Math. Program., Ser. A (2014) 145:403–450
DOI 10.1007/s10107-013-0652-1

FULL LENGTH PAPER

Lifted inequalities for 0−1 mixed-integer bilinear covering sets

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Received: 1 March 2011 / Accepted: 21 February 2013 / Published online: 17 April 2013

Abstract In this paper, we study 0−1 mixed-integer bilinear covering sets. We derive several families of facet-defining inequalities via sequence-independent lifting techniques. We then show that these sets have a polyhedral structure that is similar to that of a certain fixed-charge single-node flow set. As a result, we also obtain new facet-defining inequalities for the single-node flow set that generalize well-known lifted flow cover inequalities from the integer programming literature.

Mathematics Subject Classification 90C11 · 90C20 · 90C30 · 90C57

1 Introduction and motivation

Nonlinear branch-and-bound is a method to solve mixed-integer nonlinear programming (MINLP) problems to global optimality; see [10,16]. This method has been implemented in commercial solvers such as BARON [25] and LINDO Global [17]. It requires that convex relaxations of the problem be recursively solved over smaller

This work was supported by NSF CMMI Grants 0856605 and 0900065.

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and smaller subsets of the feasible region obtained by branching on variables. Most existing commercial software use a method proposed in [20] to obtain these convex relaxations for factorable problems. McCormick's relaxation is an instance of a more general technique that relaxes (nonconvex) constraints of the form $g(x) \geq r$ into (convex) constraints of the form $\bar{g}(x) \geq r$ where $\bar{g}(x)$ is a concave overestimator of $g(x)$. This technique does not use the right-hand-side of the inequality in the process. As a result, the relaxation obtained is typically not the strongest possible.

Some of the functional forms that appear most frequently in the formulation of nonlinear programs are probably multilinear inequalities and equalities. In particular, bilinear inequalities of the covering type

$$\sum_{j=1}^{n} a_j x_j y_j \geq d, \tag{1}$$

where $a_j > 0$, $x_j \in S \subseteq \mathbb{R}_+$, and $y_j \in S' \subseteq \mathbb{R}_+$ are among the simplest nonconvex inequalities that can be studied. Therefore, sets of the form (1) provide an important test bed for the derivation of new and stronger convexification methods that use right-hand-side information.

In this paper, we study the convex hull of feasible solutions to (1) when variables are bounded. In particular, we consider $0-1$ mixed-integer bilinear covering sets of the form

$$B = \left\{ (x, y) \in \{0, 1\}^n \times [0, 1]^n \,\middle|\, \sum_{j=1}^{n} a_j x_j y_j \geq d \right\},$$

where $n \in \mathbb{Z}_{++}$, $a_j > 0\ \forall j \in N := \{1, \ldots, n\}$, and $d > 0$. We will throughout refer to the convex hull of B, $\text{conv}(B)$, as PB. Although our focus in this paper will be on the study of PB, similar results can be obtained for sets defined through constraints of the form $\sum_{j=1}^{n}(a_j x_j y_j + b_j x_j + c_j y_j) \geq d$, where $(a_j, b_j, c_j) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ and $a_j + \min\{b_j, c_j\} \geq 0$ for all $j \in N$. Through scaling and translation, this generalization allows us to extend the applicability of our study to problems where the bounds on y are not 0 and 1 and, in addition, to problems where some of the x variables are fixed. Our proofs extend easily to such a setup because the two sets share strong relationships that are described in Proposition 24 and the discussion following it. The set B and its more general variant discussed above appear in a variety of application contexts. Consider, for example, the linearization strategy of [6,26] for $x^T Q x \geq d$, where $x \in \{0, 1\}^n$ and $Q \in \mathbb{R}_+^{n \times n}$. The authors define $z = Qx$ and then replace the original constraint with $x^T z \geq d$, where $0 \leq z \leq Q\mathbf{1}$ and $\mathbf{1}$ is the vector in $\mathbb{R}^n$ whose components are all equal to 1. Let $a = Q\mathbf{1}$ and $y_i = \frac{z_i}{a_i}$. Then, $x^T Q x \geq d$ reduces to the constraint defining B. The set B also appears as an objective function cut for problems involving maximization of bilinear functions of the form $\sum_{j=1}^{n} a_j x_j y_j$ where $x_j \in \{0, 1\}$ and $y_j \in [0, 1]$; see [23] for an application to fixed-charge network flow problems and see [27] for an application to shortest path interdiction with asymmetric information. We also mention that B is related to several packing and covering sets including those discussed in [4,14,31,34].

In order to guarantee that B is not empty, we impose

Assumption 1 $\sum_{j=1}^{n} a_j \geq d$.

When faced with the problem of constructing a convex relaxation of B, two existing techniques can be used. The first technique reformulates B as $B^E \cap \mathcal{H}$ where $\mathcal{H} = \{(x, y, u) \in \mathbb{R}^{2n+1} | u \geq d\}$ and $B^E = \left\{(x, y, u) \in \{0, 1\}^n \times [0, 1]^n \times \mathbb{R} | \sum_{j \in N} a_j\ x_j y_j \geq u\right\}$. It then relaxes $\text{conv}(B)$ as $\text{proj}_{(x,y)}(\text{conv}(B^E) \cap \mathcal{H})$, where $\text{proj}_{(x,y)} S$ denotes the projection of S onto the space of (x, y) variables. It is clear that $\text{conv}(B^E)$ can be obtained directly by computing the concave envelope of $\sum_{j \in N} a_j x_j y_j$ over $\{0, 1\}^n \times [0, 1]^n$. Further, it follows from the separability of $\sum_{j \in N} a_j x_j y_j$ over $j \in N$ that this concave envelope is obtained directly from the concave envelope of each bilinear term $a_j x_j y_j$ over $\{0, 1\} \times [0, 1]$; see [1]. Because the concave envelope of each bilinear term is known to be obtained through McCormick constraints, we conclude that the tightest relaxation of the type $\bar{g}(x) \geq d$, where $\bar{g}(x)$ is a concave overestimator of $\sum_{j \in N} a_j x_j y_j$ restricted to $\{0, 1\}^n \times [0, 1]^n$ over $[0, 1]^{2n}$, is the relaxation that uses McCormick constraints. Observe that this relaxation contains an exponential number of linear constraints that can be separated in polynomial time.

The second technique requires that upper bounds on the variables be relaxed, thereby yielding a bilinear covering set $B^U = \{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n \mid \sum_{j \in N} a_j x_j y_j \geq d\}$. It is shown in [30] that the convex hull of B^U can be obtained explicitly using a variant of disjunctive programming. The inequalities again are linear and can be separated in polynomial time.

In this paper, we are interested in studying stronger relaxation techniques for B that will take both the right-hand-side d and upper bounds on the variables into account. Even though we show in Proposition 1 that the simultaneous presence of upper bounds and right-hand-sides makes it NP-hard to optimize a linear function over B and hence to develop a separation oracle for its convex hull, many new strong inequalities that take advantages of both features can still be derived.

On the practical side, we are interested in studying B as a way to obtain improved convex relaxations for problems in variables $(x, z) \in \mathbb{R}^n \times \mathbb{R}^p$ with constraints of the form $\sum_{j=1}^{n} f_j(z) x_j \geq d$, where $f_j : \mathbb{R}^p \to [b_j, b_j + a_j]$ and $(a_j, b_j) \in \mathbb{R}_+^2$ for all $j \in N$. It is easy to see that these constraints can be reformulated as $\sum_{j=1}^{n} (a_j y_j + b_j) x_j \geq d$ through the introduction of new variables $y_j = \frac{f_j(z) - b_j}{a_j}$ where $y_j \in [0, 1]$. Convex relaxations stronger than those currently used in commercial solvers can then be constructed through strong inequalities of the convex hull of the 0-1 mixed integer bilinear covering set defined by the constraint $\sum_{j=1}^{n} (a_j y_j + b_j) x_j \geq d$, and through concave/convex envelopes of the functions $\frac{f_j(z) - b_j}{a_j}$ for each $j \in N$. We are also interested in studying B because of its relations to the fixed-charge single-node flow set without inflows

$$F = \left\{ (x, y) \in \{0, 1\}^n \times [0, 1]^n \left| \sum_{j=1}^{n} a_j y_j \geq d,\ x_j \geq y_j\ \forall j \in N \right. \right\}, \tag{2}$$

an important mixed-integer linear set whose convex hull, $\text{conv}(F)$, we denote by PF. In particular, we will show in Lemma 3 that the set B is a relaxation of F, thereby establishing that valid inequalities for B are also valid for F. Further, we will show in Sect. 4 that facet-defining inequalities of either PF or PB can be easily identified if facet-defining inequalities for the other set are known. As a result, the inequalities we derive for PB reveal new families of facet-defining inequalities for PF which are structurally different from those described in the literature.

We next argue that it is typically difficult to find globally optimal solutions to problems containing B as a constraint by showing that it is NP-hard to optimize a linear function over B. To this end, consider the following optimization problem (Q):

$$\min \left\{ \sum_{j=1}^{n} \eta_j x_j + \sum_{j=1}^{n} \kappa_j y_j \;\middle|\; (x, y) \in B \right\} \tag{Q}$$

where $\eta \in \mathbb{R}^n$ and $\kappa \in \mathbb{R}^n$.

Proposition 1 *Problem (Q) is NP-hard.*

Proof The proof is by reduction from the $0-1$ knapsack problem, which is proven to be NP-hard in [11]. Consider the following $0-1$ knapsack instance:

$$z^K = \min \left\{ \sum_{j=1}^{n} \eta_j x_j \;\middle|\; \sum_{j=1}^{n} a_j x_j \geq d,\ x_j \in \{0, 1\}\ \forall j \in N \right\}. \tag{K}$$

We define a corresponding instance of (Q) by setting $\kappa_j = -1$ for all $j \in N$, *i.e.*,

$$z^P = \min \left\{ \sum_{j=1}^{n} \eta_j x_j - \sum_{j=1}^{n} y_j \;\middle|\; \sum_{j=1}^{n} a_j x_j y_j \geq d,\ x_j \in \{0, 1\},\ y_j \in [0, 1]\ \forall j \in N \right\}. \tag{P}$$

The reduction from (K) to (P) is clearly polynomial. Observe further that if x^* is a feasible solution to (K), then $(x^*, \mathbf{1})$ is feasible to (P), therefore showing that $z^P \leq z^K - n$. Similarly, if (x^*, y^*) is an optimal solution to (P), then x^* is feasible to (K) as $\sum_{j=1}^{n} a_j x_j^* \geq \sum_{j=1}^{n} a_j x_j^* y_j^* \geq d$. Therefore, $z^K \leq z^P + \mathbf{1}^{\mathbf{T}} y^* \leq z^P + n$. We conclude that $z^P = z^K - n$ and that x^* is an optimal solution to (K) if and only if $(x^*, \mathbf{1})$ is an optimal solution to (P). □

In this paper, we are interested in studying PB. Since B is a finite union of polytopes, PB is polyhedral.

Proposition 2 *PB is a polytope.*

It follows from Proposition 2 that, when studying PB, it is sufficient to consider linear inequalities. Proposition 1 suggests that finding a complete closed-form expression for PB is difficult. As a result, we will focus our efforts on constructing families of strong cutting planes for optimization problems containing the constraints of B by studying PB. To construct these inequalities, we will use lifting. Lifting is a well-known integer programming technique that generates strong inequalities for a given set by transforming an inequality valid for a restricted subset of the feasible region into a globally valid constraint. Early work on lifting in integer programming can be found in [32,33]. A generalization to nonlinear programming is given in [24]. In particular, lifting is said to be *sequence-independent* if the order in which the restrictions are removed does not change the derived inequality. Subadditivity of a certain perturbation function, called the *lifting function*, is a sufficient condition for lifting to be sequence-independent when the restrictions involve fixing the variables at their bounds; see Proposition 13 and [24]. In this paper, we derive large families of facet-defining inequalities for PB by performing sequence-independent lifting. These results illustrate that lifting can successfully use bounds on variables in the generation of cuts for MINLPs. Further, the results have implications for fixed-charge flow models, a family of theoretically and practically important problems in mixed-integer linear programming.

The paper is structured as follows. In Sect. 2, we derive basic polyhedral results about PB. We provide necessary and sufficient conditions for trivial inequalities to be facet-defining. Then, we derive a linear description of PB for the special case where $n = 2$. This result is used to identify the seed inequalities that will be used in lifting procedures. In Sect. 3, we review lifting techniques and present two families of subadditive functions. Then, we use sequence-independent lifting techniques, to derive, in closed-form, three families of facet-defining inequalities for PB. One family is derived using a subadditive approximation of the lifting function. In Sect. 4, we prove that there are some tight connections between the facet-defining inequalities of PB and those of PF. In particular, we show that the lifted inequalities developed for PB generalize certain families of flow cover cuts and yield new facet-defining inequalities for the fixed-charge single-node flow set without inflows, F, as defined in (2). We summarize the contributions of our work and conclude with directions of future research in Sect. 5.

2 Basic polyhedral results

In this section, we analyze the polyhedral structure of PB. The omitted proofs are relatively straightforward and can be found in [9]. First, we provide necessary and sufficient conditions for PB to be full-dimensional.

Proposition 3 *PB is a full-dimensional polytope if and only if $\sum_{j=1}^{n} a_j - a_i \geq d$ for all $i \in N$.* □

In the remainder of this paper, we will assume that PB is full-dimensional.

Assumption 2 $\sum_{j=1}^{n} a_j - a_i \geq d$ *for all* $i \in N$.

Observe that Assumption 2 strictly dominates Assumption 1 and implies that $n \geq 2$. We next identify some basic characteristics of the facet-defining inequalities of PB.

Proposition 4 *Let $\sum_{j=1}^{n} \alpha_j x_j + \sum_{j=1}^{n} \beta_j y_j \geq \delta$ be a facet-defining inequality for PB that is not a scalar multiple of $x_i \leq 1$ for $i \in N$ or $y_i \leq 1$ for $i \in N$. Then, (i) $\alpha_i \geq 0,\ \forall i \in N$, (ii) $\beta_i \geq 0,\ \forall i \in N$, and (iii) $\delta \geq 0$.* □

The following proposition further studies facet-defining inequalities whose right-hand-sides are zero.

Proposition 5 *Let* $\sum_{j=1}^{n} \alpha_j x_j + \sum_{j=1}^{n} \beta_j y_j \geq 0$ *be a facet-defining inequality for PB. Then, this inequality is a scalar multiple of* $x_j \geq 0$ *for* $j \in N$ *or of* $y_j \geq 0$ *for* $j \in N$. □

We now focus on these inequalities that play a special role in Propositions 4 and 5 and characterize when they are facet-defining for *PB*. We refer to these inequalities as *bound inequalities*.

Proposition 6 *The upper bound inequalities* $x_i \leq 1$, $y_i \leq 1$ *are facet-defining for PB for all* $i \in N$. *Further, for* $i \in N$, *the lower bound inequalities* $x_i \geq 0$, $y_i \geq 0$ *are facet-defining for PB if and only if* $\sum_{j=1}^{n} a_j - a_i - a_{l(i)} \geq d$ *where* $l(i) \in \operatorname{argmax}\{a_j \mid j \in N \backslash \{i\}\}$. □

We mention that the above results are also valid when $y_i \in \{0, 1\}$ instead of $y_i \in [0, 1]$ for some subset $J \subseteq N$. We next study another simple facet-defining inequality for *PB*.

