Siberian Mathematical Journal, Vol. 50, *No.* 4, *pp.* 603–608, 2009
Original Russian Text Copyright © 2009 *Buskin N. V.*

ECONOMICAL SEPARABILITY IN FREE GROUPS

N. V. Buskin UDC 512.543.14

Abstract: Consider the rank n free group F_n with basis X. Bogopol′skiĭ conjectured in [1, Problem 15.35] that each element $w \in F_n$ of length $|w| \geq 2$ with respect to X can be separated by a subgroup $H \leq F_n$ of index at most $C \log |w|$ with some constant C. We prove this conjecture for all w outside the commutant of F_n, as well as the separability by a subgroup of index at most $\frac{|w|}{2} + 2$ in general.

Keywords: separability by a subgroup

§ 1. Introduction

Given some group G, say that $g \in G$ *is separated by a subgroup* $H \leq G$ if $w \notin H$. Consider the rank n free group $F_n = F(x_1, \dots, x_n)$. Bogopol′skiĭ put forth the following

Conjecture [1, Problem 15.35]. *An element $w \in F_n$ of length $|w| \geq 2$ is separated by some subgroup of index at most $C \log |w|$, where the constant C depends only on n.*

Other versions of this problem are known as well. A rather recent article is devoted to the economical separability of *normal* subgroups. The results of [2] imply that an element w of the free group F_n, with $n \geq 2$, is separated by some normal subgroup of index $O(|w|^3)$. Rivin claims [3] that if w lies in $\gamma_k F_n \setminus \gamma_{k+1} F_n$ then w is separated by some normal subgroup of index $O(\log^k |w|)$. Moreover, it is proved in [4] that $k = O(\sqrt{|w|})$.

This article studies the economical separability of arbitrary subgroups of finite index, but the techniques differ from those used in [2, 3]. The estimate obtained here is much weaker than the conjecture; nevertheless, this is a new result constituting the main content of the present article.

Theorem. *Each element $w \in F_n$ with $w \neq 1$ is separated by some subgroup of index $i \leq \frac{|w|}{2} + 2$.*

We prove the theorem in Section 2. In the proof we will assume that the reader is familiar with the description of the subgroups of F_n as the fundamental groups of the marked graphs covering the bouquet of n circles (see [5] for instance).

Let us prove the conjecture under the assumption that $w \notin [F_n, F_n]$. To start off, consider an example with w of odd length. The group F_n includes an index 2 subgroup consisting precisely of all elements $w \in F_n$ of even length which is called the *subgroup of even words*. Therefore, each element of odd length is separated by the subgroup of even words.

Consider the general case. Since $w \notin [F_n, F_n]$, it follows that for one of the generators $a \in \{x_1, \dots, x_n\}$ its total degree $\sigma_a(w)$ in $w = w(x_1, \dots, x_n)$ differs from zero. Define a homomorphism $\varphi : F_n \to \mathbb{Z}$ as follows: $\varphi(a) = 1$ and $\varphi(x_j) = 0$ for every generator $x_j \neq a$.

Define the function $d : \mathbb{Z} \setminus \{0\} \to \mathbb{N}$ by putting $d(t)$ to be equal to the smallest positive integer not dividing t. For instance, $d(1) = 2$, $d(2) = 3$, and $d(2k + 1) = 2$, where k is an arbitrary integer.

Take the canonical homomorphism $\psi : \mathbb{Z} \to \mathbb{Z}_{d(\sigma_a(w))}$. Then the image of w under $\psi \circ \varphi$ is nontrivial. Therefore, w is separated by some normal subgroup $H = \operatorname{Ker} \psi \circ \varphi$ of index $d(\sigma_a(w))$ in F_n. Since $|\sigma_a(w)| \leq |w|$, it suffices to prove that $d(t)$ is dominated by $C \log |t|$, which would follow from the existence of two constants $C_1 > 0$ and $C_2 > 0$ with $d(t) \leq C_1 \log |t| + C_2$ for $t \in \mathbb{Z} \setminus \{0\}$. Indeed, given

The author was supported by the Program "Development of the Scientific Potential of Higher School" of the Russian Federal Agency for Education (Grant 2.1.1.419).

Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 50, No. 4, pp. 765–771, July–August, 2009. Original article submitted April 21, 2009.

this, we can choose a constant C such that $d(\sigma_a(w)) \le C_1 \log|\sigma_a(w)| + C_2 \le C \log|w|$ for every w with $\sigma_a(w) \ne 0$ and $|w| \ge 2$. Here we will need the following results of analysis and elementary number theory.

Lemma 1. *$\left(\frac{k}{e}\right)^k \le k!$ for every $k \in \mathbb{N}$.*

It is not difficult to see that this lemma follows from Stirling's formula.

Lemma 2. *Denote by $\pi(m)$ the number of primes at most m, where $m \ge 2$. There exists a constant $c > 0$ such that $c\frac{m}{\log m} < \pi(m)$.*

The claim of this lemma is one part of Chebyshev's double inequality: $c_1\frac{m}{\log m} < \pi(m) < c_2\frac{m}{\log m}$ (see [6] for instance).

Let us prove the required inequality for $d(t)$. List the primes $p_1, \dots, p_k$ not exceeding $m = d(t) - 1$. Then, since these primes are strictly less than $d(t)$, it follows that $p_1, p_2, \dots, p_k | t$, and so $p_1 p_2 \dots p_k | t$, whence $k! \le p_1 p_2 \dots p_k \le |t|$. This inequality and Lemma 1 imply that $\left(\frac{k}{e}\right)^k \le |t|$. Taking the logarithms, we obtain

$$k(\log k - 1) \le \log|t|. \qquad (*)$$

Lemma 2 implies that $c\frac{m}{\log m} < k$. Replacing k by $c\frac{m}{\log m}$ in $(*)$, whose left-hand side is a strictly increasing function of k, we find that $c\frac{m}{\log m}(\log c + \log m - \log(\log m) - 1) \le \log|t|$. For large m the left-hand side of the last inequality is of order cm; hence, for every $\varepsilon \in (0, c)$ there is $N(\varepsilon) \in \mathbb{N}$ such that $(c - \varepsilon)m \le \log|t|$ for all $m \ge N(\varepsilon)$. Thereby, $(c-\varepsilon)m \le \log|t| + (c - \varepsilon)N(\varepsilon)$ for every m, which implies the required inequality for $d(t)$.

§ 2. Proof of the Theorem

We will prove our theorem in the case $n = 2$, writing $F_2 = F(a, b)$. All arguments translate easily to the general case. Take a bouquet $B(a, b)$ of two circles marked with the letters a and b. We will work in the category of graphs marked with a and b, having a vertex designated as a basepoint and defining a finite-sheeted covering of $B(a, b)$.

We can assume first that w belongs to the subgroup of even words (otherwise, for w the claim of the theorem is obvious), and consequently it is of even length: $|w| = 2(i - 1)$ for suitable i. Let us prove that there exists a subgroup of index at most $i + 1$ not containing w or, which is equivalent, there exists a marked graph Γ defining a covering of $B(a, b)$ with at most $i + 1$ sheets such that the path marked with w beginning at the basepoint of Γ is not closed. Furthermore, we sometimes refer to an arbitrary marked graph which defines a finite-sheeted covering of $B(a, b)$ as a *covering graph*.

As usual, denote by $v(\Gamma)$ and $e(\Gamma)$ the vertex and edge sets of a graph Γ. Given a marked graph, refer as a *c-edge* to every edge marked with either c or c^{-1}. Given an edge e, denote by $\bar{e}$ the same edge with the opposite orientation; and, similarly, given a path γ, define the path $\bar{\gamma}$. For an arbitrary loop e (an edge whose beginning and end coincide) naturally put $e^k = e \dots e$ (with k factors) for $k > 0$ and $e^k = \bar{e} \dots \bar{e}$ (with $|k|$ factors) for $k < 0$. Denote a path marked with w by γ_w independently of the ambient graph (for marked graphs corresponding to coverings, the path beginning at a fixed vertex is uniquely determined by its label). Agree to call a *direct* edge an edge that is not a loop.

Let us sketch how we seek a graph defining a covering of $B(a, b)$ with at most $i + 1$ sheets such that the path marked with w beginning at the basepoint is not closed. Start with introducing the necessary operations I and II which enable us to obtain the new covering graphs from the old.

Take a covering graph Γ and a c-edge $e \in e(\Gamma)$ for some $c \in \{a, b\}$.

