Communications in Statistics—Theory and Methods, 41: 1813–1832, 2012

ISSN: 0361-0926 print/1532-415X online
DOI: 10.1080/03610926.2010.551451

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On-Line Process Control using Attributes with Misclassification Errors: An Economical Design for Short-Run Production

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On-line process control consists of inspecting a single item for every m (integer and $m \geq 2$) produced items. Based on the results of the inspection, it is decided whether the process is in-control (the fraction of conforming items is p_1; State I) or out-of-control (the fraction of conforming items is $p_2 < p_1$; State II). If the inspected item is non conforming, it is determined that the process is out-of-control, and the production process is stopped for an adjustment; otherwise, production continues. As most designs of on-line process control assume a long-run production, this study can be viewed as an extension because it is concerned with short-run production and the decision regarding the process is subject to misclassification errors. The probabilistic model of the control system employs properties of an ergodic Markov chain to obtain the expression of the average cost of the system per unit produced, which can be minimised as a function of the sampling interval, m. The procedure is illustrated by a numerical example.

Keywords Economic design; Markov chain; Misclassification errors; On-line process control using attributes; Short-run production.

Mathematics Subject Classification 62P30; 60J05; 60J20; 60J10.

1. Introduction

Taguchi et al. (1989) proposed an economical procedure to monitor on-line process control with attributes as well as variables. The inspection system is automatic and allows for sampling of only a single item at a time. In general, the proposed system can be implemented in high-speed electronics manufacturing facilities, where the testing equipment is connected to a central computer, and data are taken automatically as produced items are being tested.

Received February 17, 2010; Accepted December 23, 2010
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The procedure of on-line control used to monitor a process has been studied by many authors, such as Nayebpour and Woodall (1993), Gong and Tang (1997), Borges et al. (2001), Wang and Yue (2001), Dasgupta (2003), Trindade et al. (2007), Dasgupta and Mandal (2008), and Quinino et al. (2010).

In the case of process control using attributes, the process starts as in-control [fraction of conformance equal to p_1 (State I)]. Following the occurrence of a special event (the process is out-of-control), the fraction of conformance shifts to p_2 (State II), $0 \leq p_2 < p_1$. The production continues under this condition until an adjustment is made. After an adjustment, the process restarts at its initial condition (fraction of conformance equal to p_1). The procedure consists of inspecting a single item for every m items produced. The problem involves finding the optimum sampling interval, m that minimizes the average cost of the system of control.

Taguchi et al. (1989), via simplifications and approximations, obtained a closed expression for the optimum sampling interval (m), but no explicit probabilistic mechanism of the occurrence of the special cause was assumed.

Nayebpour and Woodall (1993) developed a model for on-line process control by attributes with an explicit failure mechanism. They assumed that time unit a shift from State I to State II follows a geometric distribution with $\mathrm{P(shift)} = \pi$. This approach is more realistic, but due to probabilistic assumptions, a closed expression for the optimum sampling interval (m^0) was not available and direct search procedures were used to determine it.

About imperfect inspections, many authors have contributed on this subject. Wang and Sheu (2003) determined the optimal production–maintenance policy with inspection errors. Quinino et al. (2010) suggested the repetitive classifications to minimize the impact of the inspection errors. Ferrell and Chhoker (2002) studied the design of economically optimal acceptance sampling plans with inspection error. Borges et al. (2001) studied the impact of the misclassification errors ((α), the probability of classifying a conforming item as non-conforming, and (β), the probability of classifying a non-conforming item as conforming) in Taguchi's model. It was concluded that the cost of the optimum policy is sensitive to the misclassification errors (even small ones around of 1%), compromising the determination of the optimum sampling interval.

Hillier (1969) stated the need for short-run control charts, mainly in new processes, in the start-up of a process that restarts in terms of statistical control, and for processes in which the volume of production is not large enough for the usage of conventional control systems. Castillo et al. (1996) classified short-run production into two types: repetitive, meaning that the production is repetitive and many small lots of similar parts are produced by the same machine with non-significant changes in the equipment set-up, and non repetitive, when completely different set-ups are required in different equipment to produce different lots. The first type can be found in adjusted processes of just-in-time production and the second in custom-made job shop manufacturing. Earlier articles about short-run production include works by Hillier (1964, 1967, 1969), Yang and Hillier (1970), and Quesenberry (1991a,b,c).

Concerning on-line process control approaches, most articles (Taguchi et al., 1989; Nayebpour and Woodall, 1993; Srivastava and Wu, 1991, 1995; Trindade et al., 2007; Ho et al., 2007; Quinino et al., 2010) present economical designs for long-run production. In contrast, Ho and Trindade (2009) proposed an on-line economical X (variable) control chart model for short-run production. These authors obtained the average cost per manufactured item as a summation over the costs incurred across all inspection cycles made.

In this study, an on-line control system determined by attributes with misclassification errors for short-run production is proposed because earlier papers focus on long-run production. In addition, the analytical model developed in this article is different from Ho and Trindade's 2009 model. The average cost per manufactured item is obtained using Markov chains in a manner similar to that used for long-run production. That is, the transition probabilities of each state of the chain in a short-run process are calculated and used as weights to obtain the average cost of each state. Moreover, our proposed model allows us to show the convergence of short-run production to long-run production as the volume of the production increases.

This article is organized as follows. The properties of a Markov chain used to develop the probabilistic model are presented in Sec. 2, and the average cost of the proposed system is presented in Sec. 3. Numerical examples followed by a discussion about the impact of the parameters and different costs on the optimum design are given in Sec. 4. Section 5 closes with conclusions and suggestions for future work.

2. Probabilistic Model

The production of a lot of τ items starts with the process being in-control (State I). The duration of the process under this condition is usually modelled by the exponential distribution for a continuous case. The geometric distribution behaves similarly to the exponential distribution, but it is typically used for the discrete case in which the duration is measured by the number of units produced before the shift. Following previous articles (Nayebpour and Woodall, 1993; Nandi and Sreehari, 1997; Jiang and Tsui, 2000; Borges et al., 2001; Ho et al., 2007; Trindade et al., 2007; Dasgupta and Mandal, 2008; Ding and Gong, 2008), this study relies on a geometric distribution with parameter π, $0 < \pi < 1$, to describe the random time shift from State I to State II (out-of-control).

