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Efficient carrier frequency offset estimation for orthogonal frequency-division multiple access uplink with an arbitrary number of subscriber stations

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Abstract: An efficient method is proposed to estimate the carrier frequency offsets (CFOs) in the orthogonal frequency-division multiple access (OFDMA) uplink. The conventional alternating projection method is accelerated by utilising the inherited properties of the matrices involved. The multiplication of large sparse projection matrices can be elegantly transformed to a series of products involving small dense matrices, and the inverse operation of these large matrices can be substituted by direct computations. Hence, the computational cost is significantly reduced without compromising the accuracy of the CFO estimation.

1 Introduction

Orthogonal frequency-division multiple access (OFDMA) has been widely used in satellite communications [1], cable TV [2], IEEE 802.16e-2005 [3] and so on. The orthogonality among subcarriers is the key feature to render the OFDMA scheme suitable for multi-user diversity, adaptive control of service quality and low complexity of equalisation.

Each subscriber station (SS) may suffer from carrier frequency offset (CFO) in the downlink because of clock mismatch, Doppler shift, among other reasons. The downlink CFO can be estimated in the same way as the OFDM scheme [4]. On the other hand, since subcarriers are allocated to different SS's in the uplink, estimation of the CFO's from different SS's at the base station (BS) becomes a complicated task.

In general, the range of each CFO can be narrowed down with downlink synchronisation procedure. The residual CFO's can then be estimated and compensated to reduce inter-carrier interference (ICI) [5, 6]. Several methods have been proposed to estimate the CFO's [7–15]. In [10, 11], minimum mean-squared-error (MMSE) and least-square method are used to minimise the difference between the received signal and the reconstructed signal, without resorting to a grid search. These methods require less computational load, but the Cramer–Rao lower bound (CRLB) can be reached only at high signal-to-noise ratio (SNR).

Iteration methods such as space-alternating generalised expectation-maximisation (SAGE) and its variations [7–9], as well as alternating projection (AP) method have been proposed. In [7], a modified SAGE algorithm is improved in convergence rate with a windowing technique in the frequency domain to decrease the ICI, and its computational complexity to estimate the CFO is also reduced by using the cyclic prefix (CP) similar to the OFDM scheme. Its computational load is relatively low since no grid search is needed, but its performance is degraded at low SNR.

In [12], a closed-form solution of multi-dimensional CFO estimation procedure is derived based on an optimisation theorem. The computational load is reduced to $O(N^2)$ by reducing the matrix to a block diagonal form. Although it is more efficient than the conventional AP method, but its computational load is still a way too high.

Although the AP method can reach the CRLB of CFO's and can be applied over a wide SNR range, its computational load is much higher than the other approaches [11, 12]. Variants of AP method have been developed to overcome this drawback. For example, approximate AP frequency estimator and divide-and-update frequency estimator bear less computational complexity, but at the cost of degraded performance [13]. The variant in [14] is claimed to approach the performance of AP with large number of SS's.

In this paper, an efficient AP method is proposed to reduce the computational load of conventional AP methods by several orders, while maintaining the same accuracy. It starts with a proof that the matrix product, $\bar{\bar{C}}_B^{\dagger} \cdot \bar{\bar{C}}_B$, representing a column space, can be reduced to a constant identity matrix. Hence, its inverse can be derived using a more efficient method, and its eigenvalues can be derived using direct algebraic computation, without resorting to matrix multiplications. It is also observed that other matrices appearing in the CFO estimator can be decomposed into various constant identity submatrices, of which the eigenvalues can be derived without using matrix multiplications. As a consequence, the computational cost is dramatically reduced. The conventional AP method is briefly reviewed in Section 2. The efficient AP method is presented in Section 3 for four SS's or less, and for any number of SS's in Section 4, followed by the conclusion.

2 Brief review of conventional AP method

2.1 Signal model

Assume there are N subcarriers assigned to K SS's. Each preamble symbol is transmitted with pilot tones of equal amplitude at regular intervals. The preamble symbol of the kth SS can be expressed in the digital-frequency domain as

$$x_k[m] = \begin{cases} c_k[u], & m = uD + \Delta_k,\ u = 0,\ \ldots,\ N_\mathrm{p} - 1 \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

where N_p is the number of pilot tones, $D = N/N_\mathrm{p}$, $c_k[u]$ is a binary phase-shift keying data stream carried by the kth SS, $\Delta_k = \lfloor (k-1)D/K \rfloor$ is the index offset of pilot tones of the kth SS and $\lfloor \alpha \rfloor$ stands for the integer part of α.

The preamble symbol of the kth SS in the time domain is

$$\bar{s}_k = \frac{1}{\sqrt{N}} \bar{\bar{F}}^\dagger \cdot \bar{x}_k$$

where $\{\bar{\bar{F}}\}_{m,n} = \sqrt{N}^{-1} \mathrm{e}^{-\mathrm{j}2\pi mn/N}$, $\bar{x}_k = [x_k[0],\ x_k[1],\ \ldots,\ x_k[N-1]]^t$, and $\bar{\bar{F}}^\dagger$ means the Hermitian of $\bar{\bar{F}}$. The preamble symbol amended with CP of length N_g becomes $\bar{u}_k = [s_k[N - N_\mathrm{g}],\ \ldots,\ s_k[N-1],\ s_k[0],\ \ldots,\ s_k[N-1]]^t$.

In the presence of CFO and timing error, the received signal at the BS can be represented as

$$r'[n] = \sum_{k=1}^{K} \mathrm{e}^{\mathrm{j}\omega_k n} \sum_{\ell=0}^{L_k - 1} h_k[\ell] u_k[n - \ell - \mu_k] + v[n]$$

where $-N_g \le n \le N-1$ and $\omega_k = 2\pi\Delta f_k/N$, with Δf_k as the normalised frequency offset, L_k is the number of channel taps, μ_k is the integer part of timing error, all associated with the kth SS, the fractional part of timing error is absorbed into the channel impulse response (CIR), $\bar{\boldsymbol{h}}_k$ and $v[n]$ is a complex white Gaussian noise with zero-mean and covariance matrix $\sigma^2 \bar{\bar{I}}_N$. To avoid inter-symbol interference, choose $N_\mathrm{g} \ge \max\{\mu_k + L_k\}$ for all SS's. Fig. 1 shows the schematic of the system. After removing the CP part from $\bar{r}'$, the received signal can be represented in a matrix form as

$$\bar{r} = \sum_{k=1}^{K} \bar{r}_k + \bar{v}$$
$$= \left[\bar{\bar{\Gamma}}(\omega_1) \cdot \bar{\bar{A}}'_1,\ \ \ldots,\ \ \bar{\bar{\Gamma}}(\omega_K) \cdot \bar{\bar{A}}'_K \right] \begin{bmatrix} \bar{\boldsymbol{h}}_1 \\ \vdots \\ \bar{\boldsymbol{h}}_K \end{bmatrix} + \bar{v} \quad (2)$$

where $\bar{\bar{\Gamma}}(\omega_k) = \mathrm{diag}\{1,\ \mathrm{e}^{\mathrm{j}\omega_k},\ \ldots,\ \mathrm{e}^{\mathrm{j}(N-1)\omega_k}\}$, $\bar{r}_k = \bar{\bar{\Gamma}}(\omega_k) \cdot \bar{\bar{A}}'_k \cdot \bar{h}_k$, $\bar{h}_k = [h_k[0],\ h_k[1],\ \ldots,\ h_k[L_k - 1]]^t$, and

$$\bar{\bar{A}}'_k = \begin{bmatrix} u_k[-\mu_k] & \cdots & u_k[-\mu_k - L_k + 1] \\ u_k[1 - \mu_k] & \cdots & u_k[-\mu_k - L_k + 2] \\ \vdots & \ddots & \vdots \\ u_k[N - 1 - \mu_k] & \cdots & u_k[N - \mu_k - L_k] \end{bmatrix}$$

The channel length of SS k is extended to N_g by inserting μ_k zeros in front of the CIR and padding $N_\mathrm{g} - \mu_k - L_k$ zeros after it. The extended CIR thus becomes $\bar{\boldsymbol{h}}_k^e = \left[\bar{0}^t_{\mu_k},\ \bar{h}^t_k,\ \bar{0}^t_{N_\mathrm{g} - \mu_k - L_k}\right]^t$, and $\bar{\bar{A}}'_k \cdot \bar{\boldsymbol{h}}_k$ is reorganised as $\bar{\bar{A}}_k \cdot \bar{\boldsymbol{h}}_k^e$ with

$$\bar{\bar{A}}_k = \begin{bmatrix} u_k[0] & \cdots & u_k[1 - N_\mathrm{g}] \\ u_k[1] & \cdots & u_k[2 - N_\mathrm{g}] \\ \vdots & \vdots & \vdots \\ u_k[N-1] & \cdots & u_k[N - N_\mathrm{g}] \end{bmatrix}$$

Since $u_k[-i] = u_k[N - i]$ with $1 \le i \le N_\mathrm{g}$ by definition of CP, $\bar{\bar{A}}_k$ can be further expanded to $\bar{\bar{A}}_k^{\mathrm{circ}} \cdot \bar{\bar{U}}$, where $\bar{\bar{U}} = \left[\bar{\bar{I}}_{N_\mathrm{g} \times N_\mathrm{g}},\ \bar{\bar{0}}^t_{(N - N_\mathrm{g}) \times N_\mathrm{g}}\right]^t$. The circulant matrix $\bar{\bar{A}}_k^{\mathrm{circ}}$ can be decomposed into $\bar{\bar{F}}^\dagger \cdot \bar{\bar{\Lambda}}_k \cdot \bar{\bar{F}}$, where DFT stands for discrete Fourier transform, and

$$\bar{\bar{\Lambda}}_k = \mathrm{diag}\{\mathrm{DFT}\{u_k[0],\ \ldots,\ u_k[N-1]\}\}$$
$$= \mathrm{diag}\{x_k[0],\ \ldots,\ x_k[N-1]\}$$

Explicitly

$$\bar{\bar{A}}_k = \bar{\bar{A}}_k^{\mathrm{circ}} \cdot \bar{\bar{U}} = \bar{\bar{F}}^\dagger \cdot \bar{\bar{\Lambda}}_k \cdot \bar{\bar{F}} \cdot \bar{\bar{U}} \quad (3)$$

The received signal in (2) is rewritten as

$$\bar{r} = \sum_{k=1}^{K} \bar{\bar{\Gamma}}(\omega_k) \cdot \bar{\bar{A}}_k \cdot \bar{h}_k^e + \bar{v}$$
$$= \left[\bar{\bar{\Gamma}}(\omega_1) \cdot \bar{\bar{A}}_1,\ \ldots,\ \bar{\bar{\Gamma}}(\omega_K) \cdot \bar{\bar{A}}_K\right] \cdot \bar{h}^e + \bar{v} = \bar{\bar{Q}} \cdot \bar{h}^e + \bar{v} \quad (4)$$

where $\bar{\bar{Q}} = \left[\bar{\bar{\Gamma}}(\omega_1) \cdot \bar{\bar{A}}_1,\ \ldots,\ \bar{\bar{\Gamma}}(\omega_K) \cdot \bar{\bar{A}}_K\right]$ and $\bar{h}^e = \left[[\bar{h}_1^e]^t,\ \ldots,\ [\bar{h}_K^e]^t\right]^t$.

