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Optimal Measurement Basis for Economical Phase-covariant Telecloning with Partially Entangled States

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Abstract Recently, Wang and Yang (Phys. Rev. A 79:062315, 2009) presented a scheme for economical phase-covariant telecloning of qubits with W-class entangled states. For realizing probabilistically the suboptimal telecloning in the case that the sender's subsystem and the receivers' subsystem are partially entangled, they introduced a special two-qubit measurement basis. I here study the effects of the sender's different measurements on the fidelity of the clones in such a scheme, and obtain several interesting results. The most important result is that Bell-basis is the optimal measurement basis in terms of the average fidelity of the clones, although the special-basis measurement can lead to the suboptimal fidelity with a certain probability.

Keywords Economical phase-covariant telecloning · Partially entangled states · Optimal measurement basis

1 Introduction

Quantum no-cloning theorem [1] tells us that an unknown quantum state can not be exactly cloned. Nevertheless, the question that how well one can clone an unknown quantum state has been attracting much interest [2–6] since Bužek and Hillery first introduced the concept of approximate quantum cloning [7], because it can help us to find the quantum operation limit [8], measure the amount of radiated power [9], and is related to quantum computation, quantum communication, and quantum cryptography (see e.g., [10–12]). According to whether or not ancillas are involved, quantum cloning is divided into two types, i.e., non-economical cloning [2] and economical cloning [13]. In non-economical cloning no ancilla is needed, while in economical cloning ancillas are needed.

The combination of quantum cloning and teleportation [14] leads to advent of the concept of telecloning [15, 16]. Telecloning functions as transmitting multiple copies of an unknown quantum state to distant sites, i.e., realizing one-to-many nonlocal cloning. Telecloning can

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implement remote distribution of quantum information [17, 18], reveal some entanglement properties [19, 20], and so on. It has been shown that the quantum information previously distributed by telecloning procedure from a quantum system can be remotely concentrated back to a quantum system via a suitable entanglement channel [21, 22]. Then telecloning and concentrating can also be, respectively, regarded as quantum information depositing and withdrawing, or quantum information encoding and decoding. In the past decade, telecloning was extensively studied and developed [23–28].

In most of the existing telecloning schemes, the quantum channels are the maximally entangled states in terms of the sender's subsystem and the receivers' subsystem. Recently, Wang and Yang [29] presented a scheme for economical phase-covariant telecloning [30] of qubits with W-class entangled states [31]. For realizing probabilistically the suboptimal telecloning in the case that the sender's subsystem and the receivers' subsystem are partially entangled, they introduced a special two-qubit measurement basis. In this paper, we shall show that Bell-basis measurement, instead of the special-basis measurement, on the sender's location is the optimal in terms of the average fidelity of the clones.

Let us first give a simple review for the probabilistic suboptimal telecloning scheme of Ref. [29]. The phase-covariant state to be telecloned is

$$|\phi\rangle_T^{in} = \frac{1}{2}\left(|0_T\rangle + e^{i\delta}|1_T\rangle\right), \tag{1}$$

where $\delta \in [0, 2\pi]$ and $\{|0\rangle, |1\rangle\}$ represents the computational basis of a qubit. The quantum channel is the asymmetric W-class entangled state

$$|W_{n+1}\rangle = \frac{1}{Q}\left[q|1_A\rangle\sum_{j=1}^{n}|0_{B_j}\rangle + |0_A\rangle\sum_{j=1}^{n}\left(|1_{B_j}\rangle\prod_{k=1,k\neq j}^{n}|0_{B_k}\rangle\right)\right], \tag{2}$$

where $Q = \sqrt{n+q^2}$ and q is a real number. Qubits (T, A) are held by the sender Alice and B_j $(j = 1, 2, \ldots, n)$ belongs to the jth receiver Bob$j$. In order to implement telecloning, Alice needs performing a project measurement jointly on qubits (T, A) in the basis $\{|\Psi_h^+\rangle_{TA}, |\Psi_h^-\rangle_{TA}, |\Phi_h^+\rangle_{TA}, |\Phi_h^-\rangle\}$ with

$$\begin{aligned}
|\Psi_h^+\rangle_{TA} &= \frac{1}{H}(|0_T1_A\rangle + h|1_T0_A\rangle), \\
|\Psi_h^-\rangle_{TA} &= \frac{1}{H}(h|0_T1_A\rangle - |1_T0_A\rangle), \\
|\Phi_h^+\rangle_{TA} &= \frac{1}{H}(|0_T0_A\rangle + h|1_T1_A\rangle), \\
|\Phi_h^-\rangle_{TA} &= \frac{1}{H}(h|0_T0_A\rangle - |1_T1_A\rangle),
\end{aligned} \tag{3}$$

where $H = \sqrt{1+h^2}$ and h is a real number. Wang and Yang [29] demonstrated that when h is equal to $q/\sqrt{n}$ or $\sqrt{n}/q$, the fidelity of each clone can be equal to $1/2(1+1/\sqrt{n})$ with probability $P = 2nq^2/(n+q^2)^2$. In the case $n = 2$, the fidelity is equal to the optimal one of $1 \to 2$ phase-covariant cloning [32]. Thus the fidelity is suboptimal.