After a shift, the process is out-of-control (fraction of conformance equal to p_2, $p_2 < p_1$) and returns to its initial value, p_1, only after an adjustment. For the process monitoring, a single item is inspected for each m items produced. The inspected item is submitted to a single classification and then discarded. In high-speed electronics manufacturing which is area of research that motivates this article, the inspected item is submitted to a set of tests, some of which are destructive. The case in which the inspection is not free of classification errors is considered. Let α be the probability of classifying a conforming item as non conforming and β the probability of classifying a non conforming item as conforming. If the inspected item is considered to be conforming, production continues; otherwise, it is stopped for adjustment. In the production of a lot of size τ, n inspections are made to monitor the process, $n = \left\lfloor \frac{\tau}{m-1} \right\rfloor$, with $\lfloor x \rfloor = \max\{n \in Z; n < x\}$. After n inspections, an additional residual quantity of $m_{res} = \tau - n(m-1)$ items is produced to totalize τ items to the customer, and then the production of the lot is stopped. A flowchart of the process control and decision rule is illustrated in Fig. 1.

Let Z be a random variable that indicates the state of the process when an item was produced [$Z = 1$, the process is in-control (State I); $Z = 0$, the process is out-of-control (State II)], Y is a random variable related to the result of the inspected item ($Y = 1$, it is classified as conforming; $Y = 0$, classified as non conforming), and X is a non observable random variable related to the real condition of the inspected item ($X = 1$, the item is truly conforming; $X = 0$, the item

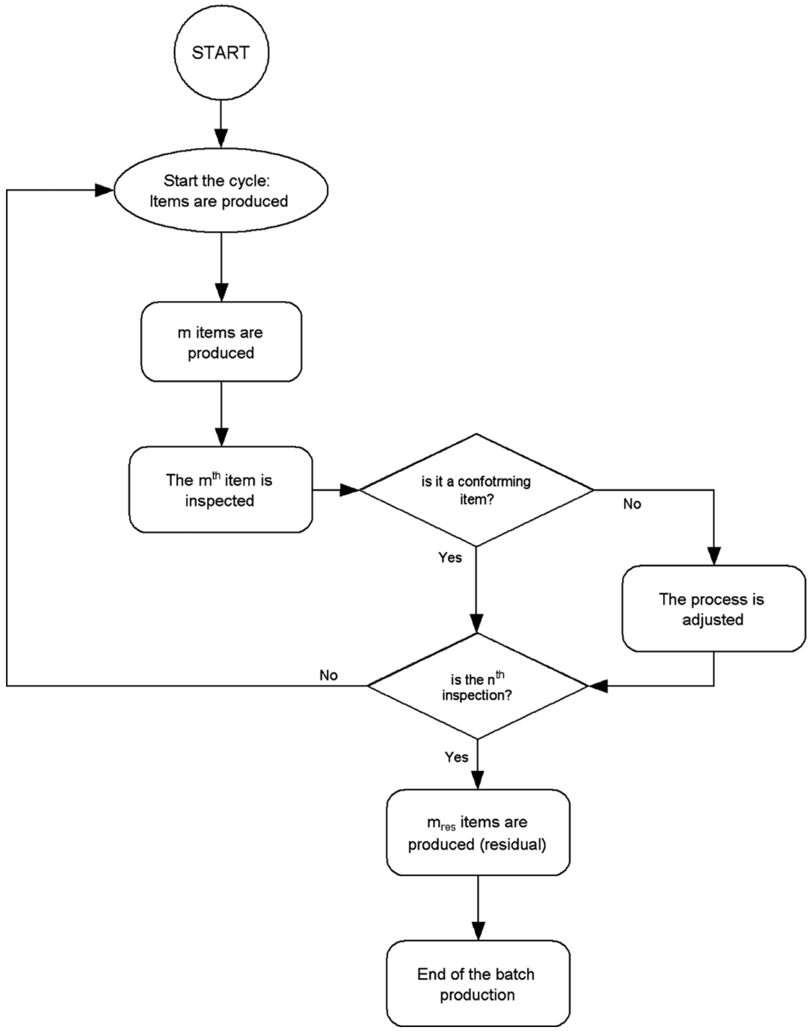

Figure 1. Flowchart of the process (intervention and control).

is truly non conforming). In this way, the conforming classification probabilities of an inspected item conditioned on the real state of the process are, respectively, p_A and p_D, when the process is in-control and out-of-control:

$$p_A = \sum_{j=0}^{1} P\{Y = 1 \cap X = j \,|\, Z = 1\} = p_1 (1 - \alpha) + (1 - p_1) \beta, \qquad (2.1)$$

$$p_D = \sum_{j=0}^{1} P\{Y = 1 \cap X = j \,|\, Z = 0\} = p_2 (1 - \alpha) + (1 - p_2) \beta. \qquad (2.2)$$

The production process, the monitoring process, and the decision rule for the adjustment of the process, can be modelled as an ergodic Markov chain in discrete time, where each step of the chain represents a cycle of inspection. The set of the states is denoted by $E = \{(w, s);\ w = 0, 1, 2 \text{ and } s = 0, 1\}$, where w indicates the condition of the process. For $w = 0$, the process is in-control until the end of the current cycle (no shift in the fraction of the conformance); $w = 1$, the process is out-of-control but the shift occurs in the current cycle; $w = 2$, the process is out-of-control but the shift occurred in earlier cycles; $s = 1$, a non adjustment decision is made; and $s = 0$, the production is stopped for adjustment. Figure 2 relates the production process to the Markov chain states.

The transition probability matrix is given by $\mathbf{P}$ in (2.3), and the transition probabilities of the states are denoted by $P_{\left(w_{i-1},s_{i-1}\right),\left(w_i,s_i\right)}$, where (w_i, s_i) is the current state at the ith monitored cycle.

$$\mathbf{P} = \begin{bmatrix} P_{(0,0),(0,0)} & P_{(0,0),(0,1)} & P_{(0,0),(1,0)} & P_{(0,0),(1,1)} & P_{(0,0),(2,0)} & P_{(0,0),(2,1)} \\ P_{(0,1),(0,0)} & P_{(0,1),(0,1)} & P_{(0,1),(1,0)} & P_{(0,1),(1,1)} & P_{(0,1),(2,0)} & P_{(0,1),(2,1)} \\ P_{(1,0),(0,0)} & P_{(1,0),(0,1)} & P_{(1,0),(1,0)} & P_{(1,0),(1,1)} & P_{(1,0),(2,0)} & P_{(1,0),(2,1)} \\ P_{(1,1),(0,0)} & P_{(1,1),(0,1)} & P_{(1,1),(1,0)} & P_{(1,1),(1,1)} & P_{(1,1),(2,0)} & P_{(1,1),(2,1)} \\ P_{(2,0),(0,0)} & P_{(2,0),(0,1)} & P_{(2,0),(1,0)} & P_{(2,0),(1,1)} & P_{(2,0),(2,0)} & P_{(2,0),(2,1)} \\ P_{(2,1),(0,0)} & P_{(2,1),(0,1)} & P_{(2,1),(1,0)} & P_{(2,1),(1,1)} & P_{(2,1),(2,0)} & P_{(2,1),(2,1)} \end{bmatrix} \tag{2.3}$$