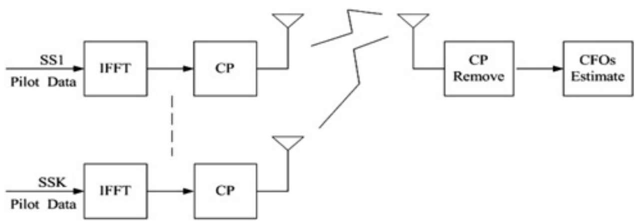

Fig. 1 *Schematic of the system*

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2.2 Conventional AP method

There are several cycles in the conventional AP method, with each cycle divided into K steps [16]. In each step, only the CFO of a specific SS is updated, whereas those of the other SS's are fixed. Without loss of generality, step k is applied to SS k.

The maximum-likelihood estimate of CFO of the kth SS can be derived as [12]

$$\left\{\hat{\boldsymbol{\omega}}_k^{(i)}\right\} = \underset{\tilde{\omega}_k}{\arg\min}\left\{\left\|\bar{r} - \bar{\bar{\boldsymbol{P}}}_Q\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right)\cdot\bar{r}\right\|^2\right\} \tag{5}$$

where $\hat{\boldsymbol{\omega}}_k^{(i)}$ denotes the estimate of parameter $\tilde{\omega}_k$ in step k of cycle i, all the previously updated CFO's except the kth one are denoted as $\hat{\bar{\boldsymbol{\omega}}}_k^{(i)} = \left[\hat{\omega}_1^{(i)},\ \ldots,\ \hat{\omega}_{k-1}^{(i)},\ \hat{\omega}_{k+1}^{(i-1)},\ \ldots,\ \hat{\omega}_K^{(i-1)}\right]$, and

$$\bar{\bar{\boldsymbol{P}}}_Q\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right) = \bar{\bar{\boldsymbol{Q}}}\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right) \cdot\left[\bar{\bar{\boldsymbol{Q}}}^{\dagger}\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right)\cdot\bar{\bar{\boldsymbol{Q}}}\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right)\right]^{-1}\cdot\bar{\bar{\boldsymbol{Q}}}^{\dagger}\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right)$$

The $\bar{\bar{\boldsymbol{P}}}_Q$ matrix projects $\bar{r}$ onto the column space of $\bar{\bar{\boldsymbol{Q}}}$, which is decomposed into [16]

$$\bar{\bar{\boldsymbol{P}}}_Q\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right) = \bar{\bar{\boldsymbol{P}}}_B\left(\hat{\bar{\omega}}_k^{(i)}\right) + \bar{\bar{\boldsymbol{P}}}_{C_B}\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right) \tag{6}$$

where

$$\bar{\bar{\boldsymbol{P}}}_B\left(\hat{\bar{\omega}}_k^{(i)}\right) = \bar{\bar{\boldsymbol{B}}}\left(\hat{\bar{\omega}}_k^{(i)}\right) \cdot\left[\bar{\bar{\boldsymbol{B}}}^{\dagger}\left(\hat{\bar{\omega}}_k^{(i)}\right)\cdot\bar{\bar{\boldsymbol{B}}}\left(\hat{\bar{\omega}}_k^{(i)}\right)\right]^{-1}\cdot\bar{\bar{\boldsymbol{B}}}^{\dagger}\left(\hat{\bar{\omega}}_k^{(i)}\right) \tag{7}$$

represents the projection onto the column space of $\bar{\bar{\boldsymbol{B}}}\left(\hat{\bar{\omega}}_k^{(i)}\right)$ $\bar{\bar{\boldsymbol{P}}}_{C_B}\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right) = \bar{\bar{\boldsymbol{C}}}_B\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right)\cdot\left[\bar{\bar{\boldsymbol{C}}}_B^{\dagger}\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right)\cdot\bar{\bar{\boldsymbol{C}}}_B\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right)\right]^{-1}\cdot\bar{\bar{\boldsymbol{C}}}_B^{\dagger}\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right)$ represents the projection onto the column space of $\bar{\bar{\boldsymbol{C}}}(\tilde{\omega}_k)$, excluding the column space of $\bar{\bar{\boldsymbol{B}}}\left(\hat{\bar{\omega}}_k^{(i)}\right)$, with

$$\bar{\bar{\boldsymbol{C}}}_B\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right) = \left[\bar{\bar{\boldsymbol{I}}}_N - \bar{\bar{\boldsymbol{P}}}_B\left(\hat{\bar{\omega}}_k^{(i)}\right)\right]\cdot\bar{\bar{\boldsymbol{C}}}(\tilde{\omega}_k)$$

where $\bar{\bar{\boldsymbol{C}}}(\tilde{\omega}_k) = \bar{\bar{\boldsymbol{\Gamma}}}(\tilde{\omega}_k)\cdot\bar{\bar{\boldsymbol{A}}}_k$ and

$$\bar{\bar{\boldsymbol{B}}}\left(\hat{\bar{\omega}}_k^{(i)}\right) = \left[\bar{\bar{\boldsymbol{\Gamma}}}\left(\hat{\omega}_1^{(i)}\right)\cdot\bar{\bar{\boldsymbol{A}}}_1,\ \ldots,\ \bar{\bar{\boldsymbol{\Gamma}}}\left(\hat{\omega}_{(k-1)}^{(i)}\right)\cdot\bar{\bar{\boldsymbol{A}}}_{(k-1)},\ \bar{\bar{\boldsymbol{\Gamma}}}\left(\hat{\omega}_{(k+1)}^{(i)}\right)\cdot\bar{\bar{\boldsymbol{A}}}_{(k+1)},\ \ldots,\ \bar{\bar{\boldsymbol{\Gamma}}}\left(\hat{\omega}_K^{(i)}\right)\cdot\bar{\bar{\boldsymbol{A}}}_K\right]$$

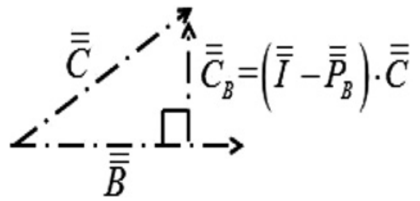

Fig. 2 *Decomposition of column space of* $\bar{\bar{\boldsymbol{Q}}}$

Note that $\bar{\bar{\boldsymbol{B}}}\left(\hat{\bar{\omega}}_k^{(i)}\right)$ and $\bar{\bar{\boldsymbol{C}}}(\tilde{\omega}_k)$ constitute

$$\bar{\bar{\boldsymbol{Q}}}\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right) = \left[\bar{\bar{\boldsymbol{\Gamma}}}\left(\hat{\omega}_1^{(i)}\right)\cdot\bar{\bar{\boldsymbol{A}}}_1,\ \ldots,\ \bar{\bar{\boldsymbol{\Gamma}}}\left(\hat{\omega}_{(k-1)}^{(i)}\right)\cdot\bar{\bar{\boldsymbol{A}}}_{(k-1)},\ \bar{\bar{C}}(\tilde{\omega}_k),\ \bar{\bar{\boldsymbol{\Gamma}}}\left(\hat{\omega}_{(k+1)}^{(i)}\right)\cdot\bar{\bar{\boldsymbol{A}}}_{(k+1)},\ \ldots,\ \bar{\bar{\boldsymbol{\Gamma}}}\left(\hat{\omega}_K^{(i)}\right)\cdot\bar{\bar{\boldsymbol{A}}}_K\right]$$

As illustrated in Fig. 2, the column space of $\bar{\bar{\boldsymbol{C}}}_B\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right)$ is orthogonal to that of $\bar{\bar{\boldsymbol{B}}}$.

Since $\bar{\bar{\boldsymbol{P}}}_B\left(\hat{\bar{\omega}}_k^{(i)}\right)$ is independent of $\tilde{\omega}_k$, (5) can be reduced to

$$\left\{\hat{\omega}_k^{(i)}\right\} = \underset{\tilde{\omega}_k}{\arg\max}\left\{\left\|\bar{\bar{\boldsymbol{P}}}_{C_B}\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right)\cdot\bar{r}\right\|^2\right\} \tag{8}$$

The matrix to be inverted in $\bar{\bar{\boldsymbol{P}}}_{C_B}\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right)$ is much smaller than that in $\bar{\bar{\boldsymbol{P}}}_B\left(\hat{\bar{\omega}}_k^{(i)}\right)$, so is the computational load. To streamline the subsequent derivations, the arguments of matrices will be ignored wherever possible, for example, $\bar{\bar{\boldsymbol{P}}}_{C_B}\left(\tilde{\omega}_k,\ \hat{\bar{\omega}}_k^{(i)}\right)\cdot\bar{r}$ is abbreviated as $\bar{\bar{\boldsymbol{P}}}_{C_B}\cdot\bar{r}$.

2.3 Some observations

A direct expansion leads to

$$\left\|\bar{\bar{\boldsymbol{P}}}_{C_B}\cdot\bar{r}\right\|^2 = \bar{r}^{\dagger}\cdot\bar{\bar{\boldsymbol{C}}}_B\cdot\left(\bar{\bar{\boldsymbol{C}}}_B^{\dagger}\cdot\bar{\bar{\boldsymbol{C}}}_B\right)^{-1}\cdot\bar{\bar{\boldsymbol{C}}}_B^{\dagger}\cdot\bar{r} \tag{9}$$

and

$$\left(\bar{\bar{\boldsymbol{C}}}_B^{\dagger}\cdot\bar{\bar{\boldsymbol{C}}}_B\right)^{-1} = \left\{\left[\left(\bar{\bar{\boldsymbol{I}}}_N - \bar{\bar{\boldsymbol{P}}}_B\right)\cdot\bar{\bar{\boldsymbol{C}}}\right]^{\dagger}\cdot\left(\bar{\bar{\boldsymbol{I}}}_N - \bar{\bar{\boldsymbol{P}}}_B\right)\cdot\bar{\bar{\boldsymbol{C}}}\right\}^{-1} = \left(\bar{\bar{\boldsymbol{C}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{C}}} - \bar{\bar{\boldsymbol{C}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{P}}}_B\cdot\bar{\bar{\boldsymbol{C}}}\right)^{-1} \tag{10}$$

where $\bar{\bar{\boldsymbol{C}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{C}}} = \bar{\bar{\boldsymbol{A}}}_k^{\dagger}\cdot\bar{\bar{\boldsymbol{A}}}_k$ is independent of the CFO's. It takes much longer time to compute $\bar{\bar{\boldsymbol{P}}}_B$ than the matrix inversion in (10), luckily the former is computed only once per step. As will be shown in the next section, the result of (10) is a constant identity matrix, and $\bar{\bar{\boldsymbol{P}}}_B$ is consisted of small blocks of constant identity matrices.

Next, expand $\bar{\bar{P}}_B$ with a given k as (see (11)) where

$$\bar{\bar{b}}_{p,q} = \left[\bar{\bar{A}}_p^\dagger \cdot \bar{\bar{\Gamma}}^\dagger(\omega_p)\right] \cdot \left[\bar{\bar{\Gamma}}(\omega_q) \cdot \bar{\bar{A}}_q\right] \tag{12}$$

Note that the ~ sign over all CFO's is removed from here on for convenience.