Operation I. Introduce a new vertex v by subdividing e into two c-edges and attaching at v a d-loop, where $d \in \{a, b\} \setminus \{c\}$ (Fig. 1).

Operation II$_k$. Given a c-loop $e \in e(\Gamma)$ with both endpoints at v, remove the loop and attach to v a cycle C (identifying v and some vertex of C) consisting of k c-edges. To every vertex of C but v attach a d-loop, with $d \in \{a, b\} \setminus \{c\}$, so that the resulting graph will also define a covering (Fig. 2).

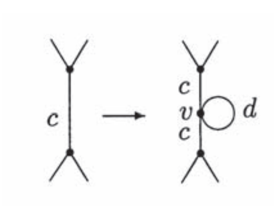

Fig. 1

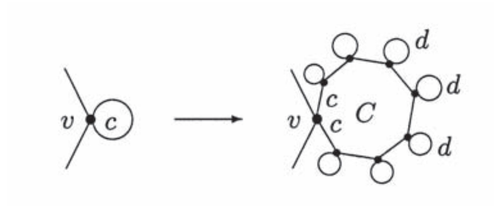

Fig. 2

We can say that operation II_k amounts to the k-fold application of operation I.

The search begins with some initial graph Γ_1 defining a covering with at most i sheets. If Γ_1 is unsuitable (γ_w is closed in Γ_1) then we consider the graphs obtained from Γ_1 using operation I. Applied to an arbitrary edge of Γ_1, operation I increases the number of its vertices by 1. If all graphs obtained using operation I are unsuitable (γ_w is closed in all these graphs, which is important) then using operation II_s for some $s > 1$, we obtain from Γ_1 either a covering graph Γ' with at most $i+1$ vertices such that the path γ_w is not closed in Γ' or a covering graph $\Gamma_2 \neq \Gamma_1$ with at most i vertices.

If the second possibility is realized then we consider the covering graphs resulting from Γ_2 by using operation I: if among them there is no suitable graph then, as at the previous step, using operation II we obtain from Γ_2 either the required graph Γ' or a graph $\Gamma_3 \notin \{\Gamma_1, \Gamma_2\}$ with at most i vertices. Step by step we construct a sequence of covering graphs $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ with $|v(\Gamma_j)| \leq i$ for $j = 1, \dots, k$. It is clear that this search is finite, and for some k either Γ_k will be the required graph or the required graph is contained among the graphs with at most $i+1$ vertices resulting from Γ_k by using operations I and II.

Proceed to a more detailed description of our algorithm. Suppose that the reduced expression for w begins with the letter a; i.e., $w = a^t b^s \dots$ with $t \neq 0$. Put $h(t) = |t| + 1$ if $|t| \leq 2$, and $h(t) = \left[\frac{|t|}{2}\right] + 1$ otherwise. It is clear that $h(t)$ does not divide t.

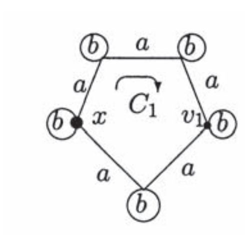

Fig. 3

The initial graph. Consider the graph Γ_1 all of whose a-edges constitute a cycle of length $h(t)$, while the b-edges are loops at the vertices of this cycle (Fig. 3 shows the example for $|t| = 8$) and x is the basepoint.

An arrow shows the direction in which γ_w traverses the cycle C_1 formed by the a-edges of Γ_1. The orientation of the edges is not fixed; therefore, Fig. 3 shows both cases $t = \pm 8$. A vertex v_1 of Γ_1 is the end of the path γ_{a^t} beginning at x (the first syllable of w). Since $h(t)$ does not divide t, it follows that $v_1 \neq x$. The graph Γ_1 defines a covering of the bouquet $B(a,b)$ with $h(t)$ sheets, and therefore it corresponds to some subgroup of index $h(t)$. If w is of syllabic length 1, i.e., $w = a^t$, then $h(t) \leq \frac{|w|}{2} + 2$ and Γ_1 is the required graph.

Assume now that the syllabic length of w is at least 2. In this case we also have an inequality $h(t) \leq \frac{|w|}{2} + 2 = i + 1$. Actually, in the case of syllabic length 2 the path γ_w is also not closed, and the case of syllabic length 3 reduces by conjugation (or an application of the construction we described to the unique b-syllable b^s) to the case of syllabic length 2 or 1. Thus, the truly interesting case is that of syllabic length at least 4.