After an adjustment, the production process returns to its initial condition (in-control). Thus, it only transitions to states indexed by $w = 0$ or $w = 1$. The probability that the process stays in-control in the current cycle is given by:

$$P\{Z_1 = Z_2 = \cdots = Z_m = 1 | Z_0 = 1\} = (1 - \pi)^m,$$

where Z_0 represents the state of the process at time zero. Employing expression (2.1), the transition probabilities of the states $(w, 0)$; $w = 0, 1, 2$ to states $(0, 0)$ and $(0, 1)$

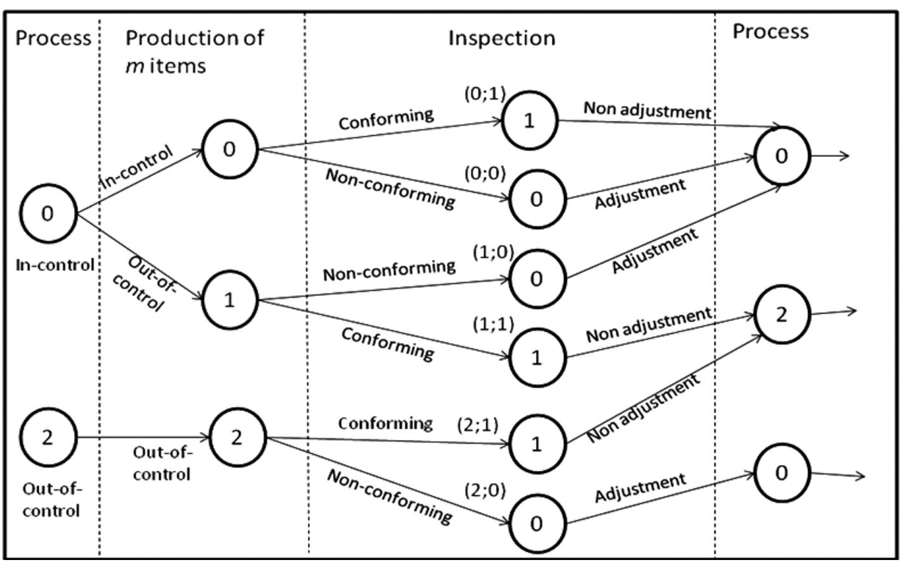

Figure 2. Diagram of the monitoring process.

are, respectively:

$$P_{(w,0),(0,0)} = (1-\pi)^m (1-p_A),$$

and

$$P_{(w,0),(0,1)} = (1-\pi)^m p_A.$$

After an adjustment, the probability of a shift in the fraction of conformance in the current cycle is given by $1-(1-\pi)^m$. Employing expression (2.2), the transition probabilities of the states $(w, 0)$; $w = 0, 1, 2$ to states (1, 0) and (1, 1) are, respectively:

$$P_{(w,0),(1,0)} = [1-(1-\pi)^m](1-p_D),$$

and:

$$P_{(w,0),(1,1)} = [1-(1-\pi)^m]\, p_D.$$

Moreover, transitions from the states $(w, 0)$; $w = 0, 1, 2$ to states $(w = 2, s)$, $s = 0, 1$ are not possible. Hence:

$$P_{(w,0),(2,s)} = 0, \quad s = 0, 1, \quad w = 0, 1, 2.$$

As detection of a non conforming item leads to an adjustment, the same results can be applied to transitions starting from the state (0,1) through the next equality:

$$P_{(0,1),(w,s)} = P_{(0,0),(w,s)}, \quad \forall (w, s) \in E.$$

To reach the states (2, 0) and (2, 1), the inspected item must be classified as conforming (no adjustment) and the process as out-of-control ($w > 0$) in the previous cycle. Thus, the following equalities hold (for $w = 1$ and $w = 2$):

$$P_{(w,1),(2,0)} = 1 - p_D,$$

and:

$$P_{(w,1),(2,1)} = p_D.$$

Consequently, the following transition probabilities are null:

$$P_{(1,1),(w,s)} = P_{(2,1),(w,s)} = 0, \quad w = 0, 1; \quad s = 0, 1.$$

With introduction of the above probabilities, matrix **P** may be written as:

$$\mathbf{P} = \begin{bmatrix} P_{(0,0),(0,0)} & P_{(0,0),(0,1)} & P_{(0,0),(1,0)} & P_{(0,0),(1,1)} & 0 & 0 \\ P_{(0,1),(0,0)} & P_{(0,1),(0,1)} & P_{(0,1),(1,0)} & P_{(0,1),(1,1)} & 0 & 0 \\ P_{(0,0),(0,0)} & P_{(0,0),(0,1)} & P_{(0,0),(1,0)} & P_{(0,0),(1,1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{(1,1),(2,0)} & P_{(1,1),(2,1)} \\ P_{(0,0),(0,0)} & P_{(0,0),(0,1)} & P_{(0,0),(1,0)} & P_{(0,0),(1,1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{(1,1),(2,0)} & P_{(1,1),(2,1)} \end{bmatrix}.$$

The matrix **P** is irreducible and aperiodic, and its stationary distribution can be obtained and denoted by the vector:

$$\pi' = [\pi(w, s), w = 0, 1, 2; \quad s = 0, 1]. \tag{2.4}$$

The probability $\pi(w, s)$, $w = 0, 1, 2$; $s = 0, 1$, can be viewed as the proportion of the time that the process remains in state (w, s) after a large number of inspections. The vector π is the solution of the linear equation system $\pi' = \pi'\mathbf{P}$, subjected to the restriction $\sum_{(w,s)\in E} \pi(w, s) = 1$. The solution may be obtained by available numerical procedures.