3 Efficient AP method for $k \leq 4$

3.1 Cyclic product $\bar{\bar{b}}_{p,q} \cdot \bar{\bar{b}}_{q,r} \ldots \bar{\bar{b}}_{\gamma,p}$

A direct substitution leads to

$$\begin{aligned}\bar{\bar{b}}_{p,q} &= \bar{\bar{A}}_p^\dagger \cdot \bar{\bar{\Gamma}}^\dagger(\omega_p) \cdot \bar{\bar{\Gamma}}(\omega_q)\bar{\bar{A}}_q = \bar{\bar{A}}_p^\dagger \cdot \bar{\bar{\Gamma}}(\omega_q - \omega_p)\bar{\bar{A}}_p \\ &= \bar{\bar{U}}^t \cdot \bar{\bar{F}}^\dagger \cdot \bar{\bar{\Lambda}}_p \cdot \bar{\bar{F}} \cdot \bar{\bar{\Gamma}}(\omega_{qp}) \cdot \bar{\bar{F}}^\dagger \cdot \bar{\bar{\Lambda}}_q \cdot \bar{\bar{F}} \cdot \bar{\bar{U}}\end{aligned} \tag{13}$$

with $\omega_{qp} = \omega_q - \omega_p$ and

$$\bar{\bar{F}} \cdot \bar{\bar{\Gamma}}(\omega_{qp}) \cdot \bar{\bar{F}}^\dagger = \begin{bmatrix} v_0 & v_{N-1} & \cdots & v_1 \\ v_1 & v_0 & \cdots & v_2 \\ \vdots & \vdots & \vdots & \vdots \\ v_{N-1} & v_{(N-2)} & \cdots & v_0 \end{bmatrix} \tag{14}$$

where v_n is the nth entry of the inverse discrete fourier transform (IDFT) of the complex conjugate of $\bar{\bar{\Gamma}}(\omega_{qp})'s$ diagonal entries, $\bar{\bar{\Lambda}}_p$ and $\bar{\bar{\Lambda}}_q$ are two diagonal matrices with diagonal elements of $x_k[m]$. Since $\bar{x}_k$ contains many zeros, $\bar{\bar{\Lambda}}_p \cdot \bar{\bar{F}} \cdot \bar{\bar{\Gamma}}(\omega_{qp}) \cdot \bar{\bar{F}}^\dagger \cdot \bar{\bar{\Lambda}}_q$ has non-zero entries in row m and column n, where m and n are the indices of non-zero entries in $\bar{\bar{\Lambda}}_p$ and $\bar{\bar{\Lambda}}_q$, respectively.

Let the value of pilot tones for SS p and SS q be c_p and c_q, respectively. Thus, we have (see (15) and (16))

where $\bar{0}_\ell$ represents a zero vector of length ℓ, Δ_p and Δ_q are offsets of the first pilot-tone of SS p and SS q, respectively, D is the pilot-tone interval. By carefully examining (13), $\bar{\bar{b}}_{p,q}$ can be calculated in a more computationally efficient way as

$$\bar{\bar{b}}_{p,q} = \frac{N_g}{N}\bar{\bar{D}}_p \cdot \bar{\bar{F}}_{N_g}^\dagger \cdot \bar{\bar{c}}_p \cdot \bar{\bar{V}}_{pq} \cdot \bar{\bar{c}}_q \cdot \bar{\bar{F}}_{N_g} \cdot \bar{\bar{D}}_q^\dagger \tag{17}$$

where

$$\begin{aligned}\bar{\bar{D}}_p &= \operatorname{diag}\left[1,\ e^{j2\pi\Delta_p/N},\ \ldots,\ e^{j2\pi(N_g-1)\Delta_p/N}\right],\quad \left\{\bar{\bar{F}}_{N_g}\right\}_{m,n} \\ &= \sqrt{N_g}^{-1} e^{-j2\pi mn/N_g}, \\ \bar{\bar{c}}_p &= \operatorname{diag}\left[c_p[0],\ c_p[1],\ \ldots,\ c_p[N_g-1]\right]\end{aligned}$$

The $\bar{\bar{V}}_{pq}$ is a circulant matrix [17] with the first column being $\left[v_{\text{mod}(N+\Delta_{pq},N)},\ v_{D+\Delta_{pq}},\ \ldots,\ v_{(N_g-1)D+\Delta_{pq}}\right]^t$.

The inverse of $\bar{\bar{B}}^\dagger \cdot \bar{\bar{B}}$ can be reduced to the products of various $\bar{\bar{b}}_{p,q}$ matrices, many of which can be reduced to an identity matrix multiplied by a constant. Consider the case $p = q$

$$\bar{\bar{b}}_{p,p} = \frac{N_g}{N}\bar{\bar{D}}_p \cdot \bar{\bar{F}}_{N_g}^\dagger \cdot \bar{\bar{c}}_p \cdot \bar{\bar{V}}_{pp} \cdot \bar{\bar{c}}_p \cdot \bar{\bar{F}}_{N_g} \cdot \bar{\bar{D}}_p^\dagger = \frac{N_g}{N}\bar{\bar{I}}_{N_g} \tag{18}$$

where $\bar{\bar{I}}_{N_g}$ is an $N_g \times N_g$ identity matrix. The inverse of $\bar{\bar{b}}_{p,p}$ is straightforward as

$$\bar{\bar{b}}_{p,p}^{-1} = \frac{N}{N_g}\bar{\bar{I}}_{N_g} \tag{19}$$

Next, consider the product

$$\bar{\bar{b}}_{p,q} \cdot \bar{\bar{b}}_{q,p} = \left(\frac{N_g}{N}\right)^2 \bar{\bar{D}}_p \cdot \bar{\bar{F}}_{N_g}^\dagger \cdot \bar{\bar{c}}_p \cdot \bar{\bar{V}}_{pq} \cdot \bar{\bar{V}}_{qp} \cdot \bar{\bar{c}}_p \cdot \bar{\bar{F}}_{N_g} \cdot \bar{\bar{D}}_p^\dagger \tag{20}$$

$$\begin{aligned}\bar{\bar{P}}_B = {} & \left[\bar{\bar{\Gamma}}(\omega_1) \cdot \bar{\bar{A}}_1,\ \ldots,\ \bar{\bar{\Gamma}}(\omega_{k-1}) \cdot \bar{\bar{A}}_{k-1},\ \bar{\bar{\Gamma}}(\omega_{k+1}) \cdot \bar{\bar{A}}_{k+1},\ \ldots,\ \bar{\bar{\Gamma}}(\omega_K) \cdot \bar{\bar{A}}_K\right] \\ & \begin{bmatrix} \bar{\bar{b}}_{1,1} & \cdots & \bar{\bar{b}}_{1,k-1} & \bar{\bar{b}}_{1,k+1} & \cdots & \bar{\bar{b}}_{1,K} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{\bar{b}}_{k-1,1} & \cdots & \bar{\bar{b}}_{k-1,k-1} & \bar{\bar{b}}_{k-1,k+1} & \cdots & \bar{\bar{b}}_{k-1,K} \\ \bar{\bar{b}}_{k+1,1} & \cdots & \bar{\bar{b}}_{k+1,k-1} & \bar{\bar{b}}_{k+1,k+1} & \cdots & \bar{\bar{b}}_{k+1,K} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{\bar{b}}_{K,1} & \cdots & \bar{\bar{b}}_{K,k-1} & \bar{\bar{b}}_{K,k+1} & \cdots & \bar{\bar{b}}_{K,K} \end{bmatrix}^{-1} \\ & \left[\bar{\bar{\Gamma}}(\omega_1) \cdot \bar{\bar{A}}_1,\ \ldots,\ \bar{\bar{\Gamma}}(\omega_{k-1}) \cdot \bar{\bar{A}}_{k-1},\ \bar{\bar{\Gamma}}(\omega_{k+1}) \cdot \bar{\bar{A}}_{k+1},\ \ldots,\ \bar{\bar{\Gamma}}(\omega_K) \cdot \bar{\bar{A}}_K\right]^\dagger\end{aligned} \tag{11}$$

$$\bar{\bar{\Lambda}}_p = \operatorname{diag}\left[\bar{0}_{\Delta_p},\ c_p[0],\ \bar{0}_{D-1},\ c_p[1],\ \bar{0}_{D-1},\ \ldots,\ c_p[N_g-1],\ \bar{0}_{D-\Delta_p-1}\right] \tag{15}$$

$$\bar{\bar{\Lambda}}_q = \operatorname{diag}\left[\bar{0}_{\Delta_q},\ c_j[0],\ \bar{0}_{D-1},\ c_q[1],\ \bar{0}_{D-1},\ \ldots,\ c_q[N_g-1],\ \bar{0}_{D-\Delta_q-1}\right] \tag{16}$$

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From the definition of $\bar{\bar{V}}_{pq}$, it can be derived that

$$\left\{\bar{\bar{V}}_{pq}\cdot\bar{\bar{V}}_{qr}\right\}_{(m,n)} = \frac{N_g}{N^2}\frac{1-e^{j\omega_{qp}}}{1-e^{j\left(\omega_{qp}-2\pi\Delta_{pq}\right)/D}}\frac{1-e^{j\omega_{rq}}}{1-e^{j\left(\omega_{rq}-2\pi\Delta_{qr}\right)/D}} \times\sum_{\ell=0}^{N_g-1}\left[e^{j\left(\omega_{rp}-2\pi\Delta_{pr}\right)/N}e^{-j2\pi(m-n)/N_g}\right]^{\ell} \tag{21}$$

In case of $r=p$

$$\left\{\bar{\bar{V}}_{pq}\cdot\bar{\bar{V}}_{qp}\right\}_{(m,n)} = \frac{N_g}{N^2}\frac{1-cos\,\omega_{pq}}{1-cos\left[\left(\omega_{pq}-2\pi\Delta_{qp}\right)/D\right]}\sum_{\ell_2=0}^{N_g-1}\left[e^{-j2\pi(m-n)/N_g}\right]^{\ell_2} \tag{22}$$

The summation in (22) is zero if $m\neq n$, and is N_g if $m=n$. Hence

$$\bar{\bar{V}}_{pq}\cdot\bar{\bar{V}}_{qp}=\frac{N_g^2}{N^2}\frac{1-cos\,\omega_{pq}}{\left\{1-cos\left[\left(\omega_{pq}-2\pi\Delta_{qp}\right)/D\right]\right\}}\bar{\bar{I}}_{N_g}$$

and the product in (20) becomes

$$\bar{\bar{b}}_{p,q}\cdot\bar{\bar{b}}_{q,p}=\beta_{p,q}\bar{\bar{I}}_{N_g}=\bar{\bar{b}}_{q,p}\cdot\bar{\bar{b}}_{p,q} \tag{23}$$

where

$$\beta_{p,q}=\left(\frac{N_g}{N}\right)^4\frac{1-cos\,\omega_{qp}}{1-cos\left[\left(\omega_{qp}-2\pi\Delta_{pq}\right)/D\right]}$$

By induction, we derive

$$\bar{\bar{V}}_{pq}\cdot\bar{\bar{V}}_{qr}\cdots\bar{\bar{V}}_{\gamma p}=\frac{1-e^{j\omega_{qp}}}{1-e^{j\left(\omega_{qp}-2\pi\Delta_{pq}\right)/D}}\frac{N_g^{K_1}}{N^{K_1}}\cdots\frac{1-e^{j\omega_{p\gamma}}}{1-e^{j\left(\omega_{p\gamma}-2\pi\Delta_{\gamma p}\right)/D}}\bar{\bar{I}}_{N_g}$$

$$\bar{\bar{b}}_{pq}\cdot\bar{\bar{b}}_{qr}\cdots\bar{\bar{b}}_{\gamma p}=\frac{N_g^{K_1}}{N^{K_1}}\bar{\bar{D}}_p\cdot\bar{\bar{F}}_{N_g}^{\dagger}\cdot\bar{\bar{c}}_p\cdot\bar{\bar{V}}_{pq}\cdot\bar{\bar{V}}_{qr}\cdots \bar{\bar{V}}_{\gamma p}\cdot\bar{\bar{c}}_p\cdot\bar{\bar{F}}_{N_g}\cdot\bar{\bar{D}}_p=\beta_{p,q,r,\ldots,\gamma}\bar{\bar{I}}_{N_g}$$

where K_1+1 is the number of $\bar{\bar{V}}$ matrices in the product, and

$$\beta_{p,q,r,\ldots,\gamma}=\frac{1}{D^2}\frac{1-e^{j\omega_{qp}}}{1-e^{j\left(\omega_{qp}-2\pi\Delta_{pq}\right)/D}}\cdots\frac{1}{D^2}\frac{1-e^{j\omega_{p\gamma}}}{1-e^{j\left(\omega_{p\gamma}-2\pi\Delta_{\gamma p}\right)/D}} \tag{24}$$

If F_g frequency grids and K SS's are considered, there are $2F_gK(K-1)$ terms of the form $\left(1-e^{j\omega}\right)/\left[D^2\left(1-e^{j(\omega-2\pi\Delta)/D}\right)\right]$, which can be computed in advance and stored. Hence, it takes only K complex multiplications to compute $\beta_{p,q,r,\ldots,\gamma}$ in (24).