If the path γ_w is not closed in Γ_1, there is nothing left to prove.

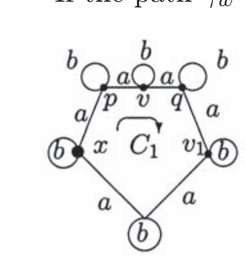

Fig. 4

Suppose that γ_w is closed. Assume that for some edge e_a marked with a the total number of occurrences of the edges e_a and $\bar{e}_a$ in γ_w is equal to 1 (i.e., exactly one of the two oriented edges e_a and $\bar{e}_a$ occurs in γ_w, and exactly once). Suppose for definiteness that e_a occurs in γ_w. Denote the beginning and end of e_a by p and q. Then $\gamma_w = \gamma_1 e_a \gamma_2$, where the subpaths γ_1 and γ_2 contain no occurrences of e_a and $\bar{e}_a$. Apply operation I to e_a and denote the new vertex by v (Fig. 4). It is clear that the new graph Γ' corresponds to a subgroup of index at most $i+1$. The equality $\gamma_w = \gamma_1 e_a \gamma_2$ yields the reduced representation $w = w_1 a w_2$ for w, where the subwords w_1 and w_2 mark the subpaths γ_1 and γ_2 respectively. In Γ'

the ends of the paths marked with w_1a and w_2^{-1} beginning at x are distinct (these are the vertices v and q). Hence, γ_w cannot be a closed path in this graph. Therefore, we have found a subgroup of index at most $i+1$ not containing w.

In exactly the same fashion we can separate w by a subgroup of index at most $i+1$ provided that the total number of occurrences of some b-loop in γ_w is equal to 1. If γ_w is a closed path in Γ_1 and every edge of Γ_1 appears in γ_w at least twice, disregarding the orientation, then we proceed to the next step of the algorithm.

Step of the algorithm. Assume that for some $k \geq 1$ we have already constructed a graph Γ_k with marked vertices $v_1, \dots, v_k$ such that $|v(\Gamma_k)| \leq i$, which is of the form depicted in Fig. 5.

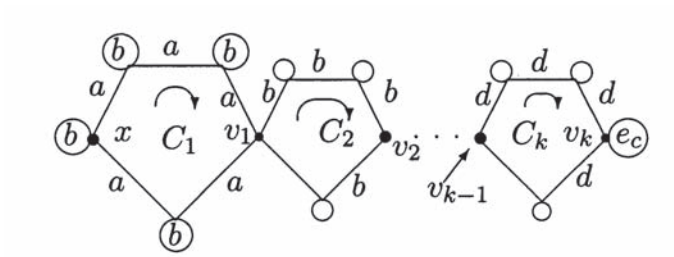

Fig. 5

The vertex v_j, where $j = 1, \dots, k$, is the end of the subpath α_j of γ_w corresponding to the first j syllables of w. The index c in the symbol e_c is one of the letters a and b, while $d \in \{a, b\} \setminus \{c\}$. For $k = 1$ there is a unique cycle C_1, and we have the graph Γ_1. A cycle C_j in this graph, where $1 \leq j \leq k$, corresponds to the jth syllable $c^{\pm s_j}$ of w, with $c \in \{a, b\}$: it consists of $h(s_j)$ c-edges, where s_j is the length of the jth syllable of w. For simplicity we illustrate a concrete situation, when the first k syllables are of length 8.

Some remark is in order: If l_j, where $1 \leq j \leq k$, is the number of distinct pairs of direct edges $\{e, \bar{e}\}$ in the cycle C_j of Γ_k such that e or $\bar{e}$ appears in the subpath β_j of γ_w corresponding to the jth syllable of w and beginning at the vertex v_{j-1} (with $v_0 = x$) then

(A) $|C_j| = h(s_j) \leq l_j + 1$.

These inequalities are easy to verify (it suffices to verify the inequality for $j = 1$, and the others follow similarly).