Moreover, the matrix **P** is diagonalisable, so a non singular matrix $\boldsymbol{\Gamma}$ exists such that:

$$\boldsymbol{\Gamma}\mathbf{P}\boldsymbol{\Gamma}^{-1} = \Lambda, \tag{2.5}$$

$\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_6)$ is a diagonal matrix, and the elements of its principal diagonal are the eigenvalues of the matrix $\mathbf{P}$, $\lambda_1, \lambda_2, \ldots, \lambda_6$. Let $\mathbf{V} = \Gamma^{-1}$ and $\mathbf{U} = \Gamma'$ be the matrices that satisfy the equalities $\mathbf{PV} = \mathbf{V}\Lambda$ and $\mathbf{U}'\mathbf{P} = \mathbf{U}'\Lambda$ where the ith column of $\mathbf{V}$ is the eigenvector $\mathbf{v}_i$ associated with the eigenvalue λ_i and the ith row of $\mathbf{U}'$ is the eigenvector $\mathbf{u}_i'$ associated with the eigenvalue λ_i such that $\mathbf{u}_i'\mathbf{v}_i = 1$. After some manipulation, the matrix $\mathbf{P}$ can be expressed as $\mathbf{P} = \mathbf{V}\Lambda\mathbf{U}'$. By Perron-Frobenius' theorem (Brémaud, 1999), the following properties hold:

$$\begin{aligned} &\lambda_1 = 1, \text{with multiplicity equal to 1;} \\ &\lambda_1 > |\lambda_i|, i = 2, \ldots, 6; \quad \text{and} \\ &v_1 u_1' = 1\pi' = \Pi, \end{aligned} \tag{2.6}$$

where **1** is a vector of ones and $\boldsymbol{\pi}$ is the stationary distribution from (2.4). Thus, the second eigenvalue of the matrix **P** may be obtained algebraically as:

$$\begin{aligned} &\lambda_2 = (1 - \pi)^m p_D, \quad \text{and} \\ &\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0. \end{aligned} \tag{2.7}$$

Besides, in this model, the following equalities hold:

$$\begin{aligned} u_1' v_1 &= u_2' v_2 = 1; \\ u_1' v_2 &= u_2' v_1 = 0. \end{aligned}$$

Let $\alpha_k(w, s)$ be the probability of the process being found at state (w, s) in the kth cycle of inspection after beginning the production of the lot. The probability distribution of the states of the chain is denoted by the vector:

$$\alpha_k' = [\alpha_k(w, s); (w, s) \in E]. \tag{2.8}$$

As the process starts in-control, let:

$$\alpha_0' = [1, 0, 0, 0, 0, 0], \tag{2.9}$$

By Chapman–Kolmogorov equations, and employing (2.9), it follows:

$$\alpha_k' = \alpha_0' \mathbf{P}^k. \tag{2.10}$$

The matrix $\mathbf{P}^k$ is the transition matrix after k cycles (Brémaud, 1999), and the probability distribution of the states of the chain after n inspections can be obtained by expression (2.10), replacing k by n. But $\mathbf{P}^k$ can be written as

$$P^k = V\Lambda^k U' = V \operatorname{diag}(1, \lambda_2^k, 0, 0, 0, 0)U'. \tag{2.11}$$

By properties (2.6), and (2.7), the spectral decomposition of (2.11) is available as:

$$\mathbf{P}^k = v_1 u_1' + \lambda_2^k v_2 u_2' = \Pi + \lambda_2^k v_2 u_2'. \tag{2.12}$$

Note that by spectral decomposition, it is not necessary to make the product of the matrices to obtain $\mathbf{P}^k$ as in (2.12).

When $\tau \to \infty$, the average cost per cycle is given by the cost of each state (w, s) weighted by $\pi(w, s)$, $w = 0, 1, 2$; $s = 0, 1$. However, in this article, as τ is finite, an adequate procedure will be presented in the next section. The costs for each state (w, s) are presented in Secs. 3.1–3.4 and the average cost per cycle of inspection when τ is finite is the subject of the Sec. 3.5.

3. Cost of the States of the Markov Chain

The structure of the cost adopted here is similar to that of Trindade et al. (2007) and other earlier articles. Three components are considered: inspection, adjustment, and non conforming items sent to the customer. We use the following notation:

- c_{insp}: cost of inspecting a produced item;
- c_{nc}: cost of sending a non conforming item to the customer or to the next stages of the process;
- c_{s_nc}: cost of scrapping a non conforming item;
- c_{s_c}: cost of scrapping a conforming item;
- c_a: cost of adjusting the process.

The costs c_{s_c} and c_{s_nc} are used when the scrapped items are submitted to a rectification (conforming or non conforming items may have different costs). It is assumed that all inspected items are discarded or rectified in the posterior stages since some tests to which the inspected item is submitted are destructive.

The cost of the state (w, s), $w = 0, 1, 2$; $s = 0, 1$ may be expressed as:

$$\varphi(w, s) = c_{insp} + \xi(w, s) + \theta(w, s) + \phi(w, s),$$

where:

- c_{insp} is the cost to inspect a single item [constant for all states (w, s), $w = 0, 1, 2$; $s = 0, 1$];
- $\xi(w, s)$ is the expected cost related to the average number of non-conforming items in $(m - 1)$ non inspected items sent to the customer or to the next stages of production;

- $\theta(w, s)$ is the expected cost related to the inspected (and discarded) item;
- $\phi(w, s)$ is the cost of adjusting the process (included in states with $s = 0$).

The vector of the average cost for the chain states is denoted by:

$$\varphi' = [\varphi(w, s), (w, s) \in E]. \tag{3.1}$$

3.1. *Cost of States* (0, 0) *and* (0, 1)

When the process reaches the states (0,0) and (0, 1), it is in-control. The expected number of non conforming (and non inspected items) sent to the customer is $(1 - p_1)(m - 1)$, and the expected cost is given by:

$$\xi(0, s) = c_{nc}(1 - p_1)(m - 1), \quad s = 0, 1.$$

The inspected item is classified as conforming, but due to misclassification errors, it can be, in reality, conforming or non conforming. The probability that the inspected item is truly conforming, given that the process is in control and inspected item is classified as conforming, is:

$$P\{X = 1 \mid Y = 1, Z = 1\} = \frac{p_1(1 - \alpha)}{p_A},$$

Moreover, the probability that the inspected item is truly conforming, given that the process is in control and inspected item is classified as non conforming, is:

$$P\{X = 1 \mid Y = 0, Z = 1\} = \frac{p_1\alpha}{1 - p_A}.$$

Then, the average cost to scrap an inspected item classified as non conforming is:

$$\theta(0, 0) = c_{s\_c}\left[\frac{p_1\alpha}{1 - p_A}\right] + c_{s\_nc}\left[\frac{(1 - p_1)(1 - \beta)}{1 - p_A}\right].$$

Similarly, the average cost to discard an inspected item classified as conforming is:

$$\theta(0, 1) = c_{s\_c}\left[\frac{p_1(1 - \alpha)}{p_A}\right] + c_{s\_nc}\left[\frac{(1 - p_1)\beta}{p_A}\right].$$

As the state (0, 0) leads to an adjustment, the costs related to the adjustment of these states are, respectively:

$$\phi(0, 0) = c_a,$$
$$\phi(0, 1) = 0.$$

Thus, the average costs related to the states (0, 0) and (0, 1) are, respectively:

$$\varphi(0, 0) = c_{insp} + \xi(0, 0) + \theta(0, 0) + c_a,$$
$$\varphi(0, 1) = c_{insp} + \xi(0, 1) + \theta(0, 1).$$