3.2 Layer-stripping approach to inverse $\bar{\bar{B}}^{\dagger}\cdot\bar{\bar{B}}$

A layer-stripping approach is proposed to compute $\left(\bar{\bar{B}}^{\dagger}\cdot\bar{\bar{B}}\right)^{-1}$ in (7). The product $\bar{\bar{B}}^{\dagger}\cdot\bar{\bar{B}}$ associated with SS's 1–3 can be represented as

$$\bar{\bar{B}}^{\dagger}\cdot\bar{\bar{B}}=\begin{bmatrix}\bar{\bar{b}}_{1,1} & \bar{\bar{b}}_{1,2} & \bar{\bar{b}}_{1,3}\\ \bar{\bar{b}}_{2,1} & \bar{\bar{b}}_{2,2} & \bar{\bar{b}}_{2,3}\\ \bar{\bar{b}}_{3,1} & \bar{\bar{b}}_{3,2} & \bar{\bar{b}}_{3,3}\end{bmatrix}=\begin{bmatrix}\bar{\bar{M}}_{1,1}^{(1)} & \bar{\bar{M}}_{1,2}^{(1)}\\ \bar{\bar{M}}_{2,1}^{(1)} & \bar{\bar{M}}_{2,2}^{(1)}\end{bmatrix} \tag{25}$$

where

$$\bar{\bar{M}}_{1,1}^{(1)}=\bar{\bar{b}}_{1,1},\quad \bar{\bar{M}}_{1,2}^{(1)}=\left[\bar{\bar{b}}_{1,2},\ \bar{\bar{b}}_{1,3}\right],$$
$$\bar{\bar{M}}_{2,1}^{(1)}=\left[\bar{\bar{b}}_{2,1}^{t},\ \bar{\bar{b}}_{3,1}^{t}\right]^{t},\quad \bar{\bar{M}}_{2,2}^{(1)}=\begin{bmatrix}\bar{\bar{b}}_{2,2} & \bar{\bar{b}}_{2,3}\\ \bar{\bar{b}}_{3,2} & \bar{\bar{b}}_{3,3}\end{bmatrix} \tag{26}$$

The superscript (1) in (25) indicates the first layer-stripping process. By applying the block inversion technique, we obtain [17]

$$\left(\bar{\bar{B}}^{\dagger}\cdot\bar{\bar{B}}\right)^{-1}=\begin{bmatrix}\bar{\bar{N}}_{1,1}^{(1)} & \bar{\bar{N}}_{1,2}^{(1)}\\ \bar{\bar{N}}_{2,1}^{(1)} & \bar{\bar{N}}_{2,2}^{(1)}\end{bmatrix}$$

where

$$\bar{\bar{N}}_{1,1}^{(1)}=\alpha_{1,1}^{(1)}\bar{\bar{I}}_{N_g},\quad \bar{\bar{N}}_{1,2}^{(1)}=\left[\bar{\bar{N}}_{1,2}^{(1)}(1,\ 1)\quad \bar{\bar{N}}_{1,2}^{(1)}(1,\ 2)\right]$$
$$\bar{\bar{N}}_{2,1}^{(1)}=\begin{bmatrix}\bar{\bar{N}}_{2,1}^{(1)}(1,\ 1)\\ \bar{\bar{N}}_{2,1}^{(1)}(2,\ 1)\end{bmatrix},\quad \bar{\bar{N}}_{2,2}^{(1)}=\alpha_{2,2}^{(1)}(2,\ 2)$$
$$\begin{bmatrix}\frac{\alpha_{2,2}^{(1)}(1,\ 1)}{\alpha_{2,2}^{(1)}(2,\ 2)}\bar{\bar{I}}_{N_g} & -\frac{\bar{\bar{b}}_{2,3}-D\bar{\bar{b}}_{2,1}\cdot\bar{\bar{b}}_{1,3}}{D^{-1}-D\beta_{2,1}}\bar{\bar{I}}_{N_g}\\ -\frac{\bar{\bar{b}}_{3,2}-D\bar{\bar{b}}_{3,1}\cdot\bar{\bar{b}}_{1,2}}{D^{-1}-D\beta_{2,1}}\bar{\bar{I}}_{N_g} & \bar{\bar{I}}_{N_g}\end{bmatrix} \tag{27}$$

where

$$\alpha_{1,1}^{(1)}=\left[D^{-1}-\alpha_{1,1}^{(2)}\left(\beta_{1,2}-2D\beta_{1,2,3}+\beta_{3,1}\right)\right]^{-1}$$
$$\alpha_{1,1}^{(2)}=\left(D^{-1}-D\beta_{3,2}\right)^{-1}$$
$$\alpha_{2,2}^{(1)}(1,\ 1)=\left[D^{-1}-D\beta_{2,1}-\left(D^{-1}-D\beta_{3,1}\right)^{-1}\times\left(\beta_{2,3}-2D\beta_{2,3,1}+D^2\beta_{1,3}\beta_{2,1}\right)\right]^{-1s}$$
$$\alpha_{2,2}^{(1)}(2,\ 2)=\left[D^{-1}-D\beta_{3,1}-\left(D^{-1}-D\beta_{2,1}\right)^{-1}\times\left(\beta_{3,2}-2D\beta_{3,2,1}+D^2\beta_{1,2}\beta_{3,1}\right)\right]^{-1}$$
$$\bar{\bar{N}}_{1,2}^{(1)}(1,\ 1)=-D\left[\alpha_{2,2}^{(1)}(1,\ 1)+\frac{\beta_{1,3}\alpha_{2,2}^{(1)}(2,\ 2)}{D^{-2}-\beta_{2,1}}\right]\bar{\bar{b}}_{1,2}+\frac{\alpha_{2,2}^{(1)}(2,\ 2)}{D^{-2}-\beta_{2,1}}\bar{\bar{b}}_{1,3}\cdot\bar{\bar{b}}_{3,2}$$

$$\bar{\bar{N}}_{1,2}^{(1)}(1,\ 2) = -D\left[\alpha_{2,2}^{(1)}(2,\ 2) + \frac{\alpha_{2,2}^{(1)}(2,\ 2)\beta_{1,2}}{D^{-2}-\beta_{2,1}}\right]\bar{\bar{b}}_{1,3} + \frac{\alpha_{2,2}^{(1)}(2,\ 2)}{D^{-2}-\beta_{2,1}}\bar{\bar{b}}_{1,2}\cdot\bar{\bar{b}}_{2,3}$$

$$\bar{\bar{N}}_{2,1}^{(1)}(1,\ 1) = -D\left[\alpha_{2,2}^{(1)}(1,\ 1) + \frac{\alpha_{2,2}^{(1)}(2,\ 2)\beta_{1,3}}{D^{-2}-\beta_{2,1}}\right]\bar{\bar{b}}_{2,1} + \frac{\alpha_{2,2}^{(1)}(2,\ 2)}{D^{-2}-\beta_{2,1}}\bar{\bar{b}}_{2,3}\cdot\bar{\bar{b}}_{3,1}$$

$$\bar{\bar{N}}_{2,1}^{(1)}(2,\ 1) = -D\left[\alpha_{2,2}^{(1)}(2,\ 2) + \frac{\alpha_{2,2}^{(1)}(2,\ 2)\beta_{1,2}}{D^{-2}-\beta_{2,1}}\right]\bar{\bar{b}}_{3,1} + \frac{\alpha_{2,2}^{(1)}(2,\ 2)}{D^{-2}-\beta_{2,1}}\bar{\bar{b}}_{3,2}\cdot\bar{\bar{b}}_{2,1}$$

Note that $\beta_{i,\ j} = \beta_{j,\ i}$, and changing the order of $\{i, j, k\}$ does not change the value of $\beta_{i,\ j,\ k}$.

3.3 Prove $\left(\bar{\bar{C}}_B^\dagger \cdot \bar{\bar{C}}_B\right)^{-1}$ is a constant identity matrix

Equation (12) implies that $\bar{\bar{b}}_{k,k} = \bar{\bar{C}}^\dagger \cdot \bar{\bar{C}}$, which is substituted into (18) to obtain $\bar{\bar{C}}^\dagger \cdot \bar{\bar{C}} = (N_g/N)\bar{\bar{I}}_{N_g}$. Let $K = 4$ and $k = 4$, thus $\bar{\bar{C}} = \bar{\bar{\Gamma}}(\omega_4)\cdot\bar{\bar{A}}_4$ and $\bar{\bar{B}} = \left[\bar{\bar{\Gamma}}(\omega_1)\cdot\bar{\bar{A}}_1,\ \bar{\bar{\Gamma}}(\omega_2)\cdot\bar{\bar{A}}_2,\ \bar{\bar{\Gamma}}(\omega_3)\cdot\bar{\bar{A}}_3\right]$. We will prove that $\bar{\bar{C}}^\dagger\cdot\bar{\bar{P}}_B\cdot\bar{\bar{C}}$ is a constant identity matrix by direct substitution as

$$\bar{\bar{C}}^\dagger\cdot\bar{\bar{P}}_B\cdot\bar{\bar{C}} = \bar{\bar{C}}^\dagger\cdot\bar{\bar{B}}\cdot\left(\bar{\bar{B}}^\dagger\cdot\bar{\bar{B}}\right)^{-1}\cdot\bar{\bar{B}}^\dagger\cdot\bar{\bar{C}}$$
$$= \left[\bar{\bar{b}}_{4,1},\ \bar{\bar{b}}_{4,2},\ \bar{\bar{b}}_{4,3}\right]\begin{bmatrix}\bar{\bar{b}}_{1,1} & \bar{\bar{b}}_{1,2} & \bar{\bar{b}}_{1,3}\\ \bar{\bar{b}}_{2,1} & \bar{\bar{b}}_{2,2} & \bar{\bar{b}}_{2,3}\\ \bar{\bar{b}}_{3,1} & \bar{\bar{b}}_{3,2} & \bar{\bar{b}}_{3,3}\end{bmatrix}^{-1}\begin{bmatrix}\bar{\bar{b}}_{1,4}\\ \bar{\bar{b}}_{2,4}\\ \bar{\bar{b}}_{3,4}\end{bmatrix} \tag{28}$$