If the path γ_w is not closed in Γ_k then we have found the required graph. If γ_w is closed then since the vertex v_k is the end of the path α_k corresponding to the first k syllables of w, it follows that the loop e_c must appear in γ_w. Furthermore, if an edge e of Γ_k is such that the total number of occurrences of e and $\bar{e}$ in γ_w is equal to 1 (i.e., only one of the edges e and $\bar{e}$ appears in γ_w, and exactly once) then operation I applied to this edge produces a graph with at most $i+1$ vertices in which γ_w is not closed.

Suppose now that the path γ_w is closed in every graph obtained from Γ_k by operation I. In particular, the following condition holds:

(B) *For every edge e of the graph Γ_k appearing in γ_w the total number of occurrences of the edges e and $\bar{e}$ in γ_w is at least* 2.

Incidentally, this implies that $2k \leq 2(l_1 + \dots + l_k) \leq |\gamma_w| = |w|$. As we mentioned in the sketch of the algorithm starting the proof, the purpose of operation II is to construct either Γ', which will be the required graph, or Γ_{k+1}. In order to understand which of these possibilities is realized, it is important to know the number and location of the occurrences of the loops e_c and $\bar{e}_c$ in the path γ_w in Γ_k.

Select the first occurrence of the loop e_c in γ_w. There are several cases here (the total number of occurrences of e_c and $\bar{e}_c$ is certainly at least 2):

(1a) $\gamma_w = \gamma_1 e_c^{\pm 2} \gamma_2$ and the subpaths γ_1 and γ_2 avoid the loop e_c and its inverse;

(1b) $\gamma_w = \gamma_1 e_c^s \gamma_2, s \in \mathbb{Z}, |s| > 2$, and the conditions on γ_1 and γ_2 are the same as in case 1a;

(2a) $\gamma_w = \gamma_1 e_c^{\pm 1}\gamma_2$, the subpath γ_1 avoids the loop e_c and its inverse, while in the subpath γ_2 the total number of occurrences of these edges is at least 1; moreover, γ_2 does not begin with $e_c, \bar{e}_c$;

(2b) $\gamma_w = \gamma_1 e_c^{\pm 2}\gamma_2$ and the conditions on γ_1 and γ_2 are the same as in case 2a;

(2c) $\gamma_w = \gamma_1 e_c^s \gamma_2, |s| > 2$, and the conditions on γ_1 and γ_2 are the same as in case 2a.

Let us show that in the cases 1a and 1b of the first group we can construct Γ', while in the cases 2a, 2b, and 2c of the second group we construct Γ_{k+1}.

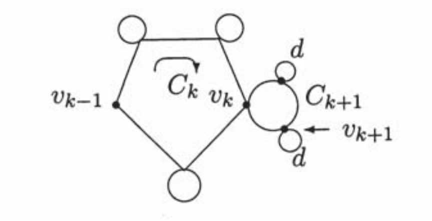

Fig. 6

Consider case (1a). Take $\gamma_w = \gamma_1 e_c e_c \gamma_2$ for definiteness. In this case condition (B) and the definition of l_j imply that $2l_1 + \cdots + 2l_k + 2 \leq |\gamma_w| = |w| = 2(i-1)$; the term 2 arises as the contribution of the subpath $e_c e_c$ to the length of γ_w. Hence, $l_1 + \cdots + l_k + 1 \leq i - 1$. Apply to the loop e_c operation $\mathrm{II}_{h(2)}$, with $h(2) = 3$ (Fig. 6). Denote the new cycle by C_{k+1}. The number of vertices of the resulting graph is equal to $|C_1| + (|C_2| - 1) + \cdots + (|C_k| - 1) + (|C_{k+1}| - 1) \leq_{(A)} (l_1 + 1) + l_2 + \cdots + l_k + 2 \leq (i-1) + 2 = i + 1$. For the word w we have $w = w_1 c^2 w_2$, where the subwords w_1 and w_2 are the labels of the subpaths γ_1 and γ_2 in Γ, and in the resulting graph the ends of the paths with the labels $w_1 c^2$ and w_2^{-1} do not coincide (these are vertices v_{k+1} and v_k). Hence, γ_w will not be a closed path in the so-constructed graph. Therefore, we have found a subgroup of index $i+1$ not containing w.