3.2. *Cost of States* (1, 0) *and* (1, 1)

When the fraction of conformance items shifts during the current cycle, at least the inspected item is produced in State II (out-of-control). The change point may occur at any instant (from the first item produced until the last one) of the current cycle. The probability that a shift occurs at time t is given by:

$$q_m(t) = \frac{(1-\pi)^{t-1}\,\pi}{1-(1-\pi)^m}, \quad t = 1, \ldots, m.$$

Thus, the average costs related to the non conforming items sent to the customer for states when $w = 1$ are:

$$\xi(1, s) = c_{nc}\left\{\sum_{t=1}^{m} q_m(t)\,[(t-1)(1-p_1) + (m-t)(1-p_2)\right\}, \quad s = 0, 1.$$

The costs of states (1, 0) and (1, 1), related to the inspected (and discarded) items, are similar to those of states when $w = 0$; the difference depends on the conformance fraction, which has changed to p_2. Thus, the new costs are, respectively:

$$\theta(1,0) = c_{s\_c}\left[\frac{p_2\alpha}{1-p_D}\right] + c_{s\_nc}\left[\frac{(1-p_2)\,(1-\beta)}{1-p_D}\right],$$
$$\theta(1,1) = c_{s\_c}\left[\frac{p_2(1-\alpha)}{p_D}\right] + c_{s\_nc}\left[\frac{(1-p_2)\,\beta}{p_D}\right].$$

Finally, the average costs of these states are:

$$\varphi(1,0) = c_{insp} + \xi(1,0) + \theta(1,0) + c_a,$$
$$\varphi(1,1) = c_{insp} + \xi(1,1) + \theta(1,1).$$

3.3. *Cost of States* (2, 0) *and* (2, 1)

The average costs of these states are similar to those presented in Sec. 3.1, except that the process has now been out-of-control since an earlier cycle. Hence, the average costs related to state (2, 0) are:

$$\xi(2,0) = c_{nc}(1-p_2)\,(m-1)\,,$$
$$\theta(2,0) = c_{s\_c}\left[\frac{p_2\alpha}{1-p_D}\right] + c_{s\_nc}\left[\frac{(1-p_2)\,(1-\beta)}{1-p_D}\right],$$
$$\phi(2,0) = c_a,$$
$$\varphi(2,0) = c_{insp} + \xi(2,0) + \theta(2,0) + c_a.$$

Similarly, the average costs related to state (2, 1) are:

$$\xi(2,1) = \xi(2,0),$$
$$\theta(2,1) = c_{s\_c}\left[\frac{p_2\,(1-\alpha)}{p_D}\right] + c_{s\_nc}\left[\frac{(1-p_2)\,\beta}{p_D}\right],$$

$$\phi(2,1) = 0,$$
$$\varphi(2,1) = c_{insp} + \xi(2,1) + \theta(2,1).$$

3.4. *Cost to Produce the Residual Amount of Items*

As stated in the previous sections, the inspected items are discarded after the examination. Thus, to complete the production of the required lot of size τ, additional production of an amount of $m_{res} = \tau - n\,(m-1)$ items is necessary, $(0 \leq m_{res} < m-1)$. The cost of the production of the additional amount depends only on the cost of sending non-conforming items to the customers because neither inspection nor adjustment is made.

Let $\varphi_{res}(w,s)$ be the average cost to produce m_{res} items when the process is at state (w,s) at the nth inspection. For the states $(w,0)$; $w = 0,1,2$, the process is in-control when it starts to produce the m_{res} items. The expected number of non conforming items sent to the customer is equal to:

$$m_{res}(1-p_1)(1-\pi)^{m_{res}} + \sum_{t=1}^{m_{res}} (1-\pi)^{t-1}\pi\,[(t-1)(1-p_1) + (m_{res}-t+1)(1-p_2)]. \tag{3.2}$$

The first component of (3.2) assumes that the process remains in-control until the production of all m_{res} items is complete. In the second component, the conforming fraction of the process shifts during the production of the additional amount of items. Thus, the average cost of the residual amount for the states $(w,0)$; $w = 0,1,2$ is:

$$\varphi_{res}(w,0) = c_{nc}\left\{ \sum_{t=1}^{m_{res}} (1-\pi)^{t-1}\pi[(t-1)(1-p_1) + (m_{res}-t+1)(1-p_2)] + m_{res}(1-p_1)(1-\pi)^{m_{res}} \right\}.$$

After n inspections, if the process is at state (0, 1), then it is in-control when it starts to produce the residual amount. Hence, the next equality holds:

$$\varphi_{res}(0,1) = \varphi_{res}(w,0), \quad w = 0,1,2.$$

Moreover, if after n inspections, the process reaches the states $(w,1)$; $w > 0$, then it starts out-of-control, and the cost related to the production of the residual amount is:

$$\varphi_{res}(w,1) = c_{nc}(1-p_2)m_{res}.$$

The vector of the average costs of the residual amount for the states of the chain is denoted by:

$$\varphi'_{res} = [\varphi_{res}(w,s),\,(w,s) \in E].$$

3.5. *Average Production Cost*

Let Φ_k be a random variable related to the cost of the control system at the kth cycle of inspection of the production process. Its conditioned expected value for the state of the chain is given by:

$$E\left(\Phi_k \mid W_k, S_k\right) = \sum_{(w,s)\in E} \varphi(w, s) 1_{\{(w,s)\}}(k).$$

$1_{\{(w,s)\}}(k)$ is a function that indicates whether the process reaches the state (w, s) at the kth inspection. Thus, the average cost of the control system for the kth cycle, μ_k, is:

$$\mu_k = E\left(\Phi_k\right) = E\left[E\left(\Phi_k \mid W_k, S_k\right)\right] = \sum_{(w,s)\in E} \varphi(w, s)\alpha_k(w, s). \tag{3.3}$$

Employing (2.8), (2.9), (2.10), and (3.1), expression (3.3) may be written as:

$$\mu_k = \alpha_k' \varphi = \alpha_0 \mathbf{P}^k \varphi. \tag{3.4}$$

The cost to monitor the production process of the lot is the sum of the costs related to each inspection:

$$\Phi = \Phi_1 + \cdots + \Phi_n.$$

The average cost of monitoring the production can be calculated as:

$$\mu = E\left(\Phi\right) = E\left(\sum_{k=1}^{n} \Phi_k\right) = \sum_{k=1}^{n} \mu_k. \tag{3.5}$$