Applying the layer-stripping approach in the last subsection with $K = 4$, (28) can be reduced to

$$\begin{aligned}\bar{\bar{C}}^\dagger\cdot\bar{\bar{P}}_B\cdot\bar{\bar{C}} = \Big\{&\alpha_{1,1}^{(1)}\beta_{4,1} - D\left[\alpha_{2,2}^{(1)}(1,\ 1) + \beta_{1,3}G\right]\beta_{4,1,2}\\ &+ G\beta_{4,1,3,2} - D\left[\alpha_{2,2}^{(1)}(2,\ 2) + \beta_{1,2}G\right]\beta_{4,1,3}\\ &+ G\beta_{4,1,2,3} - D\left[\alpha_{2,2}^{(1)}(1,\ 1) + G\beta_{1,3}\right]\beta_{4,2,1}\\ &+ G\beta_{4,2,3,1} + \alpha_{2,2}^{(1)}(1,\ 1)\beta_{4,2}\\ &- \alpha_{2,2}^{(1)}(2,\ 2)\left(D^{-1} - D\beta_{2,1}\right)^{-1}\left(\beta_{4,2,3} - D\beta_{4,2,1,3}\right)\\ &- D\left[\alpha_{2,2}^{(1)}(2,\ 2) + G\beta_{1,2}\right]\beta_{4,3,1} + G\beta_{4,3,2,1}\\ &- \alpha_{2,2}^{(1)}(2,\ 2)\left(D^{-1} - D\beta_{2,1}\right)^{-1}\left(\beta_{4,3,2} - D\beta_{4,3,1,2}\right)\\ &+\alpha_{2,2}^{(1)}(2,\ 2)\beta_{4,3}\Big\}\bar{\bar{I}}_{N_g}\end{aligned} \tag{29}$$

where $G = \alpha_{2,2}^{(1)}(2,\ 2)\left(D^{-2} - \beta_{2,1}\right)^{-1}$. Hence, $\left(\bar{\bar{C}}_B^\dagger\cdot\bar{\bar{C}}_B\right)^{-1}$ in (10) is a constant identity matrix.

3.4 Computation of $\left\|\bar{\bar{P}}_{C_B}\cdot\bar{r}\right\|^2$

Based on the results in the last subsection, (9) can be simplified as

$$\left\|\bar{\bar{P}}_{C_B}\cdot\bar{r}\right\|^2 = \frac{\bar{r}^\dagger\cdot\left(\bar{\bar{I}}_N - \bar{\bar{P}}_B\right)\cdot\bar{\bar{C}}\cdot\bar{\bar{C}}^\dagger\cdot\left(\bar{\bar{I}}_N - \bar{\bar{P}}_B\right)\cdot\bar{r}}{\left\|\bar{\bar{C}}_B^\dagger\cdot\bar{\bar{C}}_B\right\|} = \frac{\bar{R}^\dagger\cdot\bar{\bar{a}}_4\cdot\bar{R}}{\left\|\bar{\bar{C}}_B^\dagger\cdot\bar{\bar{C}}_B\right\|} \tag{30}$$

where $\bar{R} = \left(\bar{\bar{I}}_N - \bar{\bar{P}}_B\right)\cdot\bar{r}$. Next, expand $\bar{\bar{P}}_B$ as

$$\begin{aligned}\bar{\bar{P}}_B &= \left[\bar{\bar{\Gamma}}(\omega_1)\cdot\bar{\bar{A}}_1,\ \bar{\bar{\Gamma}}(\omega_2)\cdot\bar{\bar{A}}_2,\ \bar{\bar{\Gamma}}(\omega_3)\cdot\bar{\bar{A}}_3\right]\\ &\quad\cdot\begin{bmatrix}\bar{\bar{b}}_{1,1} & \bar{\bar{b}}_{1,2} & \bar{\bar{b}}_{1,3}\\ \bar{\bar{b}}_{2,1} & \bar{\bar{b}}_{2,2} & \bar{\bar{b}}_{2,3}\\ \bar{\bar{b}}_{3,1} & \bar{\bar{b}}_{3,2} & \bar{\bar{b}}_{3,3}\end{bmatrix}^{-1}\cdot\begin{bmatrix}\bar{\bar{A}}_1^\dagger\cdot\bar{\bar{\Gamma}}(\omega_1)^\dagger\\ \bar{\bar{A}}_2^\dagger\cdot\bar{\bar{\Gamma}}(\omega_2)^\dagger\\ \bar{\bar{A}}_3^\dagger\cdot\bar{\bar{\Gamma}}(\omega_3)^\dagger\end{bmatrix}\\ &= \alpha_{1,1}^{(1)}\bar{\bar{a}}_1 + \alpha_{2,2}^{(1)}\bar{\bar{a}}_2 + \alpha_{2,2}^{(1)}(2,\ 2)\bar{\bar{a}}_3\\ &\quad - D\left[\alpha_{2,2}^{(1)}(1,\ 1) + \beta_{1,3}G\right]\left(\bar{\bar{a}}_{1,2} + \bar{\bar{a}}_{2,1}\right)\\ &\quad - D\left[\alpha_{2,2}^{(1)}(2,\ 2) + \beta_{1,2}G\right]\left(\bar{\bar{a}}_{1,3} + \bar{\bar{a}}_{3,1}\right)\\ &\quad - GD^{-1}\left(\bar{\bar{a}}_{2,3} + \bar{\bar{a}}_{3,2}\right)\\ &\quad + G\left(\bar{\bar{a}}_{2,1,3} + \bar{\bar{a}}_{3,1,2} + \bar{\bar{a}}_{1,3,2} + \bar{\bar{a}}_{2,3,1} + \bar{\bar{a}}_{1,2,3} + \bar{\bar{a}}_{3,2,1}\right)\end{aligned} \tag{31}$$

with

$$\bar{\bar{a}}_k = \bar{\bar{\Gamma}}(\omega_k)\cdot\bar{\bar{A}}_k\cdot\bar{\bar{A}}_k^\dagger\cdot\bar{\bar{\Gamma}}^\dagger(\omega_k) = \frac{1}{D^2}\begin{bmatrix}\bar{\bar{I}}_{N_g} & \tilde{k}^{-1}\bar{\bar{I}}_{N_g} & \cdots & \tilde{k}^{-(D-1)}\bar{\bar{I}}_{N_g}\\ \tilde{k}\bar{\bar{I}}_{N_g} & \bar{\bar{I}}_{N_g} & \cdots & \tilde{k}^{-(D-2)}\bar{\bar{I}}_{N_g}\\ \vdots & \vdots & \vdots & \ddots\\ \tilde{k}^{D-1}\bar{\bar{I}}_{N_g} & \tilde{k}^{D-2}\bar{\bar{I}}_{N_g} & \cdots & \bar{\bar{I}}_{N_g}\end{bmatrix} \tag{32}$$

$$\begin{aligned}\bar{\bar{a}}_{k,p,\ldots,\gamma,\delta} &= \left(\frac{1}{D^2}\frac{1 - \mathrm{e}^{\mathrm{j}\omega_{pk}}}{1 - \mathrm{e}^{\mathrm{j}\left(\omega_{pk} + 2\pi\Delta_{pk}\right)/D}}\right)\left(\frac{1}{D^2}\frac{1 - \mathrm{e}^{\mathrm{j}\omega_{qp}}}{1 - \mathrm{e}^{\mathrm{j}\left(\omega_{qp} + 2\pi\Delta_{qp}\right)/D}}\right)\\ &\quad\cdots\left(\frac{1}{D^2}\frac{1 - \mathrm{e}^{\mathrm{j}\omega_{\delta\gamma}}}{1 - \mathrm{e}^{\mathrm{j}\left(\omega_{\delta\gamma} + 2\pi\Delta_{\delta\gamma}\right)/D}}\right)\frac{1}{D^2}\\ &\quad\begin{bmatrix}\bar{\bar{I}}_{N_g} & \tilde{\delta}^{-1}\bar{\bar{I}}_{N_g} & \cdots & \tilde{\delta}^{-(D-1)}\bar{\bar{I}}_{N_g}\\ \tilde{k}\bar{\bar{I}}_{N_g} & \tilde{k}\,\tilde{\delta}^{-1}\bar{\bar{I}}_{N_g} & \cdots & \tilde{k}\,\tilde{\delta}^{-(D-1)}\bar{\bar{I}}_{N_g}\\ \vdots & \vdots & \ddots & \vdots\\ \tilde{k}^{D-1}\bar{\bar{I}}_{N_g} & \tilde{k}^{D-1}\tilde{\delta}^{-1}\bar{\bar{I}}_{N_g} & \cdots & \tilde{k}^{D-1}\tilde{\delta}^{-(D-1)}\bar{\bar{I}}_{N_g}\end{bmatrix}\end{aligned} \tag{33}$$

where $\tilde{k} = \mathrm{e}^{\mathrm{j}\left(\omega_k + 2\pi\Delta_k\right)/D}$, and $\tilde{k},\ \ldots,\ \tilde{k}^{D-1}$ can be stored in a look-up table with $(D-1)F_gK$ entries. In other steps to update

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another SS, the only difference is the order of subscripts of $\bar{\bar{b}}$ in (28). The indices and parameters in (31) will also be replaced by those of the associated SS's.

The matrices $\bar{\bar{a}}_i$, $\bar{\bar{a}}_{i,j}$ and $\bar{\bar{a}}_{i,j,k}$ are constituted of $N_g \times N_g$ constant identity matrices, hence $\bar{\bar{P}}_B$ can be partitioned into constant identity matrices too. The received signal $\bar{r}$ is partitioned into D vectors, $\bar{r}_m$, each with length N_g. $\bar{\bar{P}}_B \cdot \bar{r}$ can be expanded as

$$\bar{\bar{P}}_B \cdot \bar{r}\begin{bmatrix} P_{B,(0,0)}\bar{\bar{I}}_{N_g} & \cdots & P_{B,(0,D-1)}\bar{\bar{I}}_{N_g} \\ \vdots & \ddots & \vdots \\ P_{B,(D-1,0)}\bar{\bar{I}}_{N_g} & \cdots & P_{B,(D-1,D-1)}\bar{\bar{I}}_{N_g} \end{bmatrix} \cdot \begin{bmatrix} \bar{r}_0 \\ \vdots \\ \bar{r}_{D-1} \end{bmatrix} = \begin{bmatrix} \sum_{\ell=0}^{D-1} P_{B,(0,\ell)}\bar{r}_\ell \\ \vdots \\ \sum_{\ell=0}^{D-1} P_{B,(D-1,\ell)}\bar{r}_\ell \end{bmatrix} \tag{34}$$

where $P_{B,(m,n)}$ is the constant of the (m, n)th constant identity submatrix of $\bar{\bar{P}}_B$. Similarly, $\bar{\mathcal{R}}$ can be partitioned into D vectors $\bar{\mathcal{R}}_m$, each with length N_g, and

$$\bar{\mathcal{R}}^\dagger \cdot \bar{\bar{a}}_k \cdot \bar{\mathcal{R}} = D^{-2}\sum_{m=0}^{D-1}\sum_{n=0}^{D-1}\tilde{k}^{m-n}\,\bar{\mathcal{R}}_m^\dagger \cdot \bar{\mathcal{R}}_n$$

The double summation in (34) can be further reduced to

$$\sum_{m=0}^{D-1}\mathcal{R}_{m,m} + 2\mathrm{Re}\left\{\sum_{n=1}^{D-1}\left(\sum_{m=n}^{D-1}\mathcal{R}_{m,(m-n)}\right)\tilde{k}^n\right\} \tag{35}$$

where $\mathrm{Re}\{\alpha\}$ stands for the real part of α, $\mathcal{R}_{m,n} = \bar{\mathcal{R}}_m^\dagger \cdot \bar{\mathcal{R}}_n$ can be calculated in advance at the beginning of each step, which takes $2N_gD^2$ complex multiplications, and (35) takes only D complex multiplications.

