))}$, $k_2 = 1$ (i.e. $\Lambda_1 = \Lambda_r = \sqrt{((P_1)/(P_2))}\Lambda$ and $\Lambda_2 = \Lambda$). We set $\boldsymbol{V}_1 = 0$, $\boldsymbol{D}_2 = 0$, $\gamma = 1$, $\alpha_2 = 0$, $\beta = 0$ and $\alpha_r = \alpha_1$. The encoder 1 and encoder 2 send $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ generated as follows, respectively

$$\boldsymbol{X}_1 = [-\alpha_1 \tilde{\boldsymbol{S}}_1 + \boldsymbol{D}_1]\bmod\Lambda_1; \quad \boldsymbol{X}_2 = \boldsymbol{V}_2 \quad (60)$$

where the information of user 2 is carried by $\boldsymbol{V}_2 \sim \mathrm{Unif}(\mathcal{V}_2)$ and the dither signal is denoted by $\boldsymbol{D}_1 \sim \mathrm{Unif}(\mathcal{V}_1)$. The transmitted signal has uniform distribution over $\mathcal{V}_1$, that is, $\boldsymbol{X}_1 \sim \mathrm{Unif}(\mathcal{V}_1)$, from the dithered quantisation property. The receiver calculates $\boldsymbol{Y}'$ as follows

$$\boldsymbol{Y}' = [\alpha_1 \boldsymbol{Y} - \boldsymbol{D}_1]\bmod\Lambda_1 = [\alpha_1 \boldsymbol{V}_2 - (1-\alpha_1)\boldsymbol{X}_1 + \alpha_1(\boldsymbol{S}_1 - \tilde{\boldsymbol{S}}_1) + \alpha_1 \boldsymbol{Z}]\bmod\Lambda_1 \quad (61)$$

where $\boldsymbol{X}_1$ and $\boldsymbol{V}_2$ are independent and $\alpha_1 \boldsymbol{V}_2 \sim \mathrm{Unif}(\mathcal{V}_1)$. The rate achieved by user 2 can be derived as follows

$$R_2 = \frac{1}{n} I(\boldsymbol{V}_2; \boldsymbol{Y}') = \frac{1}{n}\{h(\boldsymbol{Y}') - h(\boldsymbol{Y}'|\boldsymbol{V}_2)\} \quad (62)$$

$$\geq \frac{1}{2}\log\left(\frac{P_1}{(1-\alpha_1)^2 P_1 + \alpha_1^2(E_1 + N)}\right) - \frac{1}{2}\log(2\pi e G(\Lambda_1)) \quad (63)$$

For $P_2 = P_1 + N + E_1$, using the optimal MMSE factor for user 2, we have $\alpha_1^* = ((P_1)/(P_1 + E_1 + N)) = ((P_1)/(P_2))$. Consequently, for lattice that is good for quantisation and AWGN channel coding, the achievable rate is given by $R_2 \leq (1/2)\log(1 + ((P_1)/(E_1 + N)))$.

Similarly, the achievable rate for the second case (where $P_1 \geq P_2 + N + E_1$), can be obtained as $R_2 \leq (1/2)\log(1 + ((P_2)/(E_1 + N)))$.

10 Appendix 6

Proof of Theorem 7: Using the lattice-alignment transmission scheme, we have $k_1 = k_r = \sqrt{((P_1)/(P_2))}$, $k_2 = 1$ $\left(\Lambda_1 = \Lambda_r = \sqrt{((P_1)/(P_2))}\Lambda \text{ and } \Lambda_2 = \Lambda\right)$. We set $\boldsymbol{V}_1 = 0$, $\boldsymbol{D}_2 = 0$, $\gamma = 1$, $\alpha_2 = 0$, $\beta = 0$ and $\alpha_r = \alpha_1$. The transmitted signals by encoders are as follows.

$$\boldsymbol{X}_1 = [-\alpha_1 \tilde{\boldsymbol{S}}_1 + \boldsymbol{D}_1]\bmod\Lambda_1; \quad \boldsymbol{X}_2 = \boldsymbol{V}_2 \quad (64)$$

The receiver calculates $\boldsymbol{Y}'$ as follows

$$\boldsymbol{Y}' = [\alpha_1 \boldsymbol{Y} - \boldsymbol{D}_1]\bmod\Lambda_1 = [\alpha_1 \boldsymbol{V}_2 - (1-\alpha_1)\boldsymbol{X}_1 + \alpha_1(\boldsymbol{S}_1 - \tilde{\boldsymbol{S}}_1) + \alpha_1 \boldsymbol{Z}]\bmod\Lambda_1 \quad (65)$$

The scalar α_1 is determined such that $E\{[\alpha_1 \boldsymbol{V}_2 - (1-\alpha_1)\boldsymbol{X}_1 + \alpha_1(\boldsymbol{S}_1 - \tilde{\boldsymbol{S}}_1) + \alpha_1 \boldsymbol{Z}]^2\} = P_1$. Hence $\alpha_1^2(P_2 + N + E_1) + (1-\alpha_1)^2 P_1 = P_1$, where $\alpha_1 = ((2P_1)/(P_1 + P_2 + E_1 + N))$. For lattice Λ that is good for quantisation and AWGN channel coding, the rate achieved by user 2 is obtained as follows

$$R_2 = \frac{1}{n} I(\boldsymbol{V}_2; \boldsymbol{Y}') = \frac{1}{n}\{h(\boldsymbol{Y}') - h(\boldsymbol{Y}'|\boldsymbol{V}_2)\} \quad (66)$$

$$\geq \frac{1}{2}\log\left(\frac{P_1}{(1-\alpha_1)^2 P_1 + \alpha_1^2(E_1 + N)}\right) + \varepsilon \quad (67)$$

$$= \frac{1}{2}\log\left(1 + \frac{4P_1P_2}{\left(P_2 - P_1 + E_1 + N\right)^2 + 4P_1\left(E_1 + N\right)}\right) + \varepsilon \tag{68}$$

where (67) is because of (i) $\boldsymbol{Y}'$ is uniformly distributed over $\mathcal{V}_1$, (ii) for fixed second moment, Gaussian distribution maximises the entropy, (iii) modulo operation reduces the second moment and (v) Λ_1 is a good lattice for quantisation.

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