Proposition 7 *The inequality* $\sum_{j=1}^{n} a_j y_j \geq d$ *is facet-defining for PB.*

Proof Validity is easily verified since $\sum_{j=1}^{n} a_j y_j \geq \sum_{j=1}^{n} a_j x_j y_j \geq d$. To prove that $\sum_{j=1}^{n} a_j y_j \geq d$ is facet-defining, we present $2n$ points (x^i, y^i) in B that satisfy $\sum_{j=1}^{n} a_j y_j^i \geq d$ at equality and such that the system $\alpha x^i + \beta y^i = \delta$ for $i = 1, \ldots, 2n$ only has solutions (α, β, δ) that are scalar multiples of $(\mathbf{0}, a, d)$. Consider the $2n$ points $p^k = (\mathbf{1}, \Delta_k(\mathbf{1} - e_k))$ and $q^k = (\mathbf{1} - e_k, \Delta_k(\mathbf{1} - e_k))$ where $\Delta_k = \frac{d}{\sum_{j=1}^{n} a_j - a_k}$ for $k \in N$. Note that because of Assumption 2, $0 < \Delta_k \leq 1$ for all $k \in N$. Clearly, p^k and q^k belong to B and satisfy $\sum_{j=1}^{n} a_j y_j \geq d$ at equality. These $2n$ points yield the system:

$$\sum_{j=1}^{n} \alpha_j + \Delta_k \left(\sum_{j=1}^{n} \beta_j - \beta_k \right) = \delta \ \forall k \in N, \tag{3}$$

$$\sum_{j=1}^{n} \alpha_j - \alpha_k + \Delta_k \left(\sum_{j=1}^{n} \beta_j - \beta_k \right) = \delta \ \forall k \in N. \tag{4}$$

By subtracting (3) from (4), we obtain that $\alpha_k = 0$ for all $k \in N$. From (3) and the definition of Δ_k, we then conclude that, for all $k, l \in N$,

$$\sum_{j=1}^{n} \beta_j - \beta_k = \frac{\delta}{d} \left(\sum_{j=1}^{n} a_j - a_k \right) \text{ and } \sum_{j=1}^{n} \beta_j - \beta_l = \frac{\delta}{d} \left(\sum_{j=1}^{n} a_j - a_l \right).$$

Subtracting these expressions yields $\beta_k - \frac{\delta}{d} a_k = \beta_l - \frac{\delta}{d} a_l$. After defining $\beta_k - \frac{\delta}{d} a_k = \theta$ for $k \in N$ and using these relations in (3), we obtain that $\theta = 0$, which implies $\beta_k = \frac{\delta}{d} a_k$ for all $k \in N$. Therefore, we conclude that all solutions (α, β, δ) to the system (3) and (4) are scalar multiples of $(\mathbf{0}, a, d)$. □

In the remainder of this paper, we will often use the term *facet* to refer to a facet-defining inequality. We will also refer to inequalities $x_i \leq 1$, $y_i \leq 1$, and $\sum_{j=1}^{n} a_j y_j \geq d$ as *trivial facets* of PB. To illustrate the richness of the polyhedral structure of PB, we present an example next. The linear inequalities describing the convex hull of this set were obtained using PORTA; see [7].

Example 1 Consider the $0-1$ *mixed-integer bilinear covering set*

$$B = \left\{ (x, y) \in \{0, 1\}^4 \times [0, 1]^4 \;\middle|\; 19x_1 y_1 + 17x_2 y_2 + 15x_3 y_3 + 10x_4 y_4 \geq 20 \right\}.$$

The linear description of PB has 58 inequalities. They include:

$$50x_1 + 90x_3 + 45x_4 + 76y_1 + 153y_2 \geq 135 \quad (5)$$
$$70x_1 + 90x_2 + 27x_4 + 38y_1 + 135y_3 \geq 117 \quad (6)$$
$$19x_1 + 17x_2 + 15y_3 + 10y_4 \geq 20 \quad (7)$$
$$17x_2 + 15x_3 + 19y_1 + 10y_4 \geq 20 \quad (8)$$
$$19y_1 + 17y_2 + 15y_3 + 10y_4 \geq 20 \quad (9)$$
$$14x_1 + 10x_3 + 5x_4 + 17y_2 \geq 15 \quad (10)$$
$$12x_2 + 10x_3 + 5x_4 + 19y_1 \geq 15 \quad (11)$$
$$10x_3 + 5x_4 + 19y_1 + 17y_2 \geq 15 \quad (12)$$
$$x_1 + x_2 + x_3 + 10y_4 \geq 2 \quad (13)$$
$$x_1 + x_2 + x_3 + x_4 \geq 2 \quad (14)$$
$$x_1 \geq 0 \quad (15)$$
$$y_1 \geq 0 \quad (16)$$
$$x_1 \leq 1 \quad (17)$$
$$y_1 \leq 1. \quad (18)$$

Among the inequalities in Example 1, we recognize the upper bound inequalities (17) and (18) that are shown to be facet-defining for PB in Proposition 6. In this example, the lower bound inequalities (15) and (16) are also facet-defining, as can be established from Proposition 6. Further, (9) is the trivial facet-defining inequality studied in Proposition 7. Our goal is to now discover families of valid inequalities for PB that would explain (5–8) and (10–14).

To derive these nontrivial facet-defining inequalities, we first study the convex hull of B when $n = 2$ with the goal of identifying seed inequalities for subsequent lifting procedures. We show that the linear description of PB has at most three nontrivial inequalities. In this study, Assumption 2 requires that $a_1 \geq d$ and $a_2 \geq d$.

Proposition 8 *Let* $B^2 = \left\{ (x, y) \in \{0, 1\}^2 \times [0, 1]^2 \;\middle|\; a_1 x_1 y_1 + a_2 x_2 y_2 \geq d \right\}$, *where* $a_1 \geq d$, $a_2 \geq d$ *and* $d > 0$. *Then,*

$$conv(B^2) = X := \left\{ (x,y) \in [0,1]^2 \times [0,1]^2 \;\middle|\; \begin{array}{l} x_1 + x_2 \geq 1 \\ dx_1 + a_2 y_2 \geq d \\ a_1 y_1 + dx_2 \geq d \\ a_1 y_1 + a_2 y_2 \geq d \end{array} \right\}.$$

Proof We prove the result using disjunctive programming techniques; see [5]. We define

$$\begin{aligned}
X_{10} &:= B^2 \cap \left\{ x_1 = 1, x_2 = 0 \right\} = \left\{ (1, y_1, 0, y_2) \middle| \tfrac{d}{a_1} \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1 \right\}, \\
X_{01} &:= B^2 \cap \left\{ x_1 = 0, x_2 = 1 \right\} = \left\{ (0, y_1, 1, y_2) \middle| 0 \leq y_1 \leq 1,\ \tfrac{d}{a_2} \leq y_2 \leq 1 \right\}, \\
X_{11} &:= B^2 \cap \left\{ x_1 = 1, x_2 = 1 \right\} = \left\{ (1, y_1, 1, y_2) \middle| a_1 y_1 + a_2 y_2 \geq d,\ 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1 \right\}.
\end{aligned}$$

It is easily verified that $\text{conv}(B^2) = \text{conv}(X_{10} \cup X_{01} \cup X_{11}) = \text{conv}(X_2 \cup X_{11})$ where $X_2 := \text{conv}(X_{10} \cup X_{01})$. We first use disjunctive programming techniques to obtain a linear description of X_2 and then compute $\text{conv}(B^2)$ as $\text{conv}(X_2 \cup X_{11})$. Using Theorem 2.1 in [5], we write

$$X_2 = \text{proj}_{(x,y)} \left\{ (x_1, y_1, x_2, y_2, \bar{z}_1, \bar{z}_2, \hat{z}_1, \hat{z}_2, \lambda) \;\middle|\; \begin{array}{l} (x_1, y_1, x_2, y_2) = \left(\lambda, \bar{z}_1 + \hat{z}_1, 1 - \lambda, \bar{z}_2 + \hat{z}_2\right), \\ \frac{d}{a_1}\lambda \leq \bar{z}_1 \leq \lambda,\ 0 \leq \bar{z}_2 \leq \lambda, \\ 0 \leq \hat{z}_1 \leq 1 - \lambda,\ \frac{d}{a_2}(1-\lambda) \leq \hat{z}_2 \leq 1 - \lambda, \\ 0 \leq \lambda \leq 1 \end{array} \right\}.$$

We then use Fourier-Motzkin elimination [35] to compute the projection. We first eliminate the variables λ, $\hat{z}_1$ and $\hat{z}_2$ using the equations $\lambda = x_1$, $\hat{z}_1 = y_1 - \bar{z}_1$, and $\hat{z}_2 = y_2 - \bar{z}_2$. Then, we project $\bar{z}_1$ and $\bar{z}_2$ to obtain

$$X_2 = \text{conv}(X_{10} \cup X_{01}) = \left\{ (x_1, y_1, x_2, y_2) \;\middle|\; \begin{array}{l} x_1 + x_2 = 1,\ x_1 \geq 0,\ x_2 \geq 0, \\ \frac{d}{a_1} x_1 \leq y_1 \leq 1,\ \frac{d}{a_2} x_2 \leq y_2 \leq 1 \end{array} \right\}$$

since $x_1 \leq 1$ and $x_2 \leq 1$ are implied by $x_1 + x_2 = 1$, $x_1 \geq 0$ and $x_2 \geq 0$. Similarly, we can now derive a linear description of $\text{conv}(X_2 \cup X_{11})$ by first formulating this set as the projection of a polyhedron using disjunctive programming and then projecting the resulting formulation onto the space of x and y variables using Fourier-Motzkin elimination; see [9] for details. □

Next, we give generalizations of the nontrivial facets of $\text{conv}(B^2)$ that we prove are facet-defining for more general instances of $\text{conv}(B)$. In particular, we give a generalization of inequalities $dx_1 + a_2 y_2 \geq d$ and $a_1 y_1 + dx_2 \geq d$ in Proposition 9 and of inequality $x_1 + x_2 \geq 1$ in Proposition 11. We will use these generalizations as seed inequalities for lifting procedures in Sect. 3.

Proposition 9 *Let $L \subseteq N$ be such that $\sum_{j \in N \setminus L} a_j > d$. Define $\bar{a} = \sum_{j \in N \setminus L} a_j - \max_{i \in N \setminus L} a_i$ and assume that*
$S = \{(x, \bar{y}) \in \{0,1\}^{|L|} \times [0,1] \mid \sum_{i \in L} \min\{a_i, d\} x_i + \bar{a}\bar{y} = d\} \neq \emptyset.$

Then,

$$\sum_{j\in L}\min\{a_j,d\}x_j+\sum_{j\in N\setminus L}a_jy_j\geq d \tag{19}$$

is facet-defining for PB. In particular, (19) is facet-defining for PB if (i) $L\cap L^{\geq}\neq\emptyset$*, or (ii)* $L=\emptyset$*, or (iii)* $\bar{a}\geq\max_{i\in L}\min\{a_i,d\}$*, or as a special case (iv)* $\bar{a}\geq d$ *where* $L^{\geq}:=\{j\in N\mid a_j\geq d\}$.

Proof We first argue that inequality

$$\sum_{j\in N}\min\{dx_j,a_jx_j,a_jy_j\}\geq d \tag{20}$$

is valid for *PB*, which will directly imply that (19) is valid for *PB*. To this end, we show next that $\sum_{j\in N}\min\{dx_j,a_jx_jy_j\}\geq d$ is valid for B. Consider $(x,y)\in B$. If there exists $j\in N$ such that $dx_j<a_jx_jy_j$ then $x_j=1$ and, consequently, the inequality is satisfied. Otherwise, the inequality reduces to the defining inequality of B. Since $(x_j,y_j)\in[0,1]^2$ implies that $x_jy_j\leq\min\{x_j,y_j\}$ and $a_j\geq 0$ for $j\in N$, it follows that $\min\{dx_j,a_jx_jy_j\}\leq\min\{dx_j,a_jx_j,a_jy_j\}$ and, therefore, (20) is valid for *PB*.

We now prove that (19) is facet-defining for *PB* by providing $2n$ affinely independent points (x^i,y^i) in B that satisfy (19) at equality. Assume without loss of generality that $L=\{1,\dots,l\}$. Define $n'=|N\setminus L|$ and denote the points as $(x_L,x_{N\setminus L},y_L,y_{N\setminus L})$. Let $(x',\bar{y}')\in S$ and define $a'=\sum_{j\in N\setminus L}a_j$. Let $p^0=(0,\mathbf{1},0,\frac{d}{a'}\mathbf{1})$ and $p^j=p^0+\epsilon(0,0,0,\frac{1}{a_j}e_j-\frac{1}{a_{j+1}}e_{j+1})$ for $j=1,\dots,n'-1$. For $i\in L$, define $q^i=(e_i,\mathbf{1},e_i,\frac{d-\min\{a_i,d\}}{a'}\mathbf{1})$, $r^i=p^0+(0,0,e_i,0)$ if $a_i\leq d$, and $r^i=(e_i,\mathbf{1},\frac{d}{a_i}e_i,0)$ if $a_i>d$. For $j\in\{1,\dots,n'\}$, $s^j=(x'_L,\mathbf{1}-e_j,\mathbf{1},\bar{y}'\frac{\bar{a}}{\sum_{i\in N\setminus(L\cup\{j\})}a_i}(\mathbf{1}-e_j))$. It can be easily verified that p^0, q^i, s^j and r^i belong to B and that p^j belongs to B when ϵ is sufficiently small. We now show that the above points are affinely independent. Clearly, for $j\in\{0,\dots,n'-2\}$, $p^0,\dots,p^j$ satisfy $\sum_{i=1}^{j+1}a_i(\frac{d}{a'}-y_{l+i})=0$, whereas p^{j+1} does not. Therefore, p^j are affinely independent. Further, for $i\in L$ and $j\in\{0,\dots,n'-1\}$, q^i are affinely independent of p^j since the latter satisfy $(x_i,y_i)=(0,0)$. For $i\in L$, $j\in\{0,\dots,n'-1\}$ and $i'\in L$, r^i are independent of p^j and $q^{i'}$ since the latter satisfy $y_i=x_i$. Finally, for $j\in\{1,\dots,n\}$, $j'\in\{1,\dots,n'-1\}$, $i\in L$, and $i'\in L$, s^j are affinely independent of $p^{j'}$, q^i, $r^{i'}$ since the latter satisfy $x_{|L|+j}=1$. □

The family of inequalities described in Proposition 9 is typically exponential in size. In the case of Example 1, it contains multiple inequalities including (7)–(9).

We next relate (19) to existing relaxation techniques. In particular, we argue that it does not arise as a direct application of factorable or orthogonal disjunction principles but can be obtained through a strengthening of orthogonal disjunction results described in [29]. First, the set of solutions in $[0,1]^{2n}$ that satisfy (20) is contained in the factorable convex relaxation of B discussed in Sect. 1. In particular, when each bilinear term is outer-approximated using McCormick envelopes, we obtain $\sum_{j\in N}a_j\min\{x_j,y_j\}\geq d$, which is clearly implied by (20). Further, using orthogonal disjunctions, see [30], it can be shown that

$$O := \text{conv}\left\{(x, y) \in \mathbb{R}^{2n}_{+} \;\middle|\; \sum_{j\in N} a_j x_j y_j \geq d\right\}$$
$$= \left\{(x, y) \in \mathbb{R}^{2n}_{+} \;\middle|\; \sum_{j\in N} \sqrt{a_j x_j y_j} \geq \sqrt{d}\right\}.$$

This convex relaxation is obtained without making use of the bounds or the integrality of the variables x. It follows from the inequality relating elementary means (see Theorem 5 in [15]) that $\sqrt{da_j x_j y_j} \geq \min\{dx_j, a_j y_j\}$. Therefore, the feasible solutions to (20) are contained in O. However, when $(x, y) \in C \subsetneq \mathbb{R}^{2n}_{+}$, a procedure described in [29] allows a strengthening of the relaxation O by restricting attention to C. When one exploits the fact that $(x, y) \in C = \{0, 1\}^n \times [0, 1]^n$, this construction yields (20).

In the remainder of the paper, we will obtain strong inequalities for PB by lifting (19). To describe these liftings, we will use the following notation extensively. For $N_0, N_1 \subseteq N$ such that $N_0 \cap N_1 = \emptyset$ and $\tilde{N}_0, \tilde{N}_1 \subseteq N$ such that $\tilde{N}_0 \cap \tilde{N}_1 = \emptyset$, we let

$$B(N_0, N_1, \tilde{N}_0, \tilde{N}_1) := \left\{(x, y) \in B \;\middle|\; \begin{array}{ll} x_j = 0 \;\text{ for } j \in N_0, & x_j = 1 \;\text{ for } j \in N_1, \\ y_j = 0 \;\text{ for } j \in \tilde{N}_0, & y_j = 1 \;\text{ for } j \in \tilde{N}_1 \end{array}\right\}.$$

We also define $PB(N_0, N_1, \tilde{N}_0, \tilde{N}_1) := \text{conv}(B(N_0, N_1, \tilde{N}_0, \tilde{N}_1))$. With a slight abuse of notation, we say $B(N_0, N_1, \tilde{N}_0, \tilde{N}_1)$ is full-dimensional if its affine hull, $\text{aff}(B(N_0, N_1, \tilde{N}_0, \tilde{N}_1))$, satisfies:

$$\text{aff}\left(B(N_0, N_1, \tilde{N}_0, \tilde{N}_1)\right) = \left\{(x, y) \in \mathbb{R}^{2n} \;\middle|\; \begin{array}{ll} x_j = 0 \;\text{ for } j \in N_0, & x_j = 1 \;\text{ for } j \in N_1, \\ y_j = 0 \;\text{ for } j \in \tilde{N}_0, & y_j = 1 \;\text{ for } j \in \tilde{N}_1 \end{array}\right\}.$$

Observe that, $B(\emptyset, \emptyset, \emptyset, N)$ is equivalent to the classical $0-1$ knapsack set

$$\left\{x \in \{0, 1\}^n \;\middle|\; \sum_{j=1}^{n} a_j x_j \geq d\right\},$$

whose polyhedral structure was first studied in [4,14], and [31]. The following result, as a special case, relates the bilinear set B to the $0-1$ knapsack set $B(\emptyset, \emptyset, \emptyset, N)$.