3.5 Computational load in one cycle, $K=4$

The load to compute $\bar{\bar{P}}_B$ can be estimated on (31). In $\beta_{p,q,r,s}$ of (24), each factor of the form $\left((1-e^{j\omega_{pq}})\Big/\left(D^2\left[1-e^{j\left(\omega_{pq}-2\pi\Delta_{qp}\right)/D}\right]\right)\right)$ can be calculated in advance and stored in a table, and the product in (24) can be calculated by picking proper factors from the table. This table contains $2F_gK(K-1)$ entries to cover all possible CFO's and different combinations of SS's. Each $\beta_{p,q,r,\cdot s,\gamma}$ term containing K_1+1 factors, hence takes K_1 complex multiplications. Also note that changing the order of subscripts in $\beta_{p,q}$ or $\beta_{p,q,r}$ does not change its value.

Next, $\bar{\bar{a}}_k$ in (32) is a Toeplitz matrix with $2D-1$ different elements, which takes $2D-1$ multiplications. The second-order matrix $\bar{\bar{a}}_{k,p}$ is constructed in terms of the first-order matrices as $\bar{\bar{a}}_k \cdot \bar{\bar{a}}_p$ and takes $2D^2$ multiplications. The coefficients in (16) contain the same terms as used in generating β in (24). The third-order matrix $\bar{\bar{a}}_{k,q,p}$ with the first and the last indices the same as those in $\bar{\bar{a}}_{k,p}$. By observing (33), the matrix part of $\bar{\bar{a}}_{k,q,p}$ can be generated in the same way as constructing $\bar{\bar{a}}_{k,p}$ and takes $3D^2$ multiplications.

Tables 1 and 2 list the number of multiplications required in calculating simple and combined terms, respectively, contained in $\bar{\bar{P}}_B$ of (31). Hence, it takes $24D^2+6D+19$ complex multiplications to compute $\bar{\bar{P}}_B$.

Similarly, Table 3 lists the number of multiplications required for different terms in $\bar{\bar{C}}^\dagger \cdot \bar{\bar{P}}_B \cdot \bar{\bar{C}}$ of (29), and the total number of multiplications is 104.

It takes $2N_gD^2$ multiplications to compute $R_{m,n}$'s, D multiplications to compute $\bar{R}^\dagger \cdot \bar{\bar{a}}_k \cdot \bar{R}$, and 104 multiplications to compute $\left\|\bar{\bar{C}}_B^\dagger \cdot \bar{\bar{C}}_B\right\| = \bar{\bar{C}}^\dagger \cdot \bar{\bar{P}}_B \cdot \bar{\bar{C}}$. Hence, one complete cycle takes $K[(24D^2+6D+19)+2N_gD^2+F_g(D+104)]$ complex multiplications. In comparison, the conventional AP method takes $O(KF_g(N_g^3+N_gN^2))$ complex multiplications [12].

In the following simulations, the parameters are extracted from a scalable OFDMA system in [18]: $N=256$, $N_g=D=16$, Δf_k is uniformly distributed over $[-0.5,\ 0.5]$ with resolution $\delta f=0.001$, thus $F_g=1000$. With $K=4$, the number of multiplications are 537 804 and $O(4,\ 210,\ 688,\ 000)$ with this efficient AP method and conventional AP method, respectively. This approach reduces the computational load by about 8000 folds.

The SUI-3 model is chosen to model the channel in (2), where three taps are allocated for the CIR of each SS [19]. The integer timing error μ_k is uniformly distributed over the integer set $\{0,\ \ldots,\ 7\}$. The SNR of the kth SS is defined as

$$\mathrm{SNR}_k = \frac{E\left\{\bar{s}_k^\dagger \cdot \bar{s}_k\right\}}{N\sigma_v^2}$$

where $E\{A\}$ is the expectation value of random variable A, and σ_v^2 is the noise variance in each subcarrier.

Table 2 Number of multiplications in calculating combined terms in (31)

Terms	Numbers
$D\left(\alpha_{2,2}^{(1)}(1,\ 1)+\beta_{1,3}G\right)\left(\bar{\bar{a}}_{1,2}+\bar{\bar{a}}_{1,2}^\dagger\right)$	$3D^2$
$D\left(\alpha_{2,2}^{(1)}(2,\ 2)+\beta_{1,2}G\right)\left(\bar{\bar{a}}_{1,3}+\bar{\bar{a}}_{1,3}^\dagger\right)$	$3D^2$
$G/D\left(\bar{\bar{a}}_{2,3}+\bar{\bar{a}}_{2,3}^\dagger\right)$	$2D^2$
$G=\left(\bar{\bar{a}}_{2,1,3}+\bar{\bar{a}}_{2,1,3}^\dagger+\bar{\bar{a}}_{1,3,2}+\bar{\bar{a}}_{1,3,2}^\dagger+\bar{\bar{a}}_{1,2,3}+\bar{\bar{a}}_{1,2,3}^\dagger\right)$	D^2

Table 1 Number of multiplications in calculating simple terms in (31)

Terms	Numbers	Terms	Numbers	Terms	Numbers
$3\beta_{p,q}$s	3	$\alpha_{1,1}^{(2)}$	3	$3\bar{\bar{a}}_{p,q}$ss	$6D^2$
$\beta_{1,2,3}$	3	$\alpha_{1,1}^{(1)}$	5	$3\,\bar{\bar{a}}_{p,q,r}$s	$9D^2$
D^2	1	$\alpha_{2,2}^{(1)}(1,\ 1)$	3	G	2
D^4	1	$\alpha_{2,2}^{(1)}(2,\ 2)$	3	$3\,\bar{\bar{A}}$'s's	$6D-3$

Table 3 Number of multiplications in calculating simple terms in (29)

Terms	Numbers	Terms	Numbers	Terms	Numbers
$6\beta_{p,q}$s	6	$\alpha_{1,1}^{(2)}$	3	miscellaneous	47
$4\beta_{p,q,r}$s	12	$\alpha_{1,1}^{(1)}$	5		
$6\beta_{p,q,r}$s	24	$\alpha_{2,2}^{(1)}(1,\ 1)$	3		
D^2	1	$\alpha_{2,2}^{(1)}(2,\ 2)$	3		

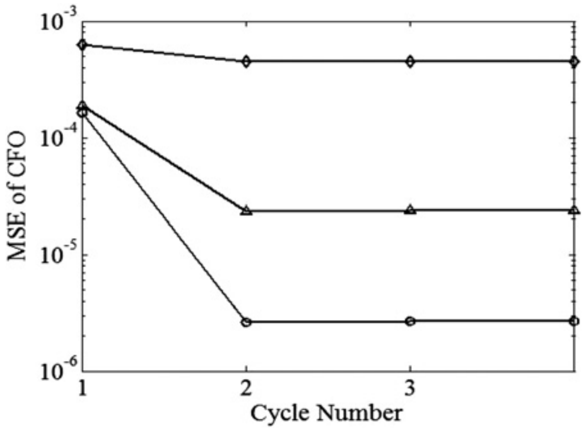

Fig. 3 *MSE of CFO, –°–:°—-: SNR$_I$ = 30 dB, —-Δ—-: SNR$_I$ = 15 dB, —-◇—-: SNR$_I$ = 0 dB*

Fig. 3 shows the MSE of CFO simulated using the efficient AP method of this work and that of conventional AP method. The MSE of CFO is defined as

$$\mathrm{MSE} = \frac{1}{M}\sum_{m=1}^{M}\left[\mathrm{CFO}_{\mathrm{est},\,m} - \mathrm{CFO}_{\mathrm{true},\,m}\right]^2 \tag{36}$$

where $M = 200$ trials are used to obtain these curves. Since no approximation has been made in deriving the efficient AP method from the conventional one, the MSE with both methods are the same.

The $\bar{\bar{\boldsymbol{P}}}_B$ matrix in (31) can be decomposed into three constituents with different orders of magnitude. The second-order matrices $\bar{\bar{\boldsymbol{a}}}_{k,p}$ are smaller than the first-order matrices $\bar{\bar{\boldsymbol{a}}}_k$ by an order of D^2 and the third-order matrices $\bar{\bar{\boldsymbol{a}}}_{k,p,q}$ are smaller than $\bar{\bar{\boldsymbol{a}}}_k$ by an order of D^4. By keeping only the first- and the second-order matrices, the computational load can be further reduced, especially when K is large.

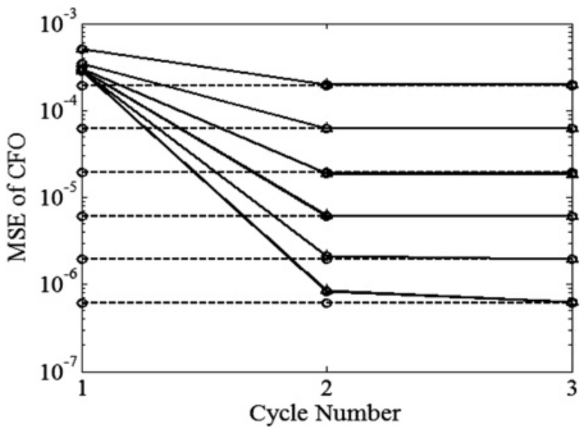

Fig. 4 *MSE of CFO with SNR$_I$ = 5, 10, 15, 20, 25, 30 dB (from top to bottom), —-°—-: efficient AP method, —-Δ—-: efficient AP method with approximate $\bar{\bar{\boldsymbol{P}}}_B$, –°–: CRLB*

Fig. 4 shows the simulation results using both the complete and the approximate version of efficient AP method. The MSE of both versions reach the CRLB of MSE in two cycles, with $M = 1000$ trials.

4 Efficient AP method for any number of SS's

4.1 Isomorphism between $\bar{\bar{\boldsymbol{B}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{B}}}$ and its inverse

As embedded in (28) and (29), the inverse of $\bar{\bar{\boldsymbol{B}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{B}}}$ is a key step to derive $\bar{\bar{\boldsymbol{C}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{P}}}_B\cdot\bar{\bar{\boldsymbol{C}}}$ and $\bar{\bar{\boldsymbol{C}}}_B^{\dagger}\cdot\bar{\bar{\boldsymbol{C}}}_B$, which dominate the computational load of both the numerator and the denominator of the CFO estimator, $\left\|\bar{\bar{\boldsymbol{P}}}_{C_B}\cdot\bar{r}\right\|^2$, in (30). The layer-stripping method presented in the last section is suitable for small K. An alternative method to compute $\bar{\bar{\boldsymbol{B}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{B}}}$, suitable for large K, is introduced here.

It will be proved by induction that a square matrix of the following form has the same structure with, or is isomorphic to, its inverse, namely

$$\begin{bmatrix} c_{1,1}\bar{\bar{\boldsymbol{b}}}_{1,1} & \cdots & c_{1,M}\bar{\bar{\boldsymbol{b}}}_{1,M} \\ \vdots & \ddots & \vdots \\ c_{M,1}\bar{\bar{\boldsymbol{b}}}_{M,1} & \cdots & c_{M,M}\bar{\bar{\boldsymbol{b}}}_{M,M} \end{bmatrix}^{-1} = \begin{bmatrix} d_{1,1}\bar{\bar{\boldsymbol{b}}}_{1,1} & \cdots & d_{1,M}\bar{\bar{\boldsymbol{b}}}_{1,M} \\ \vdots & \ddots & \vdots \\ d_{M,1}\bar{\bar{\boldsymbol{b}}}_{M,1} & \cdots & d_{M,M}\bar{\bar{\boldsymbol{b}}}_{M,M} \end{bmatrix} \tag{37}$$

where M is an arbitrary integer. Detailed derivation is presented in the Appendix.