In the following, we show that the average fidelity of the clones hits to the maximum when h is equal to one, instead of $q/\sqrt{n}$ and $\sqrt{n}/q$. The entire state of the $n+2$ qubits can

Table 1 The probabilities of getting, respectively, the four possible outcomes in Alice's joint measurement, the corresponding local operations Bobs performed on their qubits, and the corresponding fidelity of the clones. Here, I is the identity operator and $\sigma^{x,z}$ are the conventional Pauli operators

Measurement outcomes	Probability	Bob's operations	Clones' fidelity
$\vert\Psi^+\rangle_{TA}$	$\frac{X^2}{2Q^2H^2}$	I	$\frac{1}{2}+\frac{qh}{X^2}$
$\vert\Psi^-\rangle_{TA}$	$\frac{Y^2}{2Q^2H^2}$	σ^z	$\frac{1}{2}+\frac{qh}{Y^2}$
$\vert\Phi^+\rangle_{TA}$	$\frac{Y^2}{2Q^2H^2}$	σ^x	$\frac{1}{2}+\frac{qh}{Y^2}$
$\vert\Phi^-\rangle_{TA}$	$\frac{X^2}{2Q^2H^2}$	$\sigma^z\sigma^x$	$\frac{1}{2}+\frac{qh}{X^2}$

be expanded as

$$\begin{aligned}
|\psi\rangle_{total} &= |\phi\rangle_T^{in}\otimes|W_{n+1}\rangle \\
&= \frac{1}{\sqrt{2}QH}\Bigg\{X|\Psi_h^+\rangle_{TA}\frac{1}{X}\Bigg[q\prod_{j=1}^n|0_{B_j}\rangle + he^{i\delta}\sum_{j=1}^n\Bigg(|1_{B_j}\rangle\prod_{k=1,k\neq j}^n|0_{B_k}\rangle\Bigg)\Bigg] \\
&\quad + Y|\Psi_h^-\rangle_{TA}\frac{1}{Y}\Bigg[qh\prod_{j=1}^n|0_{B_j}\rangle - e^{i\delta}\sum_{j=1}^n\Bigg(|1_{B_j}\rangle\prod_{k=1,k\neq j}^n|0_{B_k}\rangle\Bigg)\Bigg] \\
&\quad + Y|\Phi_h^+\rangle_{TA}\frac{1}{Y}\Bigg[\sum_{j=1}^n\Bigg(|1_{B_j}\rangle\prod_{k=1,k\neq j}^n|0_{B_k}\rangle\Bigg) + qhe^{i\delta}\prod_{j=1}^n|0_{B_j}\rangle\Bigg] \\
&\quad + X|\Phi_h^-\rangle_{TA}\frac{1}{X}\Bigg[h\sum_{j=1}^n\Bigg(|1_{B_j}\rangle\prod_{k=1,k\neq j}^n|0_{B_k}\rangle\Bigg) - qe^{i\delta}\prod_{j=1}^n|0_{B_j}\rangle\Bigg]\Bigg\},
\end{aligned} \tag{4}$$

where $X=\sqrt{q^2+nh^2}$ and $Y=\sqrt{q^2h^2+n}$. The telecloning procedure is as follows. (i) Alice performs a joint measurement on qubits (T, A) in the basis $\{|\Psi_h^+\rangle_{TA}, |\Psi_h^-\rangle_{TA}, |\Phi_h^+\rangle_{TA}, |\Phi_h^-\rangle\}$, and sends the outcome to Bobs through classical channels. The probability of getting each outcome can be easily figured out, shown in Table 1. (ii) After receiving Alice's measurement outcome, Bobs perform, respectively, corresponding local operations on their qubits as shown in Table 1 to get the desired clones.

We now discuss the second step in detail. As an example, we assume that Alice's measurement outcome is $|\Psi_h^+\rangle_{TA}$. Then the state of the other qubits collapses into

$$|\phi\rangle^{out} = \frac{1}{X}\Bigg[q\prod_{j=1}^n|0_{B_j}\rangle + he^{i\delta}\sum_{j=1}^n\Bigg(|1_{B_j}\rangle\prod_{k=1,k\neq j}^n|0_{B_k}\rangle\Bigg)\Bigg]. \tag{5}$$

The state-density operator of each qubit is

$$\rho = \frac{q^2+(n-1)h^2}{X^2}|0\rangle\langle 0| + \frac{h^2}{X^2}|1\rangle\langle 1| + e^{i\delta}\frac{hq}{X^2}|1\rangle\langle 0| + e^{-i\delta}\frac{hq}{X^2}|0\rangle\langle 1|. \tag{6}$$