Proposition 10 *Let $I \subseteq N$. Assume that*

$$\sum_{j\in N} \alpha_j x_j + \sum_{j\in I} \beta_j y_j \geq \delta \tag{21}$$

is an inequality for $PB(\emptyset, \emptyset, \emptyset, N\backslash I)$ that is not a scalar multiple of a bound inequality. Then, (21) is facet-defining for $PB(\emptyset, \emptyset, \emptyset, N\backslash I)$ if and only if (21) is facet-defining for PB.

Proof We first prove that if (21) is facet-defining for $PB(\emptyset, \emptyset, \emptyset, N\backslash I)$, then (21) is facet-defining for PB. To show that (21) is valid for B, we assume for a contradiction that there exists a point $(x', y') \in B$ with $\sum_{j\in N} \alpha_j x'_j + \sum_{j\in I} \beta_j y'_j < \delta$.

Since $(x', y') \in B$, we have that $\sum_{j \in N} a_j x'_j y'_j \geq d$. Next, we define $(\bar{x}, \bar{y})$ as $\bar{x} = x'$, $\bar{y}_j = y'_j$ for $j \in I$, and $\bar{y}_j = 1$ for $j \in N \backslash I$. Observe that $(\bar{x}, \bar{y}) \in B(\emptyset, \emptyset, \emptyset, N \backslash I)$ as $\sum_{j \in I} a_j \bar{x}_j \bar{y}_j + \sum_{j \in N \backslash I} a_j \bar{x}_j \geq \sum_{j \in N} a_j x'_j y'_j \geq d$. Since (21) is valid for $B(\emptyset, \emptyset, \emptyset, N \backslash I)$, $(\bar{x}, \bar{y})$ satisfies $\sum_{j \in N} \alpha_j x'_j + \sum_{j \in I} \beta_j y'_j = \sum_{j \in N} \alpha_j \bar{x}_j + \sum_{j \in I} \beta_j \bar{y}_j \geq \delta$. This is the desired contradiction.

Next, we show that (21) is facet-defining for PB. Since (21) is facet-defining for $PB(\emptyset, \emptyset, \emptyset, N \backslash I)$ and $\delta \neq 0$ as (21) is not a bound, there exist $n + |I|$ linearly independent points in $B(\emptyset, \emptyset, \emptyset, N \backslash I)$, call them (x^k, y^k), that satisfy (21) at equality. Clearly, these points belong to B and satisfy (21) at equality. Now, for each $j \in N \backslash I$, we construct one new point in $B \backslash B(\emptyset, \emptyset, \emptyset, N \backslash I)$ that satisfies (21) at equality. Choose j arbitrarily in $N \backslash I$. Since (21) is not a scalar multiple of $x_j \leq 1$, there exists $k_j \in \{1, \ldots, n + |I|\}$ such that $x_j^{k_j} = 0$. Now define $(\bar{x}^{k_j}, \bar{y}^{k_j})$ such that $\bar{x}_i^{k_j} = x_i^{k_j}$ $\forall i \in N$, $\bar{y}_i^{k_j} = y_i^{k_j}$ $\forall i \in N \backslash \{j\}$ and $\bar{y}_j^{k_j} = 0$. Clearly, the point $(\bar{x}^{k_j}, \bar{y}^{k_j})$ belongs to B and satisfies (21) at equality. Further, it is easily seen that the points (x^k, y^k) and $(\bar{x}^{k_j}, \bar{y}^{k_j})$ for $j \in N \backslash I$ are linearly independent and therefore show that (21) is facet-defining for PB.

To prove the reverse implication, we assume that (21) is a facet-defining inequality for PB that is not a scalar multiple of a bound. Validity is trivial since $B(\emptyset, \emptyset, \emptyset, N \backslash I) \subseteq B$. Now, we show that (21) is facet-defining for $PB(\emptyset, \emptyset, \emptyset, N \backslash I)$. Since $\delta \neq 0$ as (21) is not a bound (see Proposition 5), the origin does not satisfy (21) at equality. Therefore, any $2n$ affinely independent points, say (x^k, y^k) in B for $k = 1, \ldots, 2n$, that satisfy (21) at equality must also be linearly independent. In other words,

$$\begin{vmatrix} x_1^1 & \ldots & x_n^1 & y_1^1 & \ldots & y_n^1 \\ x_1^2 & \ldots & x_n^2 & y_1^2 & \ldots & y_n^2 \\ & \ldots & & & \ldots & \\ x_1^{2n} & \ldots & x_n^{2n} & y_1^{2n} & \ldots & y_n^{2n} \end{vmatrix} \neq 0. \tag{22}$$

Without loss of generality, assume that $I = \{1, \ldots, |I|\}$. Consider the submatrix formed by the first $n + |I|$ columns of (22). There exist $n + |I|$ rows of the submatrix that are linearly independent. For each of these rows, define a point by setting $y_j = 1$ for $j > |I|$. The resulting points are linearly independent, feasible to $B(\emptyset, \emptyset, \emptyset, N \backslash I)$, and satisfy (21) at equality. Therefore, (21) is facet-defining for $PB(\emptyset, \emptyset, \emptyset, N \backslash I)$. □

Using a proof technique similar to that of Proposition 10, it can also be shown that facet-defining inequalities of PB that are of the form $\sum_{i \in J} \alpha_i x_i + \sum_{i \in I} \beta_i y_i \geq \delta$ are also facet-defining for $PB(\emptyset, N \backslash J, \emptyset, N \backslash I)$. In particular, if $I \cap J = \emptyset$ then there is a mixed 0−1 knapsack whose facet is also defined by the inequality. Furthermore, Proposition 10 implies that all nontrivial facets of the pure 0−1 knapsack polytope can be found in PB and that it is sufficient to study the facets of PB to obtain the facets of the 0−1 knapsack polytope. Next, we use Proposition 10 to generalize the inequality $x_1 + x_2 \geq 1$ of Proposition 8 into an inequality that will be used as a seed for lifting procedures in Sect. 3.4.

Proposition 11 *Assume that* $\sum_{j \in N} a_j - a_k - a_m < d$ *for all* $k, m \in N$ *with* $k \neq m$. *The clique inequality*

$$\sum_{j \in N} x_j \geq |N| - 1 \tag{23}$$

is facet-defining for PB.

Proof By Proposition 10, it is sufficient to prove that (23) is facet-defining for $PB(\emptyset, \emptyset, \emptyset, N)$. The remaining result follows from Proposition II.2.2.3(b) in [21] after using the transformation $\bar{x}_i = 1 - x_i$ for all $i \in N$. □

3 Lifted inequalities

In this section, we derive three families of strong valid inequalities for PB via lifting. The first two families are obtained using sequence-independent lifting from (19) and are facet-defining for PB. In this case, lifting is simple since the lifting function is subadditive. The third family is obtained by lifting (23). Although the lifting function associated with this seed inequality is not subadditive, we obtain lifted inequalities using approximate lifting. We then identify conditions under which these inequalities are facet-defining for PB.

3.1 Sequence-independent lifting for bilinear covering sets

Sequence-independent lifting is a well-known technique to construct strong valid inequalities for mixed-integer linear programs; see [13] and [33]. We next give a brief description of how this technique can be used to derive strong valid inequalities for PB. A more general treatment of lifting in nonlinear programming is given in [24].

Given $\emptyset \neq S \subsetneq N$, we consider $B(S, \emptyset, S, \emptyset)$, which is the restriction of B obtained when all variables (x_j, y_j) for $j \in S$ are fixed to $(0, 0)$. Without loss of generality, let $S = \{s, \ldots, n\}$ for some $s \geq 2$ and define $S_i = \{i + 1, \ldots, n\}$ for $i \in S$. Assume that the inequality

$$\sum_{j=1}^{s-1} \alpha_j x_j + \sum_{j=1}^{s-1} \beta_j y_j \geq \delta \tag{24}$$

is facet-defining for $PB(S, \emptyset, S, \emptyset)$. In sequential lifting, we reintroduce the variables (x_j, y_j) for $j \in S$ one at a time in (24). Assuming that variables (x_j, y_j) have already been lifted in the order $j = s, \ldots, i - 1$, we next review how to lift variables (x_i, y_i) in the inequality

$$\sum_{j=1}^{i-1} \alpha_j x_j + \sum_{j=1}^{i-1} \beta_j y_j \geq \delta, \tag{25}$$

which is assumed to be facet-defining for $PB(S_{i-1}, \emptyset, S_{i-1}, \emptyset)$. To perform this lifting, we first compute the *lifting function*

$$P^i(w) = \max \delta - \left\{ \sum_{j=1}^{i-1} \alpha_j x_j + \sum_{j=1}^{i-1} \beta_j y_j \right\}$$
$$s.t. \sum_{j=1}^{i-1} a_j x_j y_j \geq d - w$$
$$x_j \in \{0, 1\}, \ \ y_j \in [0, 1] \quad j = 1, \ldots, i - 1.$$

Once the lifting function $P^i(w)$ is computed, the lifting coefficients (α_i, β_i) are obtained from $P^i(w)$ as follows.

Proposition 12 (Richard and Tawarmalani [24]) *Let (25) be a valid inequality for $B(S_{i-1}, \emptyset, S_{i-1}, \emptyset)$. Assume that there exist $(\alpha_i, \beta_i) \in \mathbb{R}^2$ such that*

$$\alpha_i x_i + \beta_i y_i \geq P^i(a_i x_i y_i) \quad \text{for } (x_i, y_i) \in \{0, 1\} \times [0, 1] \backslash \{0, 0\}. \tag{26}$$

Then, the inequality $\sum_{j=1}^{i} \alpha_j x_j + \sum_{j=1}^{i} \beta_j y_j \geq \delta$ is valid for $B(S_i, \emptyset, S_i, \emptyset)$. □

The result of Proposition 12 can be applied recursively to construct a valid inequality for *PB* from (24). Note that, at each step, the lifting function $P^i(w)$ must be recomputed to account for the changes in the lifted inequality. Further, if $B(S, \emptyset, S, \emptyset)$ is full-dimensional, the seed inequality (24) is facet-defining for $PB(S, \emptyset, S, \emptyset)$, and for each $i \in S$, the lifting coefficients (α_i, β_i) of the variables (x_i, y_i) are chosen so that (26) is satisfied at equality by two points (x_i^1, y_i^1) and (x_i^2, y_i^2) such that $(0, 0)$, (x_i^1, y_i^1) and (x_i^2, y_i^2) are affinely independent (a feature we refer to as *maximal lifting*), then the final lifted inequality will be facet-defining for *PB*. In this scheme, (re)computing the lifting functions $P^i(w)$ for each $i \in S$ is often the most computationally demanding task. However, this computational work is unnecessary when the lifting function $P^s(w)$ is subadditive. This observation, first made in [33], leads to the following result.

Proposition 13 (Richard and Tawarmalani [24]) *Assume that (24) is valid for $B(S, \emptyset, S, \emptyset)$. Assume also that (i) $P^s(w)$ is subadditive over $\mathbb{R}_+$, i.e., $P^s(w_1) + P^s(w_2) \geq P^s(w_1 + w_2)$ $\forall w_1, w_2 \in \mathbb{R}_+$, and (ii) there exist $(\alpha_i, \beta_i) \in \mathbb{R}^2$ for all $i \in S$ such that*

$$\alpha_i x_i + \beta_i y_i \geq P^s(a_i x_i y_i) \quad \text{for } (x_i, y_i) \in \{0, 1\} \times [0, 1] \backslash \{0, 0\}. \tag{27}$$

Then, the inequality

$$\sum_{j=1}^{n} \alpha_j x_j + \sum_{j=1}^{n} \beta_j y_j \geq \delta \tag{28}$$

is valid for PB. Further, if (i) Inequality (24) is facet-defining for $PB(S, \emptyset, S, \emptyset)$, (ii) $B(S, \emptyset, S, \emptyset)$ is full-dimensional, and (iii) coefficients (α_i, β_i) are chosen in a way *that two linearly independent points satisfy (27) at equality, then (28) is facet-defining for PB.* □

The fundamental difference between Proposition 12 and Proposition 13 lies in equations (26) and (27). In the latter, the lifting coefficients of all variables (x_i, y_i) are obtained from the same lifting function $P^s(w)$ while in the former, they are obtained from $P^i(w)$ for $i \in S$. Although this difference might seem minor, it has important practical implications. In particular, the subadditivity of lifting functions typically permits the derivation of closed-form expressions for lifting coefficients that would otherwise be difficult to obtain. Observe also that in Proposition 13, the subadditivity of $P^s(w)$ is required only over $\mathbb{R}_+$ since all coefficients a_i in PB are assumed to be nonnegative.

Proposition 12 describes how to perform lifting when the variables (x_j, y_j) for $j \in S$ are fixed at $(0, 0)$. When variables (x_j, y_j) are fixed at $(1, 1)$, similar results are obtained when (26) is replaced by

$$\alpha_i(x_i - 1) + \beta_i(y_i - 1) \geq P^i(a_i x_i y_i - a_i) \quad \text{for } (x_i, y_i) \in \{0, 1\} \times [0, 1] \backslash \{1, 1\}. \tag{29}$$

Similarly, Proposition 13 can be adapted to allow sequence-independent lifting for variables (x_j, y_j) fixed at $(1, 1)$ by replacing $P^i(w)$ with $P^s(w)$ in (29) and by requiring that the lifting function $P^s(w)$ is subadditive over $\mathbb{R}_-$. Subadditive lifting can also be used to generate facets of PB if $B(\emptyset, S, \emptyset, S)$ is full-dimensional, the seed inequality (24) is facet-defining for $PB(\emptyset, S, \emptyset, S)$, and for each $i \in S$, the lifting coefficients (α_i, β_i) of the variables (x_i, y_i) are chosen so that (29) is satisfied at equality by two points (x_i^1, y_i^1) and (x_i^2, y_i^2) such that $(1, 1)$, (x_i^1, y_i^1) and (x_i^2, y_i^2) are affinely independent.

We show in the following proposition that all interesting lifted inequalities that can be obtained by fixing variables (x_i, y_i) at $(0, 1)$ or $(1, 0)$ can also be obtained by fixing variables (x_i, y_i) at $(0, 0)$. Intuitively this result can be understood as follows. We first observe that the projections of $PB(S_{i-1}, \emptyset, S_{i-1}, \emptyset)$, $PB(S_{i-1}, \emptyset, \emptyset, S_{i-1})$ and $PB(\emptyset, S_{i-1}, S_{i-1}, \emptyset)$ over the space of non-fixed variables are identical. Therefore, these sets share the same seed inequalities for lifting. Next, we argue that lifting a pair of variables (x_i, y_i) fixed at $(1, 0)$ yields maximal lifting coefficients (α_i, β_i) where $\alpha_i = 0$, while lifting a pair of variables (x_i, y_i) fixed at $(0, 1)$ yields maximal lifting coefficients (α_i, β_i) where $\beta_i = 0$. The result then follows by arguing that the above lifting coefficients can also be obtained by lifting the pair of variables (x_i, y_i) from $(0, 0)$.