4.2 Inversion of $\bar{\bar{\boldsymbol{B}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{B}}}$ with direct mapping

Let $k = K$, the isomorphism between $\bar{\bar{\boldsymbol{B}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{B}}}$ and $\left(\bar{\bar{\boldsymbol{B}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{B}}}\right)^{-1}$ leads to

$$\left(\bar{\bar{\boldsymbol{B}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{B}}}\right)^{-1} = \begin{bmatrix} d_{1,1}\bar{\bar{\boldsymbol{b}}}_{1,1} & \cdots & d_{1,K-1}\bar{\bar{\boldsymbol{b}}}_{1,K-1} \\ \vdots & \ddots & \vdots \\ d_{K-1,1}\bar{\bar{\boldsymbol{b}}}_{K-1,1} & \cdots & d_{K-1,K-1}\bar{\bar{\boldsymbol{b}}}_{K-1,K-1} \end{bmatrix} \tag{38}$$

with the coefficients $d_{i,j}$'s to be determined. The product of

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$\bar{\bar{\boldsymbol{B}}}^{\dagger} \cdot \bar{\bar{\boldsymbol{B}}}$ and $\left(\bar{\bar{\boldsymbol{B}}}^{\dagger} \cdot \bar{\bar{\boldsymbol{B}}}\right)^{-1}$ is an identity matrix

$$\bar{\bar{\boldsymbol{B}}}^{\dagger} \cdot \bar{\bar{\boldsymbol{B}}} \cdot \left(\bar{\bar{\boldsymbol{B}}}^{\dagger} \cdot \bar{\bar{\boldsymbol{B}}}\right)^{-1}$$

$$\begin{bmatrix} \sum_{i=1}^{K-1} d_{i,1} \bar{\bar{\boldsymbol{b}}}_{1,i} \cdot \bar{\bar{\boldsymbol{b}}}_{i,1} & \cdots & \sum_{i=1}^{K-1} d_{i,K} \bar{\bar{\boldsymbol{b}}}_{1,i} \cdot \bar{\bar{\boldsymbol{b}}}_{i,k} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{K-1} d_{i,1} \bar{\bar{\boldsymbol{b}}}_{K,i} \cdot \bar{\bar{\boldsymbol{b}}}_{i,1} & \cdots & \sum_{i=1}^{K} d_{i,K} \bar{\bar{\boldsymbol{b}}}_{K,i} \cdot \bar{\bar{\boldsymbol{b}}}_{i,K} \end{bmatrix} = \bar{\bar{\boldsymbol{I}}}_{(K-1)N_g} \tag{39}$$

If the coefficients $d_{i,j}$'s are determined by direct comparison, it will take $O((K-1)^6)$ complex multiplications, which is impractical when K is large. Instead, these coefficients are determined as follows.

Compare the nth diagonal entry in (39) to have

$$\sum_{i=1}^{K-1} d_{i,n} \bar{\bar{\boldsymbol{b}}}_{n,i} \cdot \bar{\bar{\boldsymbol{b}}}_{i,n} = \sum_{i=1}^{K-1} d_{i,n} \beta_{n,i} \bar{\bar{\boldsymbol{I}}}_{N_g} = \bar{\bar{\boldsymbol{I}}}_{N_g} \tag{40}$$

and compare the (m, n)th entry off diagonal to have

$$\sum_{i=1}^{K-1} d_{i,n} \bar{\bar{\boldsymbol{b}}}_{m,i} \cdot \bar{\bar{\boldsymbol{b}}}_{i,n} = \begin{cases} \sum_{i=1}^{K-1} d_{i,n} \gamma_{m,i,n} \bar{\bar{\boldsymbol{b}}}_{m,n}, & \omega_{nm} \neq 0 \\ \sum_{i=1}^{K-1} d_{i,n} \bar{\bar{0}}_{N_g}, & \omega_{nm} = 0 \end{cases} = \bar{\bar{0}}_{N_g} \tag{41}$$

with

$$\gamma_{m,i,n} = \frac{1}{D^2} \frac{1 - e^{j\omega_{im}}}{1 - e^{j(\omega_{im} - 2\pi\Delta_{mi})/D}} \times \frac{1 - e^{j\omega_{ni}}}{1 - e^{j(\omega_{ni} - 2\pi\Delta_{in})/D}} \left[\frac{1 - e^{j\omega_{nm}}}{1 - e^{j(\omega_{nm} - 2\pi\Delta_{mn})/D}} \right]^{-1} \tag{42}$$

based on the relation, $\bar{\bar{\boldsymbol{b}}}_{m,i} \cdot \bar{\bar{\boldsymbol{b}}}_{i,n} = \gamma_{m,i,n} \bar{\bar{\boldsymbol{b}}}_{m,n}$. The entries in the nth column of (39) involve the coefficients $\{d_{1,n}, \ldots, d_{K-1,n}\}$, and can be expanded as

$$\begin{bmatrix} \gamma_{1,1,n} & \cdots & \gamma_{1,K-1,n} \\ \vdots & \vdots & \vdots \\ \gamma_{n-1,1,n} & \cdots & \gamma_{n-1,K-1,n} \\ \beta_{n,1} & \cdots & \beta_{n,K-1} \\ \gamma_{n+1,1,n} & \cdots & \gamma_{n+1,K-1,n} \\ \vdots & \vdots & \vdots \\ \gamma_{K-1,1,n} & \cdots & \gamma_{K-1,K-1,n} \end{bmatrix} \begin{bmatrix} d_{1,n} \\ \vdots \\ d_{K-1,n} \end{bmatrix} = [0, \ldots, 0, 1, 0, \ldots, 0]^t \tag{43}$$

from which $\{d_{1,n}, \ldots, d_{K-1,n}\}$ can be determined. The other columns in (39) can be expanded in a similar manner to solve the other coefficients $d_{m,n}$'s. The total number of complex multiplications is $(K-1)O((K-1)^3) = O((K-1)^4)$.

Note that $d_{m,n} = d^*_{n,m}$ because $\left(\bar{\bar{\boldsymbol{B}}}^{\dagger} \cdot \bar{\bar{\boldsymbol{B}}}\right)^{-1}$ is Hermitian. Once the nth column of $d_{m,n}$ is obtained, the number of unknowns $d_{m,n'}$'s in the n'th column is reduced by one, and so on. Thus, the total number of complex multiplications is reduced to

$$\sum_{K'=2}^{K-1} (K'-1)^3 = \frac{1}{4} K^2 (K-1)^2$$

4.3 Simulation on MSE of CFO

All the parameters chosen in the simulation are extracted from the scalable physical layer in [18], and are the same as those illustrated in the previous section.

Fig. 5 shows MSE results and their convergence rate with four, six and eight SS's, respectively. The SNR ranges from 5

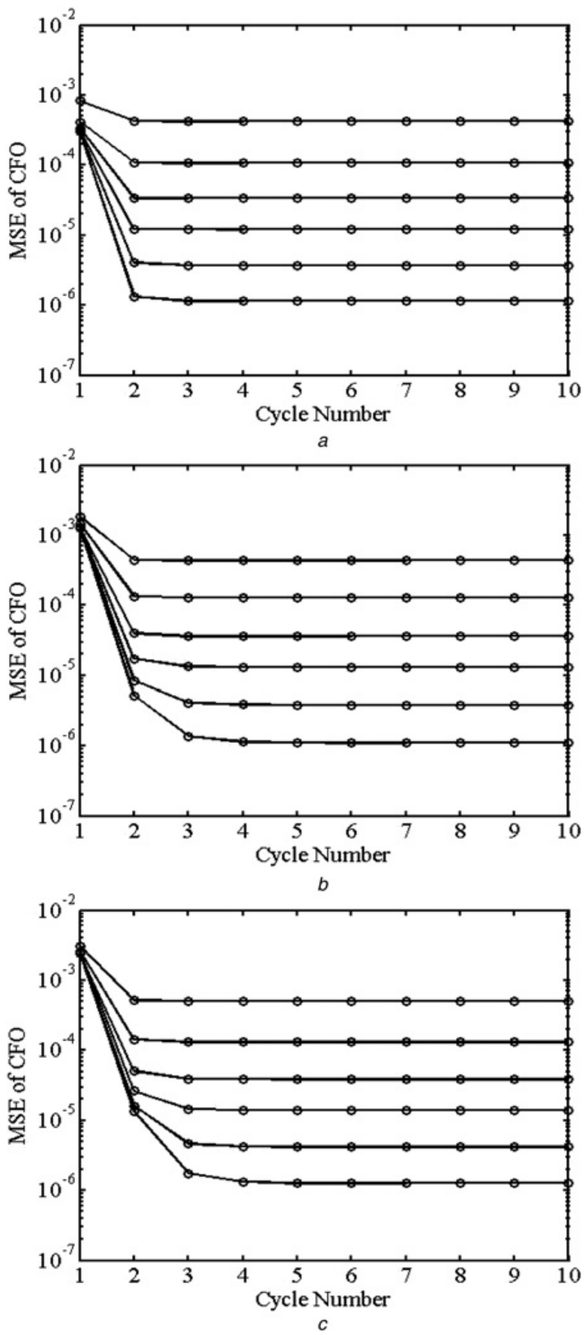

Fig. 5 *Convergence of MSE, SNR = 5, 10, 15, 20, 25, 30 dB (from top to bottom)*

a K = 4
b K = 6
c K = 8

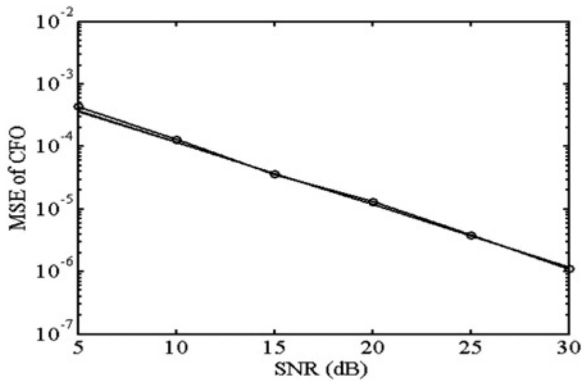

Fig. 6 *MSE of CFO, – ° – : MSE, – : CRLB, K = 6*

to 30 dB. In all cases, the MSE converges in three cycles for $K = 4$, and five cycles for $K = 6$, 8. The fair convergence properties of the conventional AP method is preserved.

The SUI-3 channel model is adopted in our paper, which is the same as that used in [7]. As shown in Figs. 4 and 5 of [7], the MSE against SNR plots with $K = 4$ and $K = 8$ look very similar. In our case, the MSE results with $K = 4$, 6 and 8 also look close to one another, under the SUI-3 channel model.