From (2.10) and (3.4), (3.5) can be rewritten as:

$$\mu = \sum_{k=1}^{n} \alpha_0' \mathbf{P}^k \varphi = \alpha_0' \left(\sum_{k=1}^{n} \mathbf{P}^k\right) \varphi. \tag{3.6}$$

Substituting (2.12) in (3.6), we find:

$$\mu = \alpha_0' \left(\sum_{k=1}^{n} \Pi + \lambda_2^k v_2 u_2'\right) \varphi = \alpha_0' \left[n\Pi + \frac{\lambda_2}{1-\lambda_2}\left(1-\lambda_2^n\right) v_2 u_2'\right] \varphi,$$

Note that:

$$\sum_{k=1}^{n} \lambda_2^k = \frac{\lambda_2}{1-\lambda_2}\left(1-\lambda_2^n\right).$$

Similarly, the average cost of the control system related to the additional residual amount is:

$$\mu_{res} = \alpha_n' \varphi_{res}. \tag{3.7}$$

Then, the total cost of the control system of the lot is given as:

$$\mu + \mu_{res}. \tag{3.8}$$

The expected cost of the control system for the produced item is the ratio of the average cost, $(\mu + \phi_{res})$, to the quantity of items produced and sent to the customer, τ, expressed as:

$$C(m) = \frac{\mu + \mu_{res}}{\tau} = \frac{\alpha_0' \left[n\Pi + \frac{\lambda_2}{1-\lambda_2} (1 - \lambda_2^n)\, v_2 u_2' \right] \varphi + \mu_{res}}{\tau}. \tag{3.9}$$

The sampling interval, m, that minimizes $C(m)$ is the solution of the following:

$$m^0 = \arg\min_m C(m). \tag{3.10}$$

It is difficult to obtain a closed expression for (3.10), so the optimal parameter is found computationally through an exhaustive search. It is important to note that expression (3.9) makes the computational search task easier since it simplifies Eq. (3.6) (which is a function of series of matrix potentials).

Thus, from expression (3.9), $\alpha_0'\Pi\phi = \pi'\phi$, and when $\tau \to \infty$:

$$\frac{\tau}{n} \to m - 1; \quad \frac{\mu_{res}}{\tau} \to 0; \quad \text{and} \quad \frac{\frac{\lambda_2}{1-\lambda_2} (1 - \lambda_2^n)}{n(m-1)} \to 0; \quad \text{so,} \quad C(m) \to \frac{\pi'\varphi}{m-1}.$$

This is the function to be optimized in the economical design for on-line process control proposed by Borges et al. (2001).

Let $\sum_{k=1}^{n} 1_{\{(w,s)\}}(k)$ be the frequency of the process at the state (w, s); $\sum_{(w,s)\in E} \sum_{k=1}^{n} 1_{\{(w,s)\}}(k) = n_{with:}$

$$E\left[1_{\{(w,s)\}}(k)\right] = \alpha_k(w, s).$$

Hence, after n cycles of monitoring, the frequency of the process at state (w, s) is, on average:

$$\bar{\alpha}(w, s) = E\left[\frac{\sum_{k=1}^{n} 1_{\{(w,s)\}}(k)}{n}\right] = \frac{\sum_{k=1}^{n} E\left[1_{\{(w,s)\}}(k)\right]}{n} = \frac{\sum_{k=1}^{n} \alpha_k(w, s)}{n}.$$

By expressions (3.4) and (3.5), the average cost of monitoring per cycle is:

$$\frac{\mu}{n} = \frac{\sum_{k=1}^{n} \alpha_k'\varphi}{n} = \frac{\sum_{(w,s)\in E} \sum_{k=1}^{n} \alpha_k(w, s)\varphi(w, s)}{n} = \sum_{(w,s)\in E} \bar{\alpha}(w, s)\varphi(w, s),$$

This equation is similar to the expression obtained for long-run production where the average cost per cycle is denoted as:

$$\sum_{(w,s)\in E} \pi(w, s)\varphi(w, s),$$

where $\pi(w, s)$ is the same as that used in (2.4), the stationary probability of the state (w, s).

4. Numerical Example and Sensitivity Analysis

The example described in this section is based on Trindade et al. (2007), Nandi and Sreehari (1999), and Taguchi et al. (1989) and real cases related to Dasgupta (2003) and Taguchi et al. (2004). This choice is motivated by simplicity and ease of adjustment to other applications. Other examples may include applications in the preventive maintenance of the production of semiconductors as well as the production of diodes in printed circuit boards and chemical processes. Generally, high-quality processes that employ some type of automatic control via collection of individual observations may be improved by the procedure discussed here.

A manufacturer of integrated circuits is facing difficulties in the soldering process and internal conditions of the metalized holes. Soldered connections may present problems such as insufficient or excess solder, lead projection, or incorrect positioning of a device or lead. Historical data allow for adoption of $p_1 = 0.999$ as the conformance probability in a welding process (State I: in control), and shifts to the out-of-control state [State II ($p_2 = 0.95$)] can be described by a geometric distribution (i.e., $\pi = 0.0001$). To evaluate the soldering problem an automated X-ray inspection system is installed in the production line. In order to determine the conditions of the holes, a small cut is made; as such, this is a destructive test. The holes are then examined in a microscope to decide whether the integrated circuit is conforming. Note that these two tasks are not free of errors such that inspection errors are assumed ($\alpha = \beta = 0.01$). The cost components are estimated as $c_{insp} =$ \$0.25; $c_{nc} =$ \$20; $c_a =$ \$100; $c_{s\_nc} =$ \$1.5; and $c_{s\_c} =$ \$2.1. An order for a lot of size $\tau = 2300$ integrated circuits is requested by a customer. To calculate the optimum value, a program in MatLab© was developed (interested readers can request a copy of the program directly from the authors). The aim is to find the optimum value (interval sampling) of m.

Figure 3 shows a plot of the expected cost versus the sampling interval m. The best policy is obtained with a sampling interval of $m^o = 330$ and an additional amount of $m_{res} = 326$. This results in an average cost per unit of \$0.122385.

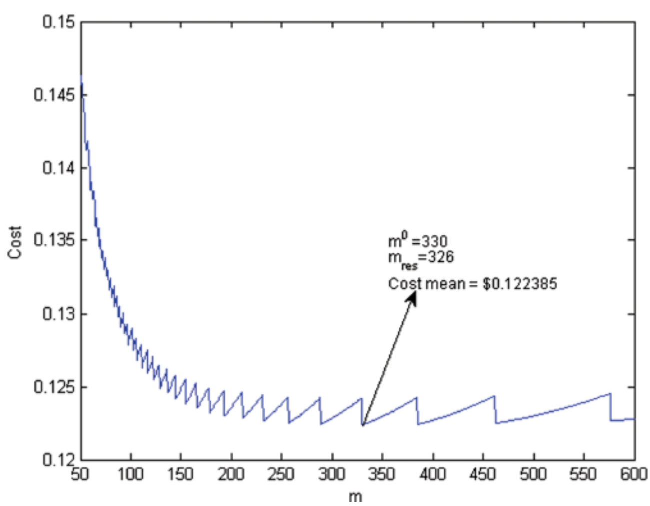

Figure 3. Plot of the average cost vs. m. (color figure available online.)