Obviously, ρ is related to the measurement basis, h. The fidelity of each clone is given by

$$F = \langle\phi^\delta|\rho|\phi^\delta\rangle = \frac{1}{2}+\frac{qh}{X^2}. \tag{7}$$

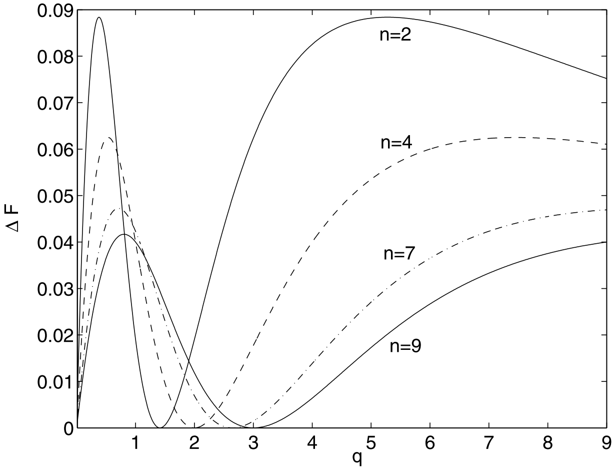

Fig. 1 $\Delta F(q,n)$ versus q with $n=2,4,7,9$

If Alice's measurement outcome is one of the other three states $\{|\Psi_h^-\rangle_{TA}, |\Phi_h^+\rangle_{TA}, |\Phi_h^-\rangle\}$, Bobs can also obtain the desired clones with a certain fidelity as shown in Table 1.

It can be seen from Table I that the average fidelity of each clone for Alice's four possible measurement outcomes is

$$\begin{aligned}\bar{F} &= 2\times\frac{X^2}{2Q^2H^2}\times\left(\frac{1}{2}+\frac{qh}{X^2}\right)+2\times\frac{Y^2}{2Q^2H^2}\times\left(\frac{1}{2}+\frac{qh}{Y^2}\right)\\ &= \frac{1}{2}+\frac{2qh}{(n+q^2)(1+h^2)}.\end{aligned} \tag{8}$$

As mentioned above, if choosing $h=q/\sqrt{n}$ or $h=\sqrt{n}/q$ suboptimal telecloning can be implemented with probability $P=2nq^2/(n+q^2)^2$. Then the average fidelity of each clone is

$$\bar{F}_{sub}=\frac{1}{2}+\frac{2q^2\sqrt{n}}{(n+q^2)^2}. \tag{9}$$

However, the maximum of the average fidelity is (can be calculated by Lagrange multipliers)

$$\bar{F}_{max}=\frac{1}{2}+\frac{q}{(n+q^2)} \tag{10}$$

with $h=1$. The difference between $\bar{F}_{max}$ and $\bar{F}_{sub}$ is

$$\Delta F(q,n)=\bar{F}_{max}-\bar{F}_{sub}=\frac{(n+q^2)q-2q^2\sqrt{n}}{(n+q^2)^2}. \tag{11}$$

The distinct dependence of $\Delta F(q,n)$ on q and n is shown in Fig. 1. From Fig. 1, I can safely deduce three conclusions as follows. (i) $\Delta F(q,n)=0$, i.e., $\bar{F}_{sub}=\bar{F}_{max}$, if and only if $q=\sqrt{n}$. Then $h=q/\sqrt{n}=\sqrt{n}/q$ is in fact equal to one and the subsystem of Alice and that of Bobs are maximally entangled. (ii) The larger n is, the smaller the peak value of $\Delta F(q,n)$ is, and the easier the curve of $\Delta F(q,n)$ is. (iii) For arbitrary n_1 and n_2 with $n_1<n_2$, the corresponding curves of $\Delta F(q,n_1)$ and $\Delta F(q,n_2)$ have two intersection points q_0 and q_1. Then the values of q can be divided into three ranges, i.e., $0<q\le q_0$, $q_0<q\le q_1$, and

$q_1 < q$. In the ranges $0 < q \leq q_0$ and $q_1 < q$, $\Delta F(q, n_1) > \Delta F(q, n_2)$; and in the range $q_0 < q \leq q_1$, $\Delta F(q, n_1) < \Delta F(q, n_2)$.

In summary, I have studied the effects of the sender's different joint measurements on the fidelity of the clones in the economical phase-covariant telecloning, and obtained several interesting results. The most important result is that Bell-basis is the optimal measurement basis in terms of the average fidelity of the clones, although a special-basis measurement can lead to the suboptimal fidelity with a certain probability. In other words, the protocol using Bell-basis measurement is more efficient than the ones using non-Bell-basis measurements in point of view of quantum information distribution.

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