Proposition 14 *Assume that (25) defines a nonempty face of $PB(S_{i-1}, \emptyset, S_{i-1}, \emptyset)$ (or equivalently of $PB(S_{i-1}, \emptyset, \emptyset, S_{i-1})$, or of $PB(\emptyset, S_{i-1}, S_{i-1}, \emptyset)$). Then any inequality obtained by maximally lifting (25) in $PB(S_{i-1}, \emptyset, \emptyset, S_{i-1})$ or $PB(\emptyset, S_{i-1}, S_{i-1}, \emptyset)$ could also have been obtained by maximally lifting (25) in $PB(S_{i-1}, \emptyset, S_{i-1}, \emptyset)$.*

Proof First, we consider the case where (x_i, y_i) is fixed at $(1, 0)$. In this situation, valid lifting coefficients must satisfy

$$\alpha_i(x_i-1)+\beta_i y_i \geq P^i(a_i x_i y_i) \quad \text{for } (x_i, y_i) \in \{0,1\} \times [0,1]. \tag{30}$$

We next show that maximal lifting coefficients (α_i, β_i) in (30) must also satisfy

$$\alpha_i x_i + \beta_i y_i \geq P^i(a_i x_i y_i) \quad \text{for } (x_i, y_i) \in \{0,1\} \times [0,1] \tag{31}$$

and be maximal for (31). This is sufficient to prove the result since restricting $(x_i, y_i) = (0, 0)$ instead of $(1, 0)$ does not change the projection in the space of the non-fixed variables and, therefore, the seed inequality is still a face of same dimension. Consider $(0, y_i^*)$ satisfying (30) at equality. Such a point exists since lifting is assumed to be maximal. Then,

$$0 \geq \alpha_i = \beta_i y_i^* \geq P^i(a_i y_i^*) \geq 0,$$

where the first inequality follows from (30) by setting $(x_i, y_i) = (0, 0)$, the equality holds since $(0, y_i^*)$ satisfies (30) at equality, the second inequality is satisfied from (30) with $(x_i, y_i) = (1, y_i^*)$ and the last inequality follows since $a_i y_i^* \geq 0$. Therefore, equality holds throughout and, in particular, $\alpha_i = 0$. It follows that $\alpha_i(x_i - 1) + \beta_i y_i = \alpha_i x_i + \beta_i y_i$ and, consequently, (α_i, β_i) is valid and maximal to (31).

Now, we fix (x_i, y_i) at $(0, 1)$. Then, we show that any (α_i, β_i) that is valid and maximal to

$$\alpha_i x_i + \beta_i(y_i - 1) \geq P^i(a_i x_i y_i) \tag{32}$$

is also valid and maximal to (31). Let $y_i^* = \min\{y_i \in [0,1] \mid \alpha_i + \beta_i(y_i - 1) = P^i(a_i y_i)\}$, *i.e.*, $(1, y_i^*)$ satisfies (32) at equality. It follows that

$$0 \leq \beta_i \left(y_i^* - 1\right) = P^i\left(a_i y_i^*\right) - \alpha_i \leq P^i\left(a_i y_i^*\right) - P^i(a_i) \leq 0,$$

where the first inequality follows from (32) by substituting $(x_i, y_i) = (0, y_i^*)$, the equality is satisfied since $(1, y_i^*)$ satisfies (32) at equality, the second inequality is verified by substituting $(1, 1)$ in (32), and the last inequality holds since $P^i(\cdot)$ is non-decreasing and $a_i y_i^* \leq a_i$. Therefore, the equality holds throughout and, in particular, $\beta_i(y_i^* - 1) = 0$. It follows that either $\beta_i = 0$ or $y_i^* = 1$. We show that $\beta_i = 0$ in the latter case as well. If $y_i^* = 1$, because lifting is assumed to be maximal and because of the definition of y_i^*, there is a $y_i' \in [0, 1)$ such that $(0, y_i')$ satisfies (32) at equality. Therefore, $\beta_i(y_i' - 1) = 0$ and so $\beta_i = 0$. It follows that $\alpha_i x_i + \beta_i(y_i - 1) = \alpha_i x_i + \beta_i y_i$ and, consequently, (α_i, β_i) is valid and maximal for (31). □

3.2 Subadditivity of lifting functions

In this section, we present two families of functions that are subadditive. These functions will appear as lifting functions of the seed inequalities described in Sect. 2. Direct proofs of these results can be found in [8]; see Propositions 5.17 and 5.22 respectively. An alternate proof technique can also be found in [9].

Corollary 1 *Let* ν *and* D_i *for* $i = 0, 1, \ldots, r$ *be nonnegative integers that satisfy* $\nu > 0$, $D_0 = 0$, *and* $D_i \geq D_{i-1} + \nu$ *for* $i = 1, \ldots, r$. *Then, the function*

$$g_1(w) := \begin{cases} 0 & \text{if } w < D_0 \\ w - i\nu & \text{if } D_i \leq w < D_{i+1} - \nu,\ i = 0, \ldots, r-1, \\ D_i - i\nu & \text{if } D_i - \nu \leq w < D_i, \quad i = 1, \ldots, r-1, \\ D_r - r\nu & \text{if } D_r - \nu \leq w \end{cases}$$

is subadditive over $\mathbb{R}$ *if and only if* $D_i + D_j \geq D_{i+j}$ *for* $0 \leq i \leq j \leq r$ *with* $i + j \leq r$. □

Corollary 1 equivalently shows the superadditivity of $w - g_1(w)$, generalizing prior similar results in the literature. In particular, see Lemmas 6 and 7 in [3] and Definition 4 in [18].

Corollary 2 *Let* λ *and* C_i *for* $i = 0, 1, \ldots, s$ *be nonnegative integers that satisfy* $\lambda > 0$, $C_0 = 0$, *and* $C_{i-1} + \lambda \leq C_i$ *for* $i = 1, \ldots, s$. *Then, the function*

$$g_2(w) = \begin{cases} 0 & \text{if } w < C_0 \\ i + \frac{w - C_i}{\lambda} & \text{if } C_i \leq w < C_i + \lambda, \quad i = 0, \ldots, s, \\ i & \text{if } C_{i-1} + \lambda \leq w < C_i,\ i = 1, \ldots, s, \\ s + 1 & \text{if } C_s + \lambda \leq w. \end{cases}$$

is subadditive over $\mathbb{R}$ *if and only if* $C_i + C_j \leq C_{i+j}$ *for* $0 \leq i \leq j \leq s$ *with* $i + j \leq s$. □

3.3 Lifted inequalities by sequence-independent lifting

In this section, we derive strong inequalities for *PB* through lifting using (19) as seed inequality. To describe the general form of these inequalities, we use the notion of a cover, which is adapted from the definition of a cover for the $0-1$ knapsack polytope; see [4,14,31]. We also use the notation $(x)^+$ to denote $\max\{x, 0\}$ and the notation $(x)^-$ to denote $\min\{x, 0\}$.

Definition 1 *Let* $C \subseteq N$. *We say that* C *is a cover for* B *if* $\sum_{j \in C} a_j > d$. *Further, we define the excess of the cover as* $\mu = \sum_{j \in C} a_j - d > 0$.

We create lifted inequalities by first partitioning the set of variables N into $(C', \{l\}, M, T)$ such that:

(A1) $C := C' \cup \{l\}$ is a cover for B with excess μ,
(A2) $a_l \geq a_j, \forall j \in C'$,
(A3) $a_l > \mu$,
(A4) $\sum_{j \in C \cup T} a_j > d + a_l$, *i.e.*, $\sum_{j \in T} a_j > a_l - \mu$.

Note that (A1) and (A3) might be reminiscent of conditions that make a cover minimal for the $0-1$ knapsack polytope. We note however that minimal covers require $a_j > \mu$ for all $j \in C$ and not simply $a_l > \mu$. Note also that (A4) implies that $T \neq \emptyset$. To obtain lifted inequalities from the partition $(C', \{l\}, M, T)$, we first fix the variables (x_j, y_j) for $j \in M$ to $(0, 0)$ and the variables (x_j, y_j) for $j \in C'$ to $(1, 1)$. The resulting (full-dimensional) set $B(M, C', M, C')$ is then defined by the inequality

$$a_l x_l y_l + \sum_{j \in T} a_j x_j y_j \geq d - \sum_{j \in C'} a_j = a_l - \mu.$$

Since $a_l > \mu$ and $\sum_{j \in T} a_j > a_l - \mu$ from Conditions (A3) and (A4), we conclude from Proposition 9(i) that

$$(a_l - \mu)x_l + \sum_{j \in T} a_j y_j \geq a_l - \mu \tag{33}$$

is facet-defining for $PB(M, C', M, C')$. We will create two different families of lifted inequalities for PB by reintroducing the variables (x_j, y_j) for $j \in M \cup C'$ in different orders. To derive both families, we use the lifting function

$$\begin{aligned} P(w) := \max (a_l - \mu) - &\left\{ (a_l - \mu)x_l + \sum_{j \in T} a_j y_j \right\} \\ s.t.\ a_l x_l y_l + &\sum_{j \in T} a_j x_j y_j \geq a_l - \mu - w \\ &x_j \in \{0, 1\},\ \ y_j \in [0, 1] \quad \forall j \in \{l\} \cup T, \end{aligned} \tag{34}$$

where $w \in \mathbb{R}$. We next derive a closed-form expression for $P(w)$.

Proposition 15 *The lifting function $P(w)$ defined in (34) takes the values*

$$P(w) = \begin{cases} -\infty & \text{if } w < -\sum_{j \in T} a_j - \mu, \\ w + \mu & \text{if } -\sum_{j \in T} a_j - \mu \leq w < -\mu, \\ 0 & \text{if } -\mu \leq w < 0, \\ w & \text{if } 0 \leq w < a_l - \mu, \\ a_l - \mu & \text{if } a_l - \mu \leq w \end{cases}$$

over $\mathbb{R}$. Further, $P(w)$ is subadditive over $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively.

Proof We first derive a closed-form expression for $P(w)$. Observe that, if (34) is feasible, there exists an optimal solution (x^*, y^*) to (34) for which $x_j^* = 1$ for $j \in T$ and $y_l^* = 1$ since the coefficients of x_j for $j \in T$ and y_l in the objective are equal to 0. Defining $\bar{a} = \sum_{j \in T} a_j$ and $\bar{y} = \frac{\sum_{j \in T} a_j y_j}{\bar{a}}$, we can simplify the formulation of $P(w)$ in (34) as:

$$\begin{aligned} P(w) = \max (a_l - \mu) - &\{(a_l - \mu)x_l + \bar{a}\bar{y}\} \\ s.t.\ a_l x_l + &\bar{a}\bar{y} \geq a_l - \mu - w \\ x_l \in &\{0, 1\},\ \ \bar{y} \in [0, 1]. \end{aligned} \tag{35}$$

When $w < -\bar{a} - \mu$, (35) is infeasible and so $P(w) = -\infty$. When $w \geq a_l - \mu$, the optimal solution is $x_l^* = 0$ and $\bar{y}^* = 0$ with $P(w) = a_l - \mu$. For $-\bar{a} - \mu \leq w < a_l - \mu$, there are two cases. When $-\bar{a} - \mu \leq w < a_l - \bar{a} - \mu$, every feasible solution $(x_l^*, \bar{y}^*)$ has $x_l^* = 1$. Further, the optimal solution has $\bar{y}^* = \max\{\frac{-\mu - w}{\bar{a}}, 0\}$. It follows that $P(w) = \min\{w + \mu, 0\}$. When $a_l - \bar{a} - \mu \leq w \leq a_l - \mu$, an optimal solution must be found among the solutions $(1, \frac{(-\mu-w)^+}{\bar{a}})$ and $(0, \frac{a_l - \mu - w}{\bar{a}})$. It follows that $P(w) = \max\{(w + \mu)^-, w\}$ from which we obtain the desired expression for $P(w)$ after considering both the cases where $a_l - \bar{a} < 0$ and $a_l - \bar{a} \geq 0$.

Subadditivity of $P(w)$ over $\mathbb{R}_-$ and $\mathbb{R}_+$ follows from Karamata/Hardy-Littlewood-Polya inequality [15], concavity of $P(w)$ over these domains, and $P(0) = 0$. □

We note that, although $P(w)$ is subadditive over $\mathbb{R}_+$ and over $\mathbb{R}_-$, $P(w)$ is not subadditive over $\mathbb{R}$ as $P(2a_l - \mu) + P(-a_l) = (a_l - \mu) + (-a_l + \mu) = 0 < a_l - \mu = P(a_l - \mu)$.

3.3.1 Lifted bilinear cover inequalities

To obtain lifted bilinear cover inequalities, we will lift first the variables (x_i, y_i) for $i \in C'$ from $(1, 1)$ and then lift the variables (x_i, y_i) for $i \in M$ from $(0, 0)$. Since $P(w)$ is subadditive over $\mathbb{R}_-$, we can apply sequence-independent lifting for the variables (x_i, y_i) for $i \in C'$.

Proposition 16 *Under Conditions* (*A*1), (*A*2), (*A*3) *and* (*A*4),

$$\sum_{j \in C} (a_j - \mu)^+ x_j + \sum_{j \in T} a_j y_j \geq \sum_{j \in C} (a_j - \mu)^+ \tag{36}$$

is facet-defining for $PB(M, \emptyset, M, \emptyset)$.

Proof The seed inequality (33) is facet-defining for the full-dimensional polytope $PB(M, C', M, C')$. Since $P(w)$ is subadditive over $\mathbb{R}_-$, we obtain from (29) that the lifting coefficients (α_i, β_i) for (x_i, y_i) for $i \in C'$ are valid if they satisfy

$$\alpha_i(x_i - 1) + \beta_i(y_i - 1) \geq P(a_i x_i y_i - a_i) \quad \text{for } (x_i, y_i) \in \{0, 1\} \times [0, 1] \backslash \{1, 1\}. \tag{37}$$

This condition can be also written as

$$\beta_i \leq \inf_{0 \leq \phi < 1} \frac{-P(a_i \phi - a_i)}{1 - \phi}, \tag{38}$$

$$\alpha_i + \sup_{0 \leq \phi \leq 1} \beta_i(1 - \phi) \leq -P(-a_i). \tag{39}$$

From Conditions (A2) and (A4), we know that $a_i \leq a_l < \sum_{j \in T} a_j + \mu$, $\forall i \in C'$. Therefore, in (38), $a_i \phi - a_i \in (-\sum_{j \in T} a_j - \mu, 0)$ for all $\phi \in [0, 1)$. Since $P(w) \leq 0$ for $w \leq 0$, we conclude that

$$\frac{-P(a_i \phi - a_i)}{1 - \phi} \geq 0, \quad \forall\, 0 \leq \phi < 1,$$

and therefore choosing $\beta_i = 0$ for $i \in C'$ satisfies (38). Further, as $\beta_i = 0$, it is simple to verify that choosing $\alpha_i = -P(-a_i) = (a_i - \mu)^+$ satisfies (39). Finally, observe that (37) is satisfied at equality by the two points $(0, 0)$ and $\left(1, \frac{(a_i-\mu)^+}{a_i}\right)$ that are affinely independent of $(1, 1)$. Therefore, we conclude that (36) is facet-defining for $PB(M, \emptyset, M, \emptyset)$. □

Now, we lift the variables (x_j, y_j) for $j \in M$ in (36). The corresponding lifting function is

$$\begin{aligned} P^C(w) := \max \sum_{j\in C}(a_j-\mu)^+ - &\left\{\sum_{j\in C}(a_j-\mu)^+x_j + \sum_{j\in T}a_jy_j\right\} \\ s.t. \sum_{j\in C\cup T} a_jx_jy_j &\geq \sum_{j\in C} a_j - \mu - w \\ x_j \in \{0,1\},\ y_j &\in [0,1] \quad \forall j \in C\cup T. \end{aligned} \tag{40}$$

We next derive a closed-form expression for $P^C(w)$. To this end, we assume without loss of generality that $C = \{1, \ldots, p\}$ and that $a_1 \geq a_2 \geq \cdots \geq a_p$. Let $q = \max\{j \in C \mid a_j > \mu\}$. We define $A_0 = 0$ and $A_i = \sum_{j=1}^{i} a_j$ for all $i \in \{1, \ldots, q\}$.