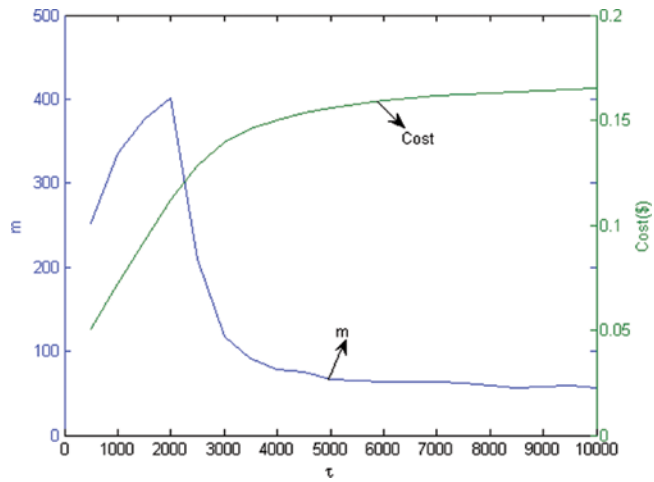

Figure 4. Plot of cost and m vs. mv vs. τ. (color figure available online.)

If the optimum parameter of a long-run production is used as proposed in Nayebpour and Woodall (1993); Borges et al. (2001); Trindade et al. (2007); Ho et al. (2007), etc., the sampling interval would be equal to 51. However, such a design will result in an average cost of \$0.144352 per unit (approximately 18% higher than the short-run production cost). Figures 4 and 5 demonstrate that the sampling interval (m) can be considerably different from that of a long-run production approach ($m = 51$ in the current example) when the size of the lot (τ) is small. As expected, if τ increases, the difference between short and long-run production costs vanishes.

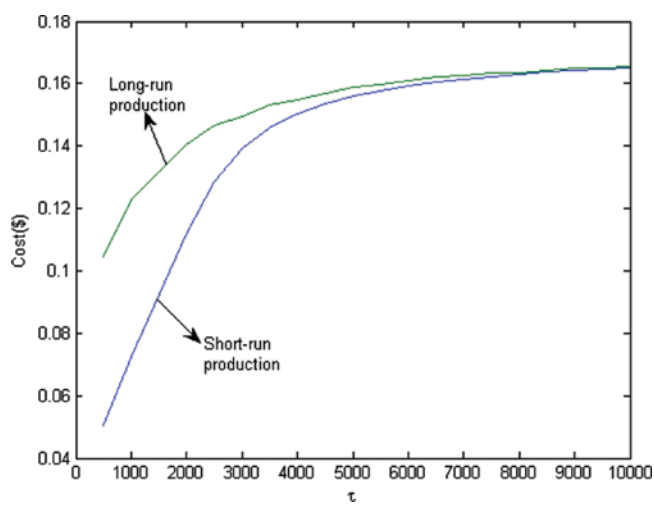

Figure 5. Plot of cost vs. t: short- and long-run production. (color figure available online.)

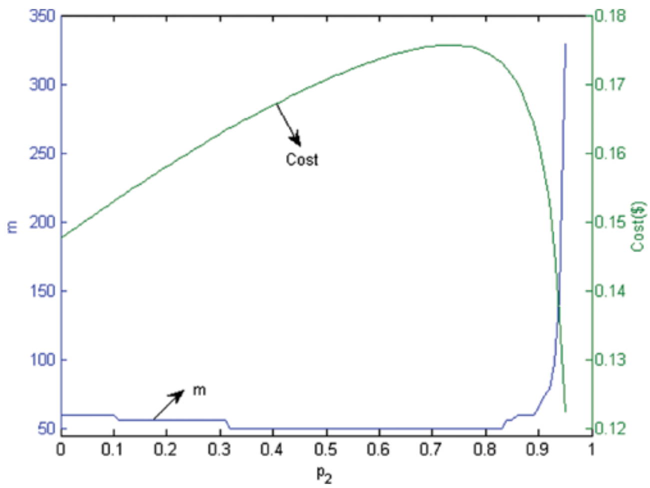

Figure 6. Plot of cost and m vs. p_2. (color figure available online.)

4.1. *The Impact of Classification Errors p_1 and p_2*

For the sensitivity analysis, each parameter is analysed over an arbitrary large range of values while the other parameters are kept equal to the numerical values presented in the beginning of this section. Plots of the optimum values of m (and the average cost) obtained by varying one parameter at a time, while keeping the other parameters equal to the values described in the numerical example presented in the beginning of this section, are shown in Figs. 6–9.

A reduction in the fraction of conforming items produced in State II (p_2) results in a decrease of the sampling interval m, as illustrated in Fig. 6. Note that a decrease in p_2 causes a decrease in the average cost, which initially seems contrary to what may be expected. Due to misclassification errors, a decrease in p_2 reduces

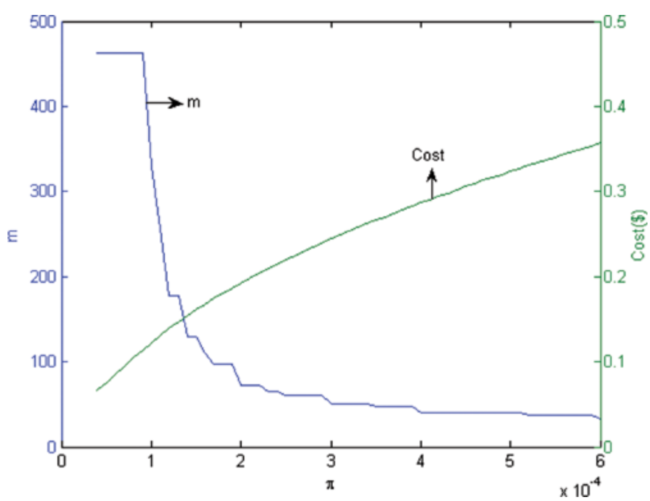

Figure 7. Plot of cost and m vs. p. (color figure available online.)