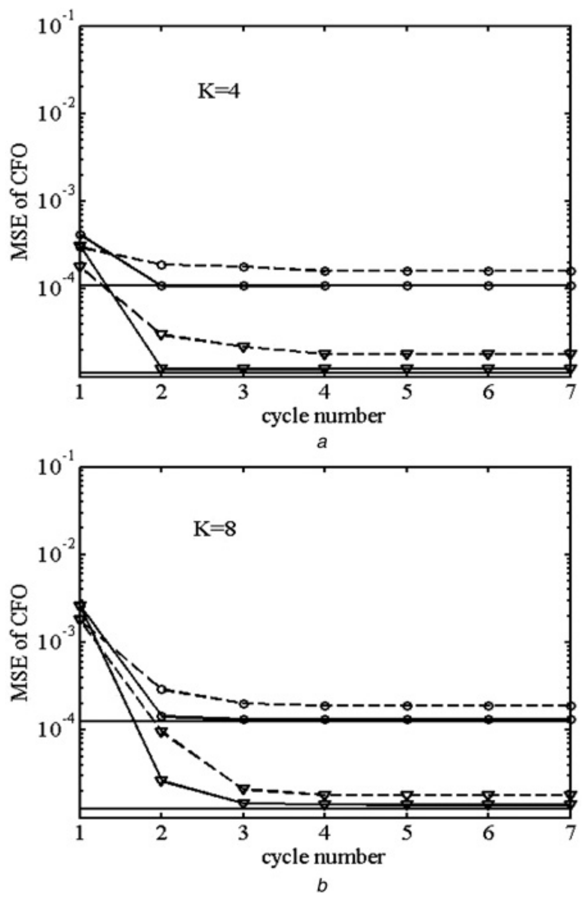

Fig. 7 *Comparison on MSE of CFO, –°–: MSAGE at SNR = 10 dB, – ° – : efficient AP at SNR = 10 dB, –∇–: MSAGE at SNR = 20 dB, – ∇ – : efficient AP at SNR = 20 dB, – : CRLB*

a $K = 4$
b $K = 8$

Fig. 6 shows the MSE with six SS's after ten cycles. The CRLB is almost reached at different SNR's.

In Fig. 7, the MSE using this method is compared with that of the modified SAGE (MSAGE) method in [7]. The efficient AP method shows a similar convergence rate as the MSAGE method when $K = 4$ and $K = 8$. In addition, the efficient AP method approaches closer to the CRLB in both cases. The MSE of the efficient AP method at SNR = 10 dB almost reaches the CRLB, and that at SNR = 20 dB is also very close to the CRLB.

5 Conclusion

An efficient AP method has been developed to estimate the CFO's of an arbitrary number of SSs. A demonstration with four SSs shows that this approach reduces the computational load of conventional AP method by about 8000 folds. Simulations confirm that the MSE of CFO with this method converges in the same manner as the conventional AP method, and also reaches the CRLB.

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8 Appendix

The induction process for derivation of isomorphism between $\bar{\bar{\boldsymbol{B}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{B}}}$ and its inverse.

First, consider a matrix with a single entry $\bar{\bar{\boldsymbol{b}}}_{1,1}$

$$c_{1,1}\bar{\bar{\boldsymbol{b}}}_{1,1}=\frac{c_{1,1}}{D}\bar{\bar{\boldsymbol{I}}}_{N_g},\ \left(c_{1,1}\bar{\bar{\boldsymbol{b}}}_{1,1}\right)^{-1}=\frac{D}{c_{1,1}}\bar{\bar{\boldsymbol{I}}}_{N_g}=d_{1,1}\bar{\bar{\boldsymbol{b}}}_{1,1}$$

where $\bar{\bar{\boldsymbol{b}}}_{1,1}=\bar{\bar{\boldsymbol{I}}}_{N_g}/D$ and $d_{1,1}=D^2/c_{1,1}$. Thus, $c_{1,1}\bar{\bar{\boldsymbol{b}}}_{1,1}$ and its inverse are isomorphic.

Next, assume the $\bar{\bar{\boldsymbol{B}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{B}}}$ matrix with $\boldsymbol{M}\times\boldsymbol{M}$ entries is isomorphic to its inverse. With the size increased to $(M+1)\times(M+1)$, the matrix $\bar{\bar{\boldsymbol{B}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{B}}}$ can be expressed as

$$\bar{\bar{\boldsymbol{B}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{B}}}=\begin{bmatrix}\bar{\bar{\boldsymbol{P}}}_{11} & \bar{\bar{\boldsymbol{P}}}_{12}\\ \bar{\bar{\boldsymbol{P}}}_{21} & \bar{\bar{\boldsymbol{P}}}_{22}\end{bmatrix} \tag{44}$$

where

$$\bar{\bar{\boldsymbol{P}}}_{11}=\begin{bmatrix}c_{1,1}\bar{\bar{\boldsymbol{b}}}_{1,1} & \cdots & c_{1,M}\bar{\bar{\boldsymbol{b}}}_{1,M}\\ \vdots & \ddots & \vdots\\ c_{M,1}\bar{\bar{\boldsymbol{b}}}_{M,1} & \cdots & c_{M,M}\bar{\bar{\boldsymbol{b}}}_{M,M}\end{bmatrix},$$

$$\bar{\bar{\boldsymbol{P}}}_{12}=\begin{bmatrix}c_{1,(M+1)}\bar{\bar{\boldsymbol{b}}}_{1,M+1}\\ \vdots\\ c_{M,M+1}\bar{\bar{\boldsymbol{b}}}_{M,M+1}\end{bmatrix}$$

$$\bar{\bar{\boldsymbol{P}}}_{21}=\begin{bmatrix}c_{M+1,1}\bar{\bar{\boldsymbol{b}}}_{M+1,1} & \cdots & c_{M+1,M}\bar{\bar{\boldsymbol{b}}}_{M+1,M}\end{bmatrix},$$

$$\bar{\bar{\boldsymbol{P}}}_{22}=c_{M+1,M+1}\bar{\bar{\boldsymbol{b}}}_{M+1,M+1}$$

Its inverse can be expressed as

$$\left(\bar{\bar{\boldsymbol{B}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{B}}}\right)^{-1}=\begin{bmatrix}\bar{\bar{Q}}_{11} & \bar{\bar{Q}}_{12}\\ \bar{\bar{Q}}_{21} & \bar{\bar{Q}}_{22}\end{bmatrix} \tag{45}$$

Comparing (44) and (45), we have

$$\bar{\bar{\boldsymbol{Q}}}_{11}=\bar{\bar{\boldsymbol{R}}}_{11}^{-1} \tag{46}$$

with

$$\bar{\bar{\boldsymbol{R}}}_{11}=\bar{\bar{\boldsymbol{P}}}_{11}-\bar{\bar{\boldsymbol{P}}}_{12}\cdot\bar{\bar{\boldsymbol{P}}}_{22}^{-1}\cdot\bar{\bar{\boldsymbol{P}}}_{21},\quad \bar{\bar{\boldsymbol{P}}}_{22}^{-1}=\frac{D}{c_{M+1,M+1}}\bar{\bar{\boldsymbol{I}}}_{N_g}$$

The (i,j)th entry of $\bar{\bar{\boldsymbol{R}}}_{11}$ can be calculated as

$$\begin{aligned}&c_{i,j}\bar{\bar{\boldsymbol{b}}}_{i,j}-\frac{D}{c_{M+1,M+1}}c_{i,M+1}c_{M+1,j}\bar{\bar{\boldsymbol{b}}}_{i,M}\,\bar{\bar{\boldsymbol{b}}}_{M+1,j}\\ &\quad=\left(c_{i,j}-\frac{D}{c_{M+1,M+1}}c_{i,M+1}c_{M+1,j}\gamma_{i,M+1,j}\right)\bar{\bar{\boldsymbol{b}}}_{i,j}\end{aligned} \tag{47}$$

where $\bar{\bar{\boldsymbol{b}}}_{i,M+1}\cdot\bar{\bar{\boldsymbol{b}}}_{M+1,j}=\gamma_{i,M+1,j}\bar{\bar{\boldsymbol{b}}}_{i,j}$. Equations. (46) and (47) imply that $\bar{\bar{\boldsymbol{Q}}}_{11}$ is isomorphic to $\bar{\bar{\boldsymbol{R}}}_{11}$, thus

$$\bar{\bar{\boldsymbol{Q}}}_{11}=\begin{bmatrix}\alpha_{1,1}\bar{\bar{\boldsymbol{b}}}_{1,1} & \cdots & \alpha_{1,M}\bar{\bar{\boldsymbol{b}}}_{1,M}\\ \vdots & \ddots & \vdots\\ \alpha_{M,1}\bar{\bar{\boldsymbol{b}}}_{M,1} & \cdots & \alpha_{M,M}\bar{\bar{\boldsymbol{b}}}_{M,M}\end{bmatrix} \tag{48}$$

Comparison of (44) and (45) also gives

$$\bar{\bar{\boldsymbol{Q}}}_{22}=\bar{\bar{\boldsymbol{R}}}_{22}^{-1} \tag{49}$$

with (see (50))

Next, substitute (50) into (49) to have

$$\bar{\bar{\boldsymbol{Q}}}_{22}=\alpha_{M+1,M+1}\bar{\bar{\boldsymbol{b}}}_{M+1,M+1} \tag{51}$$

where

$$\alpha_{M+1,M+1}=\frac{D^2}{c_{M+1,M+1}-\sum_{j=1}^{M}\sum_{i=1}^{M}c_{M+1,i}d_{i,j}\gamma_{M+1,i,j}c_{j,M+1}\gamma_{M+1,j,M+1}}$$

Similarly

$$\bar{\bar{\boldsymbol{Q}}}_{12}=\bar{\bar{\boldsymbol{P}}}_{11}^{-1}\cdot\bar{\bar{\boldsymbol{P}}}_{12}\cdot\left(-\bar{\bar{\boldsymbol{Q}}}_{22}\right)=\begin{bmatrix}\alpha_{1,M+1}\bar{\bar{\boldsymbol{b}}}_{1,M+1}\\ \vdots\\ \alpha_{M,M+1}\bar{\bar{\boldsymbol{b}}}_{M,M+1}\end{bmatrix} \tag{52}$$

$$\begin{aligned}\bar{\bar{\boldsymbol{Q}}}_{21}&=\left(-\bar{\bar{\boldsymbol{Q}}}_{22}\right)\cdot\bar{\bar{\boldsymbol{P}}}_{21}\cdot\bar{\bar{\boldsymbol{P}}}_{11}^{-1}\\ &=\begin{bmatrix}\alpha_{M+1,1}\bar{\bar{\boldsymbol{b}}}_{M+1,1} & \cdots & \alpha_{M+1,M}\bar{\bar{\boldsymbol{b}}}_{M+1,M}\end{bmatrix}\end{aligned} \tag{53}$$

By observing (48), (51)–(53), the $\bar{\bar{\boldsymbol{B}}}^{\dagger}\cdot\bar{\bar{\boldsymbol{B}}}$ matrix of dimension $(M+1)\times(M+1)$ is also isomorphic to its inverse. Thus, complete the proof by induction.

$$\bar{\bar{\boldsymbol{R}}}_{22}=\bar{\bar{\boldsymbol{P}}}_{22}-\bar{\bar{\boldsymbol{P}}}_{21}\cdot\bar{\bar{\boldsymbol{P}}}_{11}^{-1}\cdot\bar{\bar{\boldsymbol{P}}}_{12}=\left(c_{M+1,M+1}-\sum_{j=1}^{M}\sum_{i=1}^{M}c_{M+1,i}d_{i,j}\gamma_{M+1,i,j}c_{j,M+1}\gamma_{M+1,j,M+1}\right)\bar{\bar{\boldsymbol{b}}}_{M+1,M+1} \tag{50}$$