Proposition 17 *For $w \geq 0$,*

$$P^C(w) = \begin{cases} w - i\mu & \text{if } A_i \leq w < A_{i+1} - \mu,\ i = 0, \ldots, q-1, \\ A_i - i\mu & \text{if } A_i - \mu \leq w < A_i, \quad i = 1, \ldots, q-1, \\ A_q - q\mu & \text{if } A_q - \mu \leq w. \end{cases}$$

Proof First, observe that there exists an optimal solution (x^*, y^*) of (40) in which $x_j^* = 1$ for $j \in T$ and $y_j^* = 1$ for $j \in C$ since the corresponding objective coefficients are zero. It follows from the definition of q that $a_q > \mu \geq a_{q+1}$. We thus have $(a_j - \mu)^+ = 0$ for $j = q+1, \ldots, p$, which implies that we can assume $x_j^* = 1$ for $j = q+1, \ldots, p$. Defining $\bar{a} = \sum_{j\in T} a_j$ and $\bar{y} = \frac{\sum_{j\in T} a_jy_j}{\bar{a}}$, we simplify the expression of $P^C(w)$ as

$$\begin{aligned} P^C(w) = \max \sum_{j=1}^{q}(a_j-\mu) - &\left\{\sum_{j=1}^{q}(a_j-\mu)x_j + \bar{a}\bar{y}\right\} \\ s.t. \sum_{j=1}^{q} a_jx_j + \bar{a}\bar{y} &\geq A_q - \mu - w \\ x_j \in \{0,1\}, \quad \forall j = 1\ldots,q,\ &\bar{y} \in [0,1]. \end{aligned} \tag{41}$$

Next, we solve (41). When $w \geq A_q - \mu$, it is clear that $x_j^* = 0$ for $j = 1, \ldots, q$ and $\bar{y}^* = 0$ is an optimal solution for (41), showing that $P^C(w) = A_q - q\mu$. It is therefore sufficient to consider $w \in [0, A_q - \mu)$. We distinguish two cases:

1. Assume that $A_i - \mu \leq w < A_{i+1} - \mu$ for $i \in \{1\ldots, q-1\}$. Let $\theta = (A_{i+1} - \mu) - w$. Clearly, $0 < \theta \leq a_{i+1}$. Define first the solution $(x^*, \bar{y}^*)$ where $x_j^* = 0$ for $j = 1, \ldots, i+1$, $x_j^* = 1$ for $j = i+2, \ldots, q$, and $\bar{y}^* = \frac{\theta}{\bar{a}}$. When $\theta \leq \bar{a}$, $(x^*, \bar{y}^*)$ is a feasible solution to (41) with objective value $z^* = A_{i+1} - (i+1)\mu - \theta = w - i\mu$. Next consider the solution $(x', \bar{y}')$ where $x'_j = 0$ for $j = 1, \ldots, i$, $x'_j = 1$ for $j = i+1, \ldots, q$, and $\bar{y}' = 0$. Solution $(x', \bar{y}')$ is feasible to (41) and has objective value $z' = A_i - i\mu$. It is clear that $z^* \geq z'$ when $\theta \leq a_{i+1} - \mu$ and that $z' \geq z^*$ when $a_{i+1} - \mu \leq \theta \leq a_{i+1}$. Further, solution $(x^*, \bar{y}^*)$ is feasible when $\theta \leq a_{i+1} - \mu$ as $a_{i+1} - \mu \leq a_1 - \mu \leq \bar{a}$ because of Condition (A4). Therefore, we conclude that $P^C(w) \geq w - i\mu$ if $A_i \leq w \leq A_{i+1} - \mu$ and $P^C(w) \geq A_i - i\mu$ if $A_i - \mu \leq w < A_i$. We now prove that the proposed solutions are optimal. Pick any feasible solution $(x^\circ, \bar{y}^\circ)$ to (41). Define $N_1 = \{j \in \{1, \ldots, q\} \mid x_j^\circ = 1\}$. Consider first the case where $|N_1| = q - i + k$ for $k \in \{0, \ldots, i\}$. Since $\sum_{j=1}^{q} a_j x_j^\circ + \bar{a}\bar{y}^\circ \geq \sum_{j=1}^{q} a_j x_j^\circ \geq A_q - A_{i-k}$, the objective value associated with $(x^\circ, \bar{y}^\circ)$ satisfies $z^\circ = \sum_{j=1}^{q}(a_j - \mu)(1 - x_j^\circ) - \bar{a}\bar{y}^\circ \leq A_{i-k} - (i-k)\mu = A_i - i\mu - \sum_{j=i-k+1}^{i}(a_j - \mu) \leq z'$. Second, consider the case where $|N_1| = q - i - k$ for $k \in \{1, \ldots, q-i\}$. Since $\sum_{j=1}^{q} a_j x_j^\circ + \bar{a}\bar{y}^\circ \geq A_q - A_{i+1} + \theta$ from feasibility, the corresponding objective value is $z^\circ = \sum_{j=1}^{q}(a_j - \mu)(1 - x_j^\circ) - \bar{a}\bar{y}^\circ \leq A_{i+1} - \theta - (i+k)\mu \leq z^*$. Since whenever $z^* \geq z'$, the solution $(x^*, \bar{y}^*)$ corresponding to z^* is feasible, the result is proven.
2. Assume that $0 \leq w < A_1 - \mu$. An argument similar to that presented above shows that the feasible solution $x_1^* = 0$, $x_j^* = 1$ for $j = 2, \ldots, q$, and $\bar{y}^* = \frac{A_1 - \mu - w}{\bar{a}}$ is optimal for (41), which implies that $P^C(w) = w$. □

In the following result, we argue that $P^C(w)$ is subadditive. This result enables us to use Proposition 13 to perform sequence-independent lifting for the variables in M.

Corollary 3 *The lifting function $P^C(w)$ is subadditive over $\mathbb{R}_+$.*

Proof In Corollary 1, define $\nu = \mu$, $r = q$, and $D_i = A_i$. Since $a_i \geq \mu$ for $i = 1, \ldots, q$, it follows that $A_i \geq A_{i-1} + \mu$. Further, since A_i is the sum of the largest i coefficients in C, it is clear that $A_i + A_j \geq A_{i+j}$ for $0 \leq i, j \leq q$ with $i + j \leq q$. Therefore, Corollary 1 shows that $P^C(w)$ is subadditive over $\mathbb{R}_+$. □

We next illustrate the results of Proposition 16, Proposition 17, and Corollary 3 on an example.

Example 2 Consider the $0-1$ *mixed-integer bilinear covering set*

$$B = \left\{ (x, y) \in \{0, 1\}^5 \times [0, 1]^5 \;\middle|\; 21x_1y_1 + 19x_2y_2 + 17x_3y_3 + 15x_4y_4 + 10x_5y_5 \geq 20 \right\}.$$

Let $(C', \{l\}, M, T) = (\{5\}, \{4\}, \{1, 2\}, \{3\})$. *Clearly,* $(C', \{l\}, M, T)$ *satisfies Conditions (A1)–(A4) since* $C = C' \cup \{l\}$ *is a cover with* $\mu = 5$, $a_4 \geq a_5$, $a_4 > \mu$ *and* $\sum_{j \in C \cup T} a_j = 17 + 15 + 10 > 20 + 15 = d + a_l$. *By Proposition 16, the inequality*

$$17y_3 + 10x_4 + 5x_5 \geq 15 \tag{42}$$

is facet-defining for $PB(M, \emptyset, M, \emptyset)$*. Using Proposition 17, the lifting function* $P^C(w)$ *is given by*

$$P^C(w) = \begin{cases} w & \text{if } 0 \le w < 10, \\ 10 & \text{if } 10 \le w < 15, \\ w-5 & \text{if } 15 \le w < 20, \\ 15 & \text{if } 20 \le w. \end{cases}$$

Function $P^C(w)$ *is shown in Fig. 1(a). Corollary 3 shows that this function is subadditive over* $\mathbb{R}_+$*.*

We now compute the lifting coefficients of variables (x_i, y_i) for $i \in M$ from $P^C(w)$. It follows from Proposition 13 that lifting coefficients (α_i, β_i) for $i \in M$ must be chosen to satisfy

$$\alpha_i x_i + \beta_i y_i \ge P^C(a_i x_i y_i) \quad \text{for } (x_i, y_i) \in \{0, 1\} \times [0, 1] \backslash \{0, 0\}. \tag{43}$$

For the problem described in Example 2, $P^C(a_1 x_1 y_1)$ is represented in Fig. 1(b). In this figure, we observe that $P^C(a_1 x_1 y_1)$ is equal to zero when $x_1 = 0$ and is equal to $P^C(a_1 y_1)$ when $x_1 = 1$. Condition (43) requires that the lifting coefficients (α_1, β_1) be chosen in such a way that the plane $\alpha_1 x_1 + \beta_1 y_1$ (passing through the origin (0, 0)) overestimates the function $P^C(a_1 x_1 y_1)$ over $\{0, 1\} \times [0, 1]$. To obtain strong lifting coefficients, the plane created must touch the function described in Fig. 1(b) in at least two independent points that are not (0, 0). Intuitively, there are two ways of selecting these points. The first is to have the plane pass through the point (0, 1) and a point of the form $(1, y_i)$. The second is to have the plane pass through two points $(1, y_1)$ and $(1, y_1')$. In the second case, the line passing through $(y_1, P^C(a_1 y_1))$ and $(y_1', P^C(a_1 y_1'))$ must be a facet of the concave envelope of $P^C(a_1 y)$ over $0 \le y \le 1$. This is because, by (43) it overestimates $P^C(a_1 y)$ over the region. On the other hand, any linear overestimator of $P^C(a_1 y)$ is implied by the facets of the concave envelope. A similar geometric interpretation was used in [24] to obtain lifted inequalities for 0−1 mixed-integer bilinear knapsack sets. Possible overestimating planes are represented in Fig. 1(c). It is now clear from Fig. 1(c) that good overestimating planes $\alpha_1 x_1 + \beta_1 y_1$ are in direct correspondence with the concave envelope $p(w)$ of $P^C(w)$ over $[0, a_1]$. We also mention that because the concave envelope of $P^C(w)$ over $[0, a_1]$ is different from that over $[0, a_2]$, new functions $p^i(w)$ will have to be built for each new pair of variables (x_i, y_i) that is lifted. These observations motivate the following result.

Lemma 1 *For* $i \in M$*, define*

$$q_i := \begin{cases} 0 & \text{if } a_i \le A_1 - \mu, \\ j & \text{if } A_j - \mu < a_i \le A_{j+1} - \mu, \ j = 1, \dots, q-1, \\ q & \text{if } A_q - \mu < a_i. \end{cases}$$

Let $Q_0^i = 0$*,* $Q_j^i = A_j - \mu$ *for* $j = 1, \dots, q_i$ *and* $Q_{q_i+1}^i = a_i$*. Further, define* $\Delta_j^i = Q_{j+1}^i - Q_j^i$ *for* $j = 0, \dots, q_i$*. Define* $p_j^i(w) = P^C(Q_j^i) + \frac{P^C(Q_{j+1}^i) - P^C(Q_j^i)}{\Delta_j^i}(w - Q_j^i)$ *for* $j = 0, \dots, q_i$*. Then, the function*

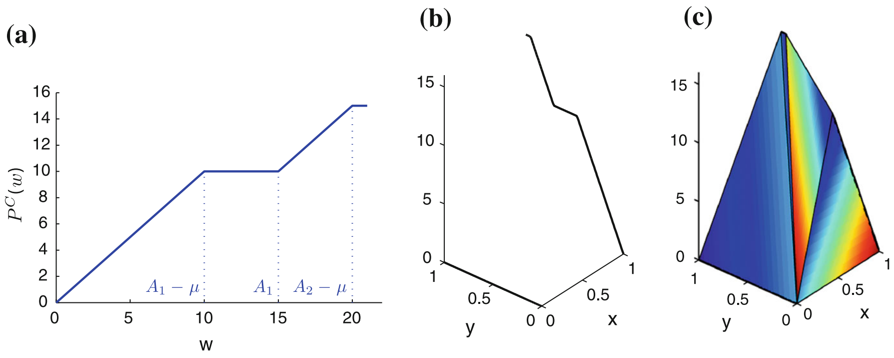

Fig. 1 Deriving lifting coefficients for Example 3

$$p^i(w) := \min\left\{p^i_j(w) \;\middle|\; j \in \{0, \ldots, q_i\}\right\} \tag{44}$$

is a concave overestimator of $P^C(w)$ *over* $[0, a_i]$.

Proof Clearly, $p^i(w)$ is concave since it is defined as the minimum of affine functions. We next verify that it overestimates $P^C(w)$, *i.e.*, $P^C(w) \le p^i_k(w)$ for all k and w. Consider $w \in [Q^i_j, Q^i_{j+1}]$. It follows from construction that $P^C(w) \le p^i_j(w)$. It remains to argue that $P^C(w) \le p^i_k(w)$ when $k \ne j$. To this end, observe that, for $j = 0, \ldots, q_i$, $p^i_j(w) \ge p^i_{j+1}(w)$ when $w \ge Q^i_{j+1}$ and $p^i_j(w) \le p^i_{j+1}(w)$ when $w < Q^i_{j+1}$. □

Observe that the concave overestimator of $P^C(w)$ derived in Lemma 1 has $q_i + 1$ linear pieces. Also note that the definition of q_i implies that $\Delta^i_j > 0$ for all $j = 0, \ldots, q_i$. Next, we compute maximal lifting coefficients for the variables (x_i, y_i) where $i \in M$ using sequence-independent lifting; see Proposition 13 and Lemma 1.

Theorem 1 *Under Conditions* (*A*1)*,* (*A*2)*,* (*A*3) *and* (*A*4)*, the lifted bilinear cover inequality*

$$\sum_{j\in C}(a_j - \mu)^+ x_j + \sum_{j\in T} a_j y_j + \sum_{i\in M} \alpha_i x_i + \sum_{i\in M} \beta_i y_i \ge \sum_{j\in C}(a_j - \mu)^+ \tag{45}$$

is facet-defining for PB if

$$(\alpha_i, \beta_i) \in \bigcup_{j=0}^{q_i} \left\{ \left(P^C\left(Q^i_j\right) - \frac{P^C(Q^i_{j+1}) - P^C\left(Q^i_j\right)}{\Delta^i_j} Q^i_j, \; \frac{P^C\left(Q^i_{j+1}\right) - P^C\left(Q^i_j\right)}{\Delta^i_j} a_i \right) \right\}$$
$$\bigcup \left\{ \left(P^C(a_i), 0\right) \right\}$$

for $i \in M$ *in* (45) *where* Q^i_j*,* Δ^i_j*, and* q_i *are as defined in Lemma* 1.

Proof Because $P^C(w)$ is subadditive over $\mathbb{R}_+$, we know that (45) is valid for *PB* if the lifting coefficients (α_i, β_i) of (x_i, y_i) for $i \in M$ are chosen to satisfy the condition

$$\alpha_i x_i + \beta_i y_i \geq P^C(a_i x_i y_i) \quad \text{for } (x_i, y_i) \in \{0, 1\} \times [0, 1] \backslash \{0, 0\}. \tag{46}$$

Condition (46) can be rewritten as

$$\beta_i \phi \geq P^C(0) \qquad \text{for} \quad 0 < \phi \leq 1, \tag{47}$$

$$\alpha_i + \beta_i \phi \geq P^C(a_i \phi) \quad \text{for} \quad 0 \leq \phi \leq 1. \tag{48}$$

To prove that (45) is facet-defining for *PB*, we also need to show two linearly independent points (x_i, y_i) for which (46) is satisfied at equality. First, consider the case where $(\alpha_i, \beta_i) = (P^C(a_i), 0)$. Condition (47) is satisfied since $\beta_i = 0$ and $P^C(0) = 0$. Condition (48) also holds because $\alpha_i = P^C(a_i)$ and $P^C(w)$ is non-decreasing over $\mathbb{R}_+$. Further, (46) is satisfied at equality at the two points, (0, 1) and (1, 1). Finally, consider

$$(\alpha_i, \beta_i) = \left(P^C\left(Q^i_j\right) - \frac{P^C\left(Q^i_{j+1}\right) - P^C\left(Q^i_j\right)}{\Delta^i_j} Q^i_j, \frac{P^C\left(Q^i_{j+1}\right) - P^C\left(Q^i_j\right)}{\Delta^i_j} a_i \right)$$

for any $j \in \{0, \ldots, q_i\}$. Clearly, (α_i, β_i) satisfies (47) since $\beta_i \geq 0$ and $P^C(0) = 0$. From Lemma 1, we have that

$$\begin{aligned} P^C(a_i \phi) &\leq P^C\left(Q^i_j\right) + \frac{P^C\left(Q^i_{j+1}\right) - P^C\left(Q^i_j\right)}{\Delta^i_j}\left(a_i \phi - Q^i_j\right) \\ &= \left(P^C\left(Q^i_j\right) - \frac{P^C\left(Q^i_{j+1}\right) - P^C\left(Q^i_j\right)}{\Delta^i_j} Q^i_j \right) \\ &\quad + \frac{P^C\left(Q^i_{j+1}\right) - P^C\left(Q^i_j\right)}{\Delta^i_j} a_i \phi \\ &= \alpha_i + \beta_i \phi, \end{aligned}$$

showing that (α_i, β_i) satisfy (48) for $j = 0, \ldots, q_i$. Further, (46) is satisfied at equality at the two points $\left(1, \frac{Q^i_j}{a_i}\right)$ and $\left(1, \frac{Q^i_{j+1}}{a_i}\right)$. Therefore, we conclude that (45) is facet-defining for *PB*. □

The concave overestimator $p^i(w)$ of Lemma 1 is in fact the concave envelope of $P^C(w)$ over $w \in [0, a_i]$. The concave envelope of $P^C(a_i x y)$ over $\{0, 1\} \times [0, 1]$ implicit in the proof of Theorem 1 can also be obtained using the technique for constructing envelopes of functions that satisfy pairwise complementarity described in [28]. We refer to Sect. 3 of [28] for definitions and, in particular, Proposition 3 therein for relevant constructions. The same construction also yields the concave envelope of $\Psi(a_i xy)$ over $\{0, 1\} \times [0, 1]$ proved later in Theorem 3 using the concave overestimator of $\Psi(w)$ derived in Lemma 2.