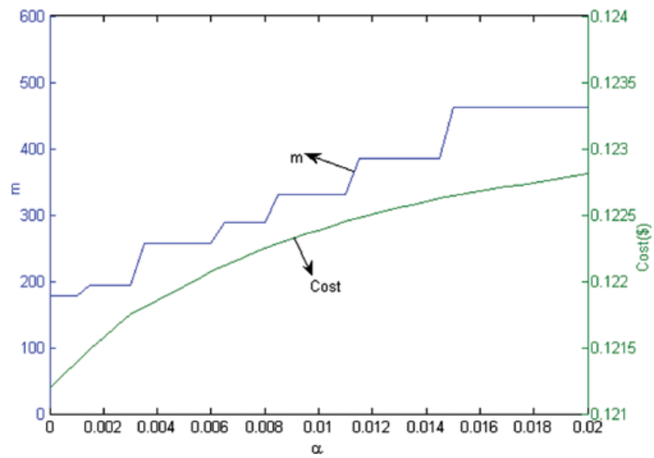

Figure 8. Plot of cost and m vs. a. (color figure available online.)

the probability of classifying an inspected item as conforming when it is truly non conforming. Consequently, fewer adjustments are required (although they are not completely eliminated). Thus, the number of expected non-conforming items sent to the customer is reduced. An increase in π (Fig. 7) causes a reduction in the sampling interval m to anticipate needed adjustments as early as possible.

In the sensitivity analysis of the misclassification errors α and β, the optimum values of m increase when they increase. However, as illustrated in Figs. 8 and 9, the increase in α has a more significant effect on the sampling interval m compared to an increase in β. The optimum sampling interval varies in the range of 178–462 (8 different sampling intervals) when α is in the interval from 0.0001–0.02. For the same range of values of β, the optimum sampling interval m is 330. This result suggests that the error α has a greater impact on the sampling interval than the error β.

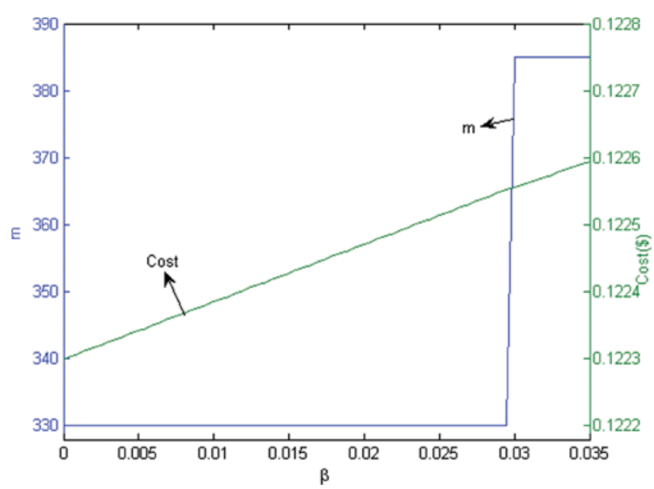

Figure 9. Plot of cost and m vs. b. (color figure available online.)

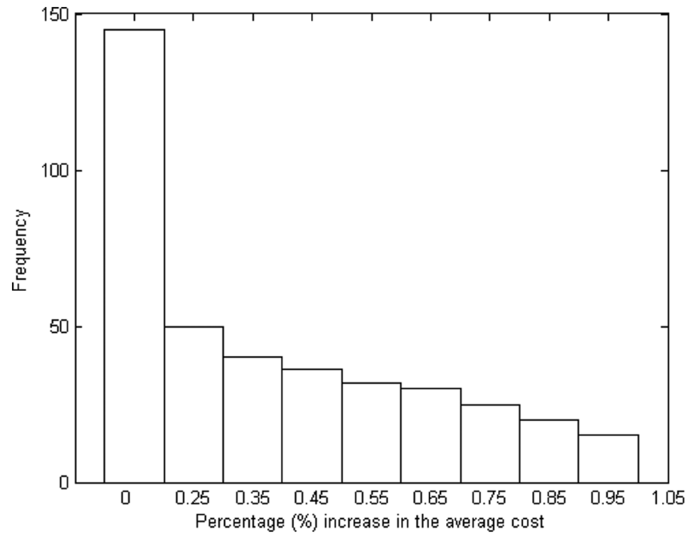

Figure 10. Percentage increase in the average cost when the sampling interval $m = 330$ is equivocally used.

4.2. *Cost Components and Constrained Optimization*

According to Dasgupta (2003), the estimation of cost components is a difficult task and can never be expected to be accurate. Thus, it is important to evaluate the impact of errors in the estimation of costs (c_{insp}; c_{nc}; c_{s_c}; c_{s_nc}; c_a) on optimum (design) policy. For this purpose, a balanced experiment was developed. Each cost (for inspecting, adjusting, scrapping, or sending conforming items) assumes seven levels: the current case, ±5%, ±10%, and ±15%. The cost used in the numerical example is the current case. It is therefore important to measure the impact on the costs (in terms of relative costs) when the optimum parameter (i.e., $m^{\circ} = 330$) is equivocally used in a case with errors in the cost. For example, if the costs (c_{insp}; c_{nc}; c_a) are 5% higher, the optimum sampling interval would be 289, with an average cost per unit of \$0.128225. If the sampling interval $m = 330$ is used, then the average cost is \$0.128231 (about 0.005% more than \$0.128225). Figure 10 illustrates the increase in the average cost (in percentage) that results if the sampling interval of $m = 330$ is employed in cases of error in cost up to 15%. Note that all values are lower than 1.05%, indicating that the proposal is robust to errors in the costs of up to 15% (c_{insp}; c_{nc}; c_{s_c}; c_{s_nc}; c_a) since significant alterations are not produced in the average cost when the sampling interval of $m = 330$ is equivocally used.

5. Conclusions and Final Remarks

In this article, the authors propose an economic design of process control for short-run production using attributes. Numerical examples are used to illustrate the proposal, and the design of short-run production is shown to depend heavily on the non-conformance cost. The main conclusion is that the optimum design of a long-run production cannot be reasonably used for a short-run production. In the example described above, a cost increase of 15% is observed if the design of a long-run production is wrongly used in the design of a short-run production. As the lot size increases, the short-run production policy converges to that of a long-run production. Hence, the proposal presented in this article can generate reasonable savings, and is recommended for application using the computer program developed for this purpose.

Future searches employing other distributions for the process failure mechanism can be developed. A natural candidate is the discrete Weibull distribution (more details, see Nakagawa and Osaki (1975) since it can take on a variety of shapes and mimic the behaviour of other distributions. Compared to the exponential/geometric distribution, the discrete Weibull distribution is more flexible in that it includes models with an increasing or decreasing hazard rate.

Acknowledgment

The authors would like to acknowledge CNPq for partial financial support. The authors would like also to acknowledge the suggestions given by the anonymous referees that certainly have contributed the improvement of this new version.

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