Consider case (1b). For convenience assume that $s > 0$. As above, condition (B) yields $2l_1 + \cdots + 2l_k + s \leq |w| = 2(i-1)$, whence $l_1 + \cdots + l_k + \left[\frac{s}{2}\right] \leq i - 1$. Apply to the loop e_c operation $\mathrm{II}_{h(s)}$ with $h(s) = \left[\frac{s}{2}\right] + 1$. Denote the new cycle by C_{k+1}. The number of vertices in the resulting graph is equal to

$$|C_1| + (|C_2| - 1) + \cdots + (|C_k| - 1) + (|C_{k+1}| - 1) \leq (l_1 + 1) + l_2 + \cdots + l_k + \left[\frac{s}{2}\right] \leq i.$$

As above, the equality $\gamma_w = \gamma_1 e_c^s \gamma_2$ yields the reduced expression $w = w_1 c^s w_2$. Since $h(s)$ does not divide s, in the new graph the paths with the labels $w_1 c^s$ and w_2^{-1} beginning at x have distinct ends v_{k+1} and v_k; consequently, the path γ_w beginning at x will not be closed. Thus, there is a subgroup of index at most i not containing w.

Let us now address the second group of cases.

Case (2a). Since the total number of occurrences of e_c and $\bar{e}_c$ in γ_w is at least 2, it follows that $2l_1 + \cdots + 2l_k + 2 \leq |w| = 2(i-1)$. Apply to the loop e_c operation $\mathrm{II}_{h(1)}$ with $h(1) = 2$. The end of the subpath α_{k+1} of γ_w (the label of this subpath is equal to $w_1 c^{\pm 1}$) is a vertex of a new cycle C_{k+1} distinct from v_k. Denote it by v_{k+1} (Fig. 7).

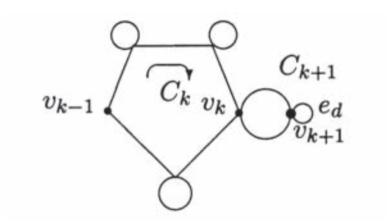

Fig. 7

The number of vertices of the constructed graph Γ is equal to $|C_1| + (|C_2| - 1) + \cdots + (|C_k| - 1) + (|C_{k+1}| - 1) \leq (l_1 + 1) + l_2 + \cdots + l_k + 1 \leq i$. Put $\Gamma_{k+1} = \Gamma$.

Consider case (2b). Since in this case the loop e_c appears in γ_w at least three times, we have an obvious inequality $2l_1 + \cdots + 2l_k + 3 \leq 2(i-1)$. Since the left-hand side here is an odd number, we actually have an even stronger inequality $2l_1 + \cdots + 2l_k + 4 \leq 2(i-1)$. Apply to the loop e_c operation $\mathrm{II}_{h(2)}$ with $h(2) = 3$, as in case (1a). Denote the new cycle by C_{k+1}. The number of vertices of the resulting graph is equal to $|C_1| + (|C_2| - 1) + \cdots + (|C_{k+1}| - 1) \leq (l_1 + 1) + l_2 + \cdots + l_k + 2 \leq i$. Denote this graph by Γ_{k+1}.

Consider case (2c). Apply to the loop e_c operation $\mathrm{II}_{h(s)}$, as in case (1b). Denote the new cycle by C_{k+1}. The number of vertices of the resulting graph is at most i. Therefore, in this case as well we can construct the graph Γ_{k+1}.

The proof is complete.

References

1. *Kourovka Notebook:* Unsolved Problems of Group Theory [in Russian], 15th edit., Inst. Mat. (Novosibirsk), Novosibirsk (2002).
2. *Khalid Bou-Rabee,* "Quantifying residual finiteness," Technical Report arXiv:0807.0862v2 [math.GR], arxiv.org, 2008.
3. *Rivin I.,* "Geodesics with one self-intersection, and other stories," Technical Report arXiv:0901.2543v3 [math.GT], arxiv.org, 2009.
4. *Malestein J. and Putman A.,* "On the self-intersections of curves deep in the lower central series of a surface group," arXiv, math.GT, January 2009; 12 pages and 2 figures.
5. *Bogopol′skiĭ O. V.*, Introduction to the Theory of Groups [in Russian], Institute of Computer Science, Moscow; Izhevsk (2002).
6. *Ireland K. and Rosen M.*, A Classical Introduction to Modern Number Theory, Springer-Verlag, New York etc. (1990).

N. V. Buskin
Novosibirsk State University, Novosibirsk, Russia
E-mail address: buskin1983@mail.ru