Recall that Fig. 1(b) depicts $P^C(a_1 x_1 y_1)$ for inequality (42). Observe that in Fig. 1(c), lifting coefficients $(0, a_1)$ define the plane passing through $(0, 0)$ and $(1, 0)$ while lifting coefficients $(P^C(a_i), 0)$ define the plane passing through $(0, 0)$ and $(0, 1)$ (which is identical to the plane obtained when $j = q_1 = 2$). Since there are several choices for the values of each of the pairs of lifting coefficients (α_i, β_i), the family of inequalities (45) contains an exponential number of members. Theorem 1 therefore provides a new illustration that sequence-independent lifting from a single seed inequality can produce exponentially large families of inequalities, a property that was discussed in a more general setting in Sect. 2 of [24]. We illustrate this characteristic of lifted bilinear cover inequalities in Example 3.

Example 3 In Example 2, we established that (42) is facet-defining for $PB(M, \emptyset, M, \emptyset)$ and described the corresponding lifting function $P^C(w)$. We compute that $q_1 = 2$ (with $Q_0^1 = 0$, $Q_1^1 = 10$, $Q_2^1 = 20$, $Q_3^1 = 21$) and $q_2 = 1$ (with $Q_0^2 = 0$, $Q_1^2 = 10$, $Q_2^2 = 19$). Applying Theorem 1, we obtain the nine inequalities

$$\left\{ \begin{array}{ll} & 21y_1 \\ 5x_1 & +\frac{21}{2}y_1 \\ 15x_1 & \end{array} \right\} + \left\{ \begin{array}{ll} & 19y_2 \\ \frac{50}{9}x_2 & +\frac{76}{9}y_2 \\ 14x_2 & \end{array} \right\} + 17y_3 + 10x_4 + 5x_5 \geq 15$$

which are all facet-defining for PB. The three possible choices for the lifting coefficients of (x_1, y_1) are depicted in Fig. 1(c). The fact that there are three possible choices for (x_2, y_2) follows similarly, with the exception that coefficient a_2 falls in the second interval $(A_1 - \mu, A_2 - \mu]$.

Another look at Fig. 1(b) also suggests that if we had fixed (x_1, y_1) at $(0, 1)$ or $(1, 0)$, we would only have been able to obtain a single lifted inequality and so fixing variables at $(0, 0)$ in this case is crucial in discovering the exponential family of lifted inequalities. This provides a graphical illustration of Proposition 14, which states that all interesting lifting coefficients that can be obtained from fixing variables at $(0, 1)$ or $(1, 0)$ can also be obtained from fixing variables at $(0, 0)$. We also note that, although there is typically an exponential number of lifted bilinear cover inequalities that can be generated from a given seed inequality, it is simple to determine the one that is most violated since lifting is sequence-independent in this case. In fact, given a fractional solution (x^*, y^*), it suffices to choose, for each $i \in M$, the coefficients (α_i, β_i) for which the quantity $\alpha_i x_i^* + \beta_i y_i^*$ is minimized.

3.3.2 Lifted reverse bilinear cover inequalities

In Theorem 1, we derived lifted bilinear cover inequalities by first lifting the variables (x_j, y_j) for $j \in C'$ and then lifting the variables (x_j, y_j) for $j \in M$. Here, we derive another family of lifted inequalities that we call *lifted reverse bilinear cover inequalities* by changing the lifting order: we start the lifting procedure with the same seed inequality (33), but we now lift the variables (x_j, y_j) for $j \in M$ before the variables (x_j, y_j) for $j \in C'$. In this case, we do not assume that $a_l \geq a_i$ for $i \in C$, *i.e.*, we do not require Condition (A2).

Proposition 18 *Under Conditions (A1), (A3), and (A4), the inequality*

$$(a_l - \mu)x_l + \sum_{j \in M} \min\{a_j, a_l - \mu\}x_j + \sum_{j \in T} a_j y_j \geq a_l - \mu \tag{49}$$

is facet-defining for $PB(\emptyset, C', \emptyset, C')$.

Proof The proof follows from Proposition 9 by letting $N = M \cup \{l\} \cup T$ and $L = M \cup \{l\}$. □

For brevity, we included a direct proof here based on Proposition 9. Proposition 18 can also be derived by lifting (33); see [9]. We emphasize that the above result does not require Condition (A2).

To obtain facet-defining inequalities for PB, we lift the remaining variables (x_j, y_j) for $j \in C'$ in (49). To this end, we first compute the function

$$\begin{aligned} P^M(w) := \max (a_l - \mu) - &\left\{ (a_l - \mu)x_l + \sum_{j \in M} \min\{a_j, a_l - \mu\}x_j + \sum_{j \in T} a_j y_j \right\} \\ s.t.\ a_l x_l y_l + &\sum_{j \in M \cup T} a_j x_j y_j \geq a_l - \mu - w \\ &x_j \in \{0, 1\},\ \ y_j \in [0, 1] \quad \forall j \in \{l\} \cup M \cup T \end{aligned} \tag{50}$$

for $w \in \mathbb{R}_-$.

Let $M = M_1 \cup M_2$ where $M_1 = \{i \in M \mid a_i > a_l - \mu\}$ and $M_2 = M \backslash M_1$. Assume without loss of generality that $\{l\} \cup M_1 = \{1, \ldots, q\}$ and $a_1 \geq a_2 \geq \cdots \geq a_q$ where $q = |M_1| + 1$. Further, define $A_0 = 0$ and $A_i = \sum_{j=1}^{i} a_j$ for $i = 1, \ldots, q$. Observe that $a_l + \sum_{j \in M \cup T} a_j = A_q + \sum_{j \in M_2} a_j + \sum_{j \in T} a_j$. We derive a closed-form expression for $P^M(w)$ in the following proposition.

Proposition 19 *The lifting function* $P^M(w)$ *defined in (50) takes the values*

$$P^M(w) = \begin{cases} -\infty & \text{if } w < -\mu - \sum_{j \in M \cup T} a_j, \\ w + A_q - q(a_l - \mu) & \text{if } -\mu - \sum_{j \in M \cup T} a_j \leq w < -A_q + (a_l - \mu), \\ -i(a_l - \mu) & \text{if } -A_{i+1} + (a_l - \mu) \leq w < -A_i, \qquad i = 0, \ldots, q-1, \\ w + A_i - i(a_l - \mu) & \text{if } -A_i \leq w < -A_i + (a_l - \mu), \qquad i = 1, \ldots, q-1, \end{cases}$$

for $w \in \mathbb{R}_-$.

Proof First, we observe that, if (50) has a feasible solution, then it has an optimal solution (x^*, y^*) that satisfies $x_j^* = 1$ for $j \in T$ and $y_j^* = 1$ for $j \in M \cup \{l\}$ since the objective coefficients corresponding to these variables are zero. Using the notation $\bar{a} = \sum_{j \in T} a_j$ and $\bar{y} = \frac{\sum_{j \in T} a_j y_j}{\bar{a}}$, we simplify the expression of $P^M(w)$ as

$$
\begin{aligned}
P^M(w) = \max(a_l - \mu) - &\left\{ \sum_{j\in\{l\}\cup M_1} (a_l-\mu)x_j + \sum_{j\in M_2} a_j x_j + \bar{a}\bar{y} \right\} \\
s.t. &\sum_{j\in\{l\}\cup M_1} a_j x_j + \sum_{j\in M_2} a_j x_j + \bar{a}\bar{y} \ge a_l - \mu - w \\
&x_j \in \{0,1\} \quad \forall j \in \{l\}\cup M_1 \cup M_2,\ \bar{y}\in[0,1].
\end{aligned} \tag{51}
$$

After introducing $\hat{a} = \sum_{j\in M_2} a_j + \bar{a}$ and $\hat{y} = \frac{\sum_{j\in M_2} a_j x_j + \bar{a}\bar{y}}{\hat{a}}$, we claim that $P^M(w)$ can be written as

$$
\begin{aligned}
P^M(w) = \max(a_l - \mu) - &\left\{ \sum_{j=1}^{q} (a_l-\mu)x_j + \hat{a}\hat{y} \right\} \\
s.t. &\sum_{j=1}^{q} a_j x_j + \hat{a}\hat{y} \ge a_l - \mu - w \\
&x_j \in \{0,1\} \quad \forall j \in \{1,\dots,q\},\ \hat{y}\in[0,1].
\end{aligned} \tag{52}
$$

We next prove that (51) and (52) are equivalent. To do so, we show that (51) has a feasible solution $(x_l^*, x_{M_1}^*, x_{M_2}^*, \bar{y}^*)$ with objective value ζ^* if and only if (52) has a feasible solution $(x_l^*, x_{M_1}^*, \hat{y}^*)$ with objective value ζ^*. On the one hand, given $(x_l^*, x_{M_1}^*, x_{M_2}^*, \bar{y}^*)$, we can obtain $(x_l^*, x_{M_1}^*, \hat{y}^*)$ directly from the definition of $\hat{y}$. The objective values of these two solutions are identical. On the other hand, let $M_2 = \{q+1,\dots,m\}$. Define $\hat{A}_0 = 0$ and $\hat{A}_i = \sum_{j=q+1}^{q+i} a_j$ for $i = 1,\dots,m-q$. Then, for a given $(x_l^*, x_{M_1}^*, \hat{y}^*)$, we build $(x_l^*, x_{M_1}^*, x_{M_2}^*, \bar{y}^*)$ as follows. Define $\hat{m} = \max\{i \in \{0,\dots,m-q\} \mid \hat{A}_i \le \hat{a}\hat{y}^*\}$ and set $x_{q+j}^* = 1$ for $j \le \hat{m}$, $x_{q+j}^* = 0$ for $j > \hat{m}$ and $\bar{y}^* = \frac{\hat{a}\hat{y}^* - \hat{A}_{\hat{m}}}{\bar{a}}$. We argue next that this solution is feasible. First observe that $\hat{a}\hat{y}^* - \hat{A}_{\hat{m}} \le a_{q+\hat{m}+1}$ when $\hat{m} \le m-q-1$ and that $\hat{a}\hat{y}^* - \hat{A}_{\hat{m}} \le \bar{a}$ when $\hat{m} = m-q$. Since $\bar{a} = \sum_{j\in T} a_j > a_l - \mu \ge a_i$ for all $i \in M_2$ because of Condition (A4) and the definition of M_2, we easily conclude that $0 \le \frac{\hat{a}\hat{y}^* - \hat{A}_{\hat{m}}}{\bar{a}} \le 1$. Also, $\sum_{j\in\{l\}\cup M_1} a_j x_j^* + \sum_{j\in M_2} a_j x_j^* + \bar{a}\bar{y}^* = \sum_{j\in\{l\}\cup M_1} a_j x_j^* + \hat{A}_{\hat{m}} + \hat{a}\hat{y}^* - \hat{A}_{\hat{m}} = \sum_{j\in\{l\}\cup M_1} a_j x_j^* + \hat{a}\hat{y}^*$. This shows that the proposed solution is feasible for (51) and has the same objective value as $(x_l^*, x_{M_1}^*, \hat{y}^*)$.

Next, we study (52). It is clear that this problem is infeasible if and only if $w < a_l - \mu - A_q - \hat{a} = -\mu - \sum_{j\in M\cup T} a_j$. Therefore, assume that $w \ge -\mu - \sum_{j\in M\cup T} a_j$. Consider now any optimal solution $(x^*, \hat{y}^*)$ for which $x_i^* < x_t^*$ and $i < t$ for some $i, t \in \{1\dots,q\}$. Then the solution $(\bar{x}, \hat{y}^*)$ where $\bar{x}_k = x_k^*$ if $k \ne i$ and $k \ne t$, $\bar{x}_i = x_t^*$, and $\bar{x}_t = x_i^*$ is also feasible for (52) since $a_i \ge a_t$ and has the same objective value as $(x^*, \hat{y}^*)$. It follows that (52) has an optimal solution that satisfies $x_j^* = 0$ for $j = 1,\dots,i$ and $x_j^* = 1$ for $j = i+1,\dots,q$ for some $i \in \{0,\dots,q\}$. Consider such a solution further. On the one hand, if $\sum_{j=1}^{i} a_j \ge a_l - \mu - w$, then $\sum_{j=1}^{i-1} a_j < a_l - \mu - w$ and $\hat{y}^* = 0$. Otherwise the solution $x_j^\circ = 1$ for $j = 1,\dots,i-1$,

$x_j^\circ = 0$ for $j = i, \ldots, q$ and $\hat{y}^\circ = 0$ would be feasible and would have a better objective value. On the other hand, if $\sum_{j=1}^{i} a_j < a_l - \mu - w$ for $i \le q - 1$ then $\sum_{j=1}^{i+1} a_j \ge a_l - \mu - w$. Otherwise the solution $x_j^\circ = 1$ for $j = 1, \ldots, i + 1$, $x_j^\circ = 0$ for $j = i + 2, \ldots, q$ and $\hat{y}^\circ = \hat{y}^* - \frac{a_{i+1}}{\hat{a}}$ would be feasible and would have an objective value $a_{i+1} - (a_l - \mu)$ larger than that of (x^*, y^*). This is a contradiction since $a_{i+1} > a_l - \mu$.

We consider two situations. First, assume $-A_q+(a_l-\mu)-\hat{a} \le w < -A_q+(a_l-\mu)$. It follows from the above discussion that there is an optimal solution $(x^*, \hat{y}^*)$ with $x^* = \mathbf{1}$. Then $\hat{y}^* = \frac{a_l-\mu-w-A_q}{\hat{a}}$. Clearly, $\hat{y}^* \in [0, 1]$ and so $P^M(w) = w + A_q - q(a_l - \mu)$. Second, assume $-A_{i+1} + (a_l - \mu) \le w < -A_i + (a_l - \mu)$ for some $i \in \{0, \ldots, q - 1\}$. It follows from the above discussion that one of the following two solutions

$$x_1^\Upsilon = x_2^\Upsilon = \cdots = x_{i+1}^\Upsilon = 1, \quad x_{i+2}^\Upsilon = \cdots = x_q^\Upsilon = 0, \quad \hat{y}^\Upsilon = 0, \text{ and}$$
$$x_1^\Lambda = x_2^\Lambda = \cdots = x_i^\Lambda = 1, \quad x_{i+1}^\Lambda = \cdots = x_q^\Lambda = 0, \quad \hat{y}^\Lambda = \frac{a_l - \mu - w - A_i}{\hat{a}}$$

with objective values $z^\Upsilon = -i(a_l - \mu)$ and $z^\Lambda = -i(a_l - \mu) + (w + A_i)$ is optimal for (52) since $a_l - \mu - w \in (A_i, A_{i+1}]$. Note that the second solution is feasible only when $a_l - \mu - w - A_i \le \hat{a}$. We now consider two cases. When $w \le -A_i$ then $z^\Upsilon \ge z^\Lambda$ and so $P^M(w) = -i(a_l - \mu)$. When $w > -A_i$, then $z^\Lambda > z^\Upsilon$. Further, solution $(x^\Lambda, \hat{y}^\Lambda)$ is feasible since $a_l - \mu - w - A_i < a_l - \mu \le \hat{a}$ because of Condition (A4). It follows that $P^M(w) = -i(a_l - \mu) + (w + A_i)$. □

To perform sequence-independent lifting for the variables (x_j, y_j) for $j \in C'$, we verify that the function $P^M(w)$ is subadditive over $\mathbb{R}_-$.

Proposition 20 *The lifting function $P^M(w)$ is subadditive over $\mathbb{R}_-$.*

Proof First, note that $P^M(w)$ is subadditive over $\mathbb{R}_-$ if it is subadditive over $I = [-\mu - \sum_{j\in M\cup T} a_j, 0]$. Consider Corollary 1 and define $D_i = A_i$, $\nu = a_l - \mu$, and $r = q$. Observe that $P^M(w) = g_1(-w) + w$. Clearly, $A_i + A_j \ge A_{i+j}$ for $0 \le i \le j \le q$ with $i + j \le q$ since A_i is the sum of the largest i coefficients in $M_1 \cup \{l\}$. It then follows from Corollary 1 that $P^M(w)$ is subadditive over I, proving the result. □

We next illustrate the results of Propositions 18, 19, and 20 via an example.

Example 4 For the set B of Example 2, consider the partition $(C', \{l\}, M, T) = (\{3\}, \{4\}, \{5\}, \{1, 2\})$. This partition satisfies Conditions (A1), (A3), and (A4) since C is a cover with $\mu = 12$, $a_4 > \mu$, and $\sum_{j\in C\cup T} a_j = 21+19+17+15 > 20+15 = d + a_l$. We obtain from Proposition 18 that

$$3x_4 + 3x_5 + 21y_1 + 19y_2 \ge 3 \tag{53}$$

is facet-defining for $PB(\emptyset, C', \emptyset, C')$. Further, the lifting function $P^M(w)$ over $\mathbb{R}_-$ is given by

$$P^M(w) = \begin{cases} -\infty & \text{if } w < -62 \\ w + 19 & \text{if } -62 \leq w < -22 \\ -3 & \text{if } -22 \leq w < -15 \\ w + 12 & \text{if } -15 \leq w < -12 \\ 0 & \text{if } -12 \leq w \leq 0, \end{cases}$$

as described in Proposition 19 since $q = 2$, $A_0 = 0$, $A_1 = 15$, *and* $A_2 = 25$.

Similar to Theorem 1, we compute the lifting coefficients for the variables (x_i, y_i) for $i \in C'$ using sequence-independent lifting; refer to the discussion following Proposition 13.

Theorem 2 *Suppose that Conditions (A1), (A3), and (A4) hold. Then, the lifted reverse bilinear cover inequality*

$$\begin{aligned} &(a_l - \mu)x_l - \sum_{j \in C'} P^M(-a_j)x_j \\ &\quad + \sum_{j \in M} \min\{a_j, a_l - \mu\}x_j + \sum_{j \in T} a_j y_j \geq (a_l - \mu) - \sum_{j \in C'} P^M(-a_j) \end{aligned} \tag{54}$$

is facet-defining for PB.

Proof Since $P^M(w)$ is subadditive over $\mathbb{R}_-$, the lifting coefficients (α_i, β_i) of the variables (x_i, y_i) for $i \in C'$ are valid if they are chosen to satisfy

$$\alpha_i(x_i - 1) + \beta_i(y_i - 1) \geq P^M(a_i x_i y_i - a_i) \quad \text{for } (x_i, y_i) \in \{0, 1\} \times [0, 1] \backslash \{1, 1\}. \tag{55}$$

Condition (55) can be rewritten as

$$\beta_i \leq \inf_{0 \leq \phi < 1} \frac{-P^M(a_i\phi - a_i)}{1 - \phi}, \tag{56}$$

$$\alpha_i + \sup_{0 \leq \phi \leq 1} \beta_i(1 - \phi) \leq -P^M(-a_i). \tag{57}$$

Because of Assumption 2, we know that $a_i \leq \sum_{j \in N} a_j - d = \sum_{j \in C \cup M \cup T} a_j - (\sum_{j \in C} a_j - \mu) = \mu + \sum_{j \in M \cup T} a_j$ for all $i \in C' \subseteq N$ and so $P^M(a_i\phi - a_i) > -\infty$ for all $\phi \in [0, 1)$. Choosing $\beta_i = 0$ satisfies (56) since $P^M(a_i\phi - a_i) \leq 0$ for all $\phi \in [0, 1)$. Moreover, as $\beta_i = 0$, it is easily verified that choosing $\alpha_i = -P^M(-a_i)$ satisfies (57). Finally, note that (55) is tight at the points $(0, 0)$ and $\left(1, \frac{(a_i - A_1 + a_l - \mu)^+}{a_i}\right)$, which proves that (54) is facet-defining for PB. □

Note that the lifted reverse bilinear cover inequality (54) we obtained through lifting is unique. This is a significant difference from lifted bilinear cover inequalities (45). We next show that not all lifted reverse bilinear cover inequalities (54) can be derived as lifted bilinear cover inequalities (45).

Example 5 *For the partition* $(C', \{l\}, M, T) = (\{3\}, \{4\}, \{5\}, \{1, 2\})$*, we established in Example 4 that (53) is facet-defining for* $PB(\emptyset, C', \emptyset, C')$*. Applying Theorem 2, we obtain the following lifted reverse bilinear cover inequality*

$$3x_3 + 3x_4 + 3x_5 + 21y_1 + 19y_2 \geq 6, \tag{58}$$

which is facet-defining for PB. Inequality (58) cannot be obtained as a lifted bilinear cover inequality (45). In fact, if (58) was of the form (45), it should be that $C \subseteq \{3, 4, 5\}$*. However, none of the four possible covers* $C_1 = \{3, 4\}$*,* $C_2 = \{3, 5\}$*,* $C_3 = \{4, 5\}$ *and* $C_4 = \{3, 4, 5\}$ *yields (58).*

3.4 Lifted inequalities by approximate lifting

We now derive another family of lifted inequalities from the seed inequality (23) developed in Proposition 11. To this end, we first identify a partition $(K, \overline{M})$ of the set of variables N that satisfies the following conditions

(C1) $\sum_{j \in K} a_j - a_k \geq d$ for all $k \in K$,
(C2) $\sum_{j \in K} a_j - a_k - a_m < d$ for all $k \neq m \in K$, i.e., $a_k + a_m > \mu$ for all $k \neq m \in K$,

where $\mu = \sum_{j \in K} a_j - d$ is the excess of K. Note that Condition (C1) implies that K is a cover. Further, Condition (C1) requires that $K \backslash \{k\}$ is also a cover for all $k \in K$ and so $a_k \leq \mu$ for all $k \in K$. It also follows from Condition (C1) that $|K| \geq 2$. We refer to a set K satisfying Conditions (C1) and (C2) as a *clique*. After fixing the variables (x_i, y_i) for $i \in \overline{M}$ to $(0, 0)$, it follows from Proposition 11 that the clique inequality

$$\sum_{j \in K} x_j \geq |K| - 1 \tag{59}$$

is facet-defining for $PB(\overline{M}, \emptyset, \overline{M}, \emptyset)$.

We now lift the remaining variables (x_i, y_i) for $i \in \overline{M}$ in two steps. We assume without loss of generality that $K = \{1, \ldots, r\}$ and that $a_1 \leq a_2 \leq \ldots \leq a_r$. We define $\mu' = a_1 + a_2 - \mu$. We assume that $a_{r+1} \leq \cdots \leq a_n$ and define p such that $\sum_{i=r+1}^{p} a_i < \mu'$ and either $p = n$ or $\mu' \leq \sum_{i=r+1}^{p+1} a_i$. Let $\widehat{M} = \{a_{r+1}, \ldots, a_p\}$. (More generally, $\widehat{M}$ can be taken to be any maximal subset of $\overline{M}$ such that $\sum_{i \in \widehat{M}} a_i < \mu'$ without altering the form of the derived inequality.) We show that (59) is facet-defining for $PB(\overline{M} \backslash \widehat{M}, \emptyset, \overline{M} \backslash \widehat{M}, \emptyset)$. First, we show by contradiction that the inequality is valid. Let (x, y) be such that $\sum_{j \in K} x_j < r - 1$. Then,

$$\sum_{j=1}^{p} a_j x_j y_j \leq \sum_{j=3}^{p} a_j = d - \mu' + \sum_{j=r+1}^{p} a_j < d,$$

where the first inequality holds since $a_1 \leq \cdots \leq a_r$ and $\sum_{j \in K} x_j < r - 1$ and the last inequality follows since $\sum_{j=r+1}^{p} a_j < \mu'$. This inequality implies that $(x, y) \notin B(\overline{M} \backslash \widehat{M}, \emptyset, \overline{M} \backslash \widehat{M}, \emptyset)$, the desired contradiction. By Proposition 10, it suffices to show that (59) is facet-defining for $PB(\overline{M} \backslash \widehat{M}, \emptyset, \overline{M} \backslash \widehat{M}, K \cup \widehat{M})$. Define $\chi \in \mathbb{R}^{|N|}$

to be an indicator vector for the elements of K, *i.e.*, $\chi_j = 1$ for $j \in K$ and $\chi_j = 0$ otherwise. Then, by (C1), $p^k = \chi - e_k$ for $k \in K$ and $q^k = \chi - e_1 + e_k$ for $k \in \widehat{M}$, are feasible. Since these points are linearly independent, (59) is facet-defining for $PB(\overline{M}\backslash\widehat{M}, \emptyset, \overline{M}\backslash\widehat{M}, \emptyset)$.

We now lift variables (x_i, y_i) for $i \in M := \overline{M}\backslash\widehat{M}$. The lifting function corresponding to (59) is defined as

$$\begin{aligned} \Phi(w) := \max\ & (|K| - 1) - \sum_{j\in K} x_j \\ s.t. & \sum_{j\in K} a_j x_j y_j + \sum_{j\in\widehat{M}} a_j x_j y_j \geq d - w \\ & x_j \in \{0, 1\},\ \ y_j \in [0, 1] \quad \forall j \in K \cup \widehat{M}. \end{aligned} \tag{60}$$

We define $a' = \sum_{j\in\widehat{M}} a_j$, $\bar{\mu} = \mu' - a'$, $B_0 = 0$, and $B_i = \sum_{j=1}^{i} a_{j+2} - a'$ for $i = 1, \ldots, r-2$. It follows from the definition of $\widehat{M}$ that $\bar{\mu} > 0$. Observe that $B_0 \leq B_1$ because $a_3 - a' \geq a_3 - \mu' = a_3 - a_1 - a_2 + \mu \geq -a_2 + \mu \geq 0$, where the last inequality follows from (C1). Also, observe that $B_{r-2} + \bar{\mu} = d - a'$ and, for all $i \in M$, $a_i \geq a_{p+1} \geq \mu' - a' = \bar{\mu}$, where the last inequality follows from the definition of $\widehat{M}$. Using the observation that, in an optimal solution to the lifting problem, clique variables with larger coefficients can always be chosen to have larger values than clique variables with smaller coefficients, we obtain the following result.

Proposition 21 *The lifting function $\Phi(w)$ defined in (60) takes the values*

$$\Phi(w) = \begin{cases} 0 & \text{if } 0 \leq w < \bar{\mu}, \\ i & \text{if } B_{i-1} + \bar{\mu} \leq w < B_i + \bar{\mu}, \quad i = 1, \ldots, r-2, \\ r-1 & \text{if } B_{r-2} + \bar{\mu} \leq w \end{cases}$$

for $w \geq 0$. □

In Sect. 3.3.1, all lifting functions were subadditive over appropriate ranges. As a result, strong valid inequalities for PB were easily obtained using sequence-independent lifting. The lifting function $\Phi(w)$ derived in Proposition 21, however, is not subadditive. To circumvent the difficulties associated with sequence-dependent lifting in such a situation, [13] proposed to use approximate lifting. Following their approach, we say that $\Psi(w)$ is a *valid subadditive approximation* of $\Phi(w)$ if $\Psi(w) \geq \Phi(w)$ for all $w \in \mathbb{R}_+$ and $\Psi(w)$ is subadditive. We say that a valid subadditive approximation $\Psi(w)$ is *nondominated* if there is no other valid subadditive approximation $\Psi'(w)$ of $\Phi(w)$ with $\Psi'(w) \leq \Psi(w)$ for all $w \in \mathbb{R}_+$ and $\Psi'(w') < \Psi(w')$ for some $w' \in \mathbb{R}_+$. We also define the notion of maximal set $E = \{w \in \mathbb{R}_+ \mid \Phi^i(w) = \Phi(w)\ \forall i \in M$, for all coefficients $a_i \in \mathbb{R}_+$and for all lifting orders$\}$, where Φ^i denotes the lifting function associated with sequentially lifting the i^{th} variable. A valid subadditive approximation $\Psi(w)$ of $\Phi(w)$ is called *maximal* if $\Psi(w) = \Phi(w)$ for all $w \in E$. It is clear that a maximal nondominated approximation of Φ leads to strong inequalities that can be obtained efficiently for PB. The approximation of $\Phi(w)$ we use has the form of $g_2(w)$ presented in Corollary 2.

We next describe in Proposition 22 a subadditive, nondominated and maximal approximation of $\Phi(w)$ over $\mathbb{R}_+$.

Proposition 22 *The function*

$$\Psi(w) := \begin{cases} i + \frac{w-B_i}{\bar{\mu}} & \text{if } B_i \leq w < B_i + \bar{\mu}, \quad i = 0, \ldots, r-2, \\ i & \text{if } B_{i-1} + \bar{\mu} \leq w < B_i, \quad i = 1, \ldots, r-2, \\ r-1 & \text{if } B_{r-2} + \bar{\mu} \leq w, \end{cases}$$

is a valid subadditive approximation of $\Phi(w)$ that is nondominated and maximal over $\mathbb{R}_+$.

Proof Note that $\Psi(w) = \Phi(w)$ when $w \in [B_{i-1} + \bar{\mu}, B_i]$ for $i \in \{1, \ldots, r-2\}$ and when $w \geq B_{r-2} + \bar{\mu}$. Further,

$$\Psi(w) = \Phi(w) + \frac{w - B_i}{\bar{\mu}} \geq \Phi(w)$$

when $w \in (B_i, B_i + \bar{\mu})$ for $i \in \{0, \ldots, r-2\}$. Next, we show that $\Psi(w)$ is subadditive over $\mathbb{R}_+$. In Corollary 2, let $s = r-2$, $C_i = B_i$ and $\lambda = \bar{\mu}$. Since B_i is the sum of the smallest i coefficients in $K\backslash\{1, 2\}$, it is clear that $B_i + B_j \leq B_{i+j}$ for $0 \leq i \leq j \leq r-2$ with $i + j \leq r-2$. Therefore, $\Psi(w)$ is subadditive over $\mathbb{R}_+$.We now argue nondominance and maximality over $\mathbb{R}_+$. To this end, we first observe that for all $w' \in \mathbb{R}_+$ there exists $w'' \in \mathbb{R}_+$ such that

$$\Psi(w') + \Psi(w'') = \Phi(w' + w''). \tag{61}$$

In particular, w'' can be chosen to be $B_i + \bar{\mu} - w'$ when $w' \in (B_i, B_i + \bar{\mu})$ and w'' can be chosen to be 0 otherwise. If Ψ' dominates Ψ strictly at w' then $\Psi'(w'+w'') \leq \Psi'(w') + \Psi'(w'') < \Psi(w') + \Psi(w'') = \Phi(w' + w'')$ yielding a contradiction to the assumption that Ψ' is an overestimator of Φ. Similarly, if $\Phi(w') < \Psi(w')$ then (61) implies that $\Phi(w') < \Psi(w') = \Phi(w' + w'') - \Psi(w'') \leq \Phi(w' + w'') - \Phi(w'')$. Therefore, $\Phi(w')$ does not yield a valid lifting coefficient for the sequential perturbation of w' after w''. □

Example 6 For the bilinear set B studied in Example 2, consider $K = \{3, 4, 5\}$. Set K satisfies Conditions (C1) and (C2) with $\mu = 22$. It follows from Proposition 11 that

$$x_3 + x_4 + x_5 \geq 2 \tag{62}$$

is facet-defining for $B(\{1, 2\}, \emptyset, \{1, 2\}, \emptyset)$. Let $\widehat{M} = \emptyset$. The lifting function of (62) obtained using Proposition 21 and its valid subadditive approximation $\Psi(w)$ obtained in Proposition 22 are given by

$$\Phi(w) = \begin{cases} 0 & \text{if } 0 \leq w < 3, \\ 1 & \text{if } 3 \leq w < 20, \\ 2 & \text{if } 20 \leq w \end{cases} \quad \text{and} \quad \Psi(w) = \begin{cases} \frac{w}{3} & \text{if } 0 \leq w < 3, \\ 1 & \text{if } 3 \leq w < 17, \\ 1 + \frac{w-17}{3} & \text{if } 17 \leq w < 20, \\ 2 & \text{if } 20 \leq w \end{cases}$$

as $r = 3$, $\bar{\mu} = 3$, $B_0 = 0$, and $B_1 = 17$.

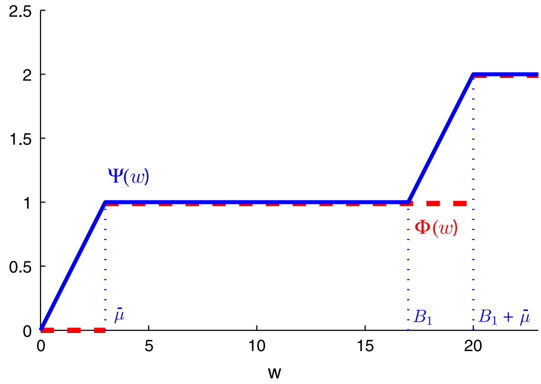

Fig. 2 A valid subadditive approximation $\Psi(w)$ of $\Phi(w)$ for Example 6

In Fig. 2, we present the lifting function $\Phi(w)$ of the clique inequality derived in Proposition 21 and its valid subadditive approximation $\Psi(w)$ obtained in Proposition 22 for the particular case of inequality (62) discussed in Example 6. The function $\Phi(w)$ is depicted with a dotted line while $\Psi(w)$ is represented using a solid line. Observe that, for $0 < w \le \bar{\mu} = 3$, the approximation is exact only when $w = \bar{\mu} = 3$, *i.e.*, $\Psi(\bar{\mu}) = \Phi(\bar{\mu})$. For $w \ge \bar{\mu} = 3$, the approximation is exact when $3 = \bar{\mu} \le w \le B_1 = 17$ and $w \ge B_1 + \bar{\mu} = 20$. Next, we obtain a concave overestimator of $\Psi(w)$ in Lemma 2, in a manner similar to Lemma 1, that we will use in Theorem 3 to compute lifting coefficients for the variables in M.

Lemma 2 *For $i \in M$, define*

$$q_i := \begin{cases} 0 & \text{if } a_i \le \bar{\mu}, \\ j+1 & \text{if } B_j + \bar{\mu} < a_i \le B_{j+1} + \bar{\mu},\ j = 0, \ldots, r-3, \\ r-1 & \text{if } B_{r-2} + \bar{\mu} < a_i. \end{cases}$$

Let $W_0^i = 0$, $W_j^i = B_{j-1} + \bar{\mu}$ for $j = 1, \ldots, q_i$ and $W_{q_i+1}^i = a_i$. Define $\Delta_j^i = W_{j+1}^i - W_j^i$ for $j = 0, \ldots, q_i$. Define also $\psi_j^i(w) = \Psi(W_j^i) + \frac{\Psi(W_{j+1}^i) - \Psi(W_j^i)}{\Delta_j^i}(w - W_j^i)$ for $j = 0, \ldots, q_i$. Then, the function

$$\psi^i(w) := \min\left\{\psi_j^i(w) \;\middle|\; j \in \{0, \ldots, q_i\}\right\} \tag{63}$$

is a concave overestimator of $\Psi(w)$ over $[0, a_i]$. □

The concave overestimator $\psi^i(w)$ of Lemma 2 can be used to obtain lifting coefficients in a manner similar to that of Theorem 1. Because of the way the concave overestimator is built, it can be observed that all of its affine pieces (except possibly $\psi_{q_i}^i$) touch the original lifting function Φ at two points and therefore can be used to generate strong lifting coefficients. To describe whether $\psi_{q_i}^i$ touches Φ in two points, we define $I(a_i)$ to be the function that returns 0 if $\Phi(a_i) = \Psi(a_i)$ and returns 1 otherwise, *i.e.*,

$$I(a_i) := \begin{cases} 0 & \text{if } B_{q_i-1} + \bar{\mu} < a_i \leq B_{q_i} \text{ or } a_i > B_{r-2} + \bar{\mu}, \\ 1 & \text{if } B_{q_i} < a_i \leq B_{q_i} + \bar{\mu}. \end{cases}$$

We observe that, when $I(a_i) = 0$, it is possible to derive maximal lifting coefficients (with respect to Φ) from all affine pieces of ψ^i since each affine piece of ψ^i touches Φ in at least two points. When $I(a_i) = 1$, however, we can only guarantee the derivation of maximal lifting coefficients (with respect to Φ) from ψ^i_j for $j = 0, \ldots, q_i - 1$. In fact, the last affine piece of ψ^i, *i.e.*, $\psi^i_{q_i}$ touches Φ in a single point. This, in turn, implies that the lifting coefficients derived from the last affine piece of ψ^i might not be maximal. Since the only two pairs of lifting coefficients derived from the last affine piece of ψ^i are (α^j_i, β^j_i) where $j = q_i$ and $j = q_i + 1$, we introduce the notation $\mathbf{1}_{\{j_i \geq q_i\}}$ to represent the indicator function that returns 1 if $j_i \geq q_i$ and 0 otherwise. Using this notation, it is clear that two new tight affinely independent points are added after lifting variables (x_i, y_i), unless $I(a_i) = 1$ and $\mathbf{1}_{\{j_i \geq q_i\}} = 1$, in which case only a single tight point is guaranteed to be added. It also follows that a facet-defining inequality of PB will be produced if $I(a_i) \times \mathbf{1}_{\{j_i \geq q_i\}} = 0$ for all $i \in M$. These intuitive observations are formalized in the following theorem.

Theorem 3 *Under Conditions (C1) and (C2),*

$$\sum_{j \in K} x_j + \sum_{i \in M} \alpha^{j_i}_i x_i + \sum_{i \in M} \beta^{j_i}_i y_i \geq |K| - 1 \tag{64}$$

defines a face of PB of dimension at least $(2n-1) - \sum_{i \in M} I(a_i) \times \mathbf{1}_{\{j_i \geq q_i\}}$ *for all* $j_i \in \{0, \ldots, q_i + 1\}$ *and for all* $i \in M$ *where*

$$\begin{aligned} \left(\alpha^j_i, \beta^j_i\right) &= \left(\Psi\left(W^i_j\right) - \frac{\Psi\left(W^i_{j+1}\right) - \Psi\left(W^i_j\right)}{\Delta^i_j} W^i_j, \frac{\Psi\left(W^i_{j+1}\right) - \Psi\left(W^i_j\right)}{\Delta^i_j} a_i\right) \\ &\text{for } j = 0, \ldots, q_i, \\ \left(\alpha^{q_i+1}_i, \beta^{q_i+1}_i\right) &= \left(\Psi(a_i), 0\right), \end{aligned} \tag{65}$$

and $\bar{\mu}$, W^i_j, Δ^i_j *and* q_i *are as defined in Lemma 2. For a given inequality of the form (64), let* $L = \{i \in M \mid j_i \geq q_i, I(a_i) = 1\}$. *Then (64) is facet-defining for PB if one of the following conditions holds:*

1. $L = \emptyset$.
2. $\exists \bar{i} \in M$ *such that* $j_{\bar{i}} = 0$.

Proof It follows from Proposition 22 that $\Psi(w)$ is a valid subadditive approximation of $\Phi(w)$ for $w \geq 0$. Hence, lifting coefficients (α_i, β_i) of (x_i, y_i) for $i \in M$ are valid if they satisfy the condition

$$\alpha_i x_i + \beta_i y_i \geq \Psi(a_i x_i y_i) \quad \text{for } (x_i, y_i) \in \{0, 1\} \times [0, 1] \backslash \{0, 0\}. \tag{66}$$

Condition (66) can be restated as

$$\beta_i \phi \geq \Psi(0) \qquad \text{for } 0 < \phi \leq 1, \tag{67}$$

$$\alpha_i + \beta_i \phi \geq \Psi(a_i \phi) \quad \text{for } 0 \leq \phi \leq 1. \tag{68}$$

To prove that (64) defines a face of PB of dimension at least $(2n-1) - \sum_{i \in M} I(a_i) \times \mathbf{1}_{\{j_i \geq q_i\}}$ when lifting coefficients are chosen according to (65), we will show that, for each $i \in M$,

$$\alpha_i x_i + \beta_i y_i = \Phi(a_i x_i y_i) \tag{69}$$

is satisfied at equality by at least $2 - I(a_i) \times \mathbf{1}_{\{j_i \geq q_i\}}$ independent points.

First, consider the case where $(\alpha_i, \beta_i) = (\Psi(a_i), 0)$. Observe that (67) is satisfied since $\beta_i = 0$ and $\Psi(0) = 0$. Further, (68) holds as $\alpha_i = \alpha_i + \beta_i \phi = \Psi(a_i) \geq \Psi(a_i \phi)$ since Ψ is a non-decreasing function. It is easily verified that (69) is satisfied at equality at the point $(0, 1)$. Further, when $I(a_i) = 0$, then (69) is also satisfied at equality at the point $(1, 1)$.

Second, consider the case where

$$(\alpha_i, \beta_i) = \left(\Psi\left(W_j^i\right) - \frac{\Psi\left(W_{j+1}^i\right) - \Psi\left(W_j^i\right)}{\Delta_j^i} W_j^i, \frac{\Psi\left(W_{j+1}^i\right) - \Psi\left(W_j^i\right)}{\Delta_j^i} a_i \right).$$

Clearly, (α_i, β_i) satisfies (67) since $\beta_i \geq 0$. From Lemma 2, we have that

$$\begin{aligned}
\Phi(a_i \phi) \leq \Psi(a_i \phi) &\leq \Psi\left(W_j^i\right) + \frac{\Psi(W_{j+1}^i) - \Psi\left(W_j^i\right)}{\Delta_j^i} \left(a_i \phi - W_j^i\right) \\
&= \left(\Psi(W_j^i) - \frac{\Psi\left(W_{j+1}^i\right) - \Psi\left(W_j^i\right)}{\Delta_j^i} W_j^i \right) + \frac{\Psi\left(W_{j+1}^i\right) - \Psi\left(W_j^i\right)}{\Delta_j^i} a_i \phi \\
&= \alpha_i + \beta_i \phi.
\end{aligned}$$

We now present points that satisfy (69) at equality. Observe first that, for $j = 0, \ldots, q_i$, the point $(x_i^*, y_i^*) = (1, \frac{W_j^i}{a_i})$ satisfies (69) at equality since $\Psi(a_i x_i^* y_i^*) = \Psi(W_j^i) = \Psi(B_{j-1} + \bar{\mu}) = \Phi(B_{j-1} + \bar{\mu})$. Similarly, for $j = 0, \ldots, q_i - 1$, the point $(x_i^*, y_i^*) = \left(1, \frac{W_{j+1}^i}{a_i}\right)$ satisfies (69) at equality. For $j = q_i$, the point $\left(1, \frac{W_{j+1}^i}{a_i}\right)$ reduces to $(1, 1)$ which satisfies (69) at equality when $\Psi(a_i) = \Phi(a_i)$, *i.e.*, when $I(a_i) = 0$. Therefore, we conclude that (64) defines a face of PB of dimension at least $(2n-1) - \sum_{i \in M} I(a_i) \times \mathbf{1}_{\{j_i \geq q_i\}}$.

We also conclude from the above derivation that when, for all $i \in M$, either $j_i < q_i$ or $I(a_i) = 0$, *i.e.*, $L = \emptyset$, then the face of PB that (64) defines has dimension $2n - 1$ showing that (64) is facet-defining for PB and proving Condition 1. Now, we show that (64) is also facet-defining if Condition 2 holds, *i.e.*, $j_{\bar{\imath}} = 0$ for some $\bar{\imath} \in M$. We first lift $(x_{\bar{\imath}}, y_{\bar{\imath}})$. Since $a_{\bar{\imath}} \geq \bar{\mu}$ (see discussion preceding Proposition 21), it follows that $(\alpha_{\bar{\imath}}^0, \beta_{\bar{\imath}}^0) = (0, \frac{a_{\bar{\imath}}}{\bar{\mu}})$. Then, we lift the variables in $M\backslash\{L \cup \{\bar{\imath}\}\}$ and choose any $j_i \leq q_i + 1$ for these variables. The above proof shows that the resulting inequality is facet-defining for $PB(L\backslash\{\bar{\imath}\}, \emptyset, L\backslash\{\bar{\imath}\}, \emptyset)$. Since $PB(L\backslash\{\bar{\imath}\}, \emptyset, L\backslash\{\bar{\imath}\}, \emptyset) \subseteq PB$, all the points tight for (64) are feasible to PB. Now, we lift a variable $i' \in L\backslash\{\bar{\imath}\}$. Let

$$F(w, a) = \left\{ (x, y) \in \{0, 1\}^n \times [0, 1]^n \;\middle|\; \sum_{i \in K} a_i x_i y_i \geq d - a' - w \text{ and } \sum_{i \in K} x_i = |K| - 1 - a \right\}.$$

We show that there exists $p \in F(B_{q_{i'}} + \mu, q_{i'} + 1)$ which is feasible to PB and tight on (64). First note that $F(B_{q_{i'}} + \mu, q_{i'} + 1) \neq \emptyset$ because $\Phi(B_{q_{i'}} + \mu) = q_{i'} + 1$. Let $p = (x', y')$. By the definition of $F(w, a)$, we are free to redefine (x_i', y_i') for $i \notin K$. Let $x_i' = y_i' = 0$ for $i \in M\backslash\{L \cup \{\bar{\imath}\}\}$ and let $x_i' = y_i' = 1$ for $i \in \widehat{M}$. Let $x_{\bar{\imath}}' = 1$ and $y_{\bar{\imath}}' = \frac{B_{q_{i'}} + \bar{\mu} - a_{i'}}{a_{\bar{\imath}}}$. Since $a_{\bar{\imath}} \geq \bar{\mu}$ and $B_{q_{i'}} < a_{i'} \leq B_{q_{i'}} + \bar{\mu}$, it follows that $0 < y_{\bar{\imath}} \leq 1$. Finally, we set $(x_{i'}', y_{i'}') = (1, 1)$. Note that $a_{\bar{\imath}} x_{\bar{\imath}}' y_{\bar{\imath}}' + a_{i'} x_{i'}' y_{i'}' = B_{q_{i'}} + \bar{\mu}$ and

$$\begin{aligned}
\alpha_{\bar{\imath}}^0 x_{\bar{\imath}}' + \beta_{\bar{\imath}}^0 y_{\bar{\imath}}' + \alpha_{i'}^{j_{i'}} x_{i'}' + \beta_{i'}^{j_{i'}} y_{i'}' &= \frac{B_{q_{i'}} + \bar{\mu} - a_{i'}}{\bar{\mu}} + q_{i'} + \frac{a_{i'} - B_{q_{i'}}}{\bar{\mu}} \\
&= q_{i'} + 1 = \Psi(B_{q_{i'}} + \mu) = \Phi(B_{q_{i'}} + \mu),
\end{aligned}$$

where the first equality holds since $(\alpha_{\bar{\imath}}^0, \beta_{\bar{\imath}}^0) = (0, \frac{a_{\bar{\imath}}}{\bar{\mu}})$, $(\alpha_{i'}^{j_{i'}}, \beta_{i'}^{j_{i'}}) = (q_{i'} - \theta \frac{B_{q_{i'}-1} + \bar{\mu}}{a_{i'}}, \theta)$ when $j_{i'} = q_{i'}$ and $\left(\alpha_{i'}^{j_{i'}}, \beta_{i'}^{j_{i'}}\right) = (\Psi(a_{i'}), 0)$ when $j_{i'} = q_{i'} + 1$ where $\theta = \frac{(\Psi(a_{i'}) - q_{i'}) a_{i'}}{a_{i'} - B_{q_{i'}-1} - \bar{\mu}}$ and $\Psi(a_{i'}) = q_{i'} + \frac{a_{i'} - B_{q_{i'}}}{\bar{\mu}}$. Therefore, $p \in PB$ and is tight for (64). For $j_{i'} = q_{i'}$, we have already demonstrated that there exists a point of PB tight for (64) that sets $(x_{i'}, y_{i'}) = \left(1, \frac{W_{j_{i'}}^{i'}}{a_{i'}}\right)$. For $j_{i'} = q_{i'} + 1$, there is a point of PB tight for (64) such that $(x_{i'}, y_{i'}) = (0, 1)$. For $j_{i'} = q_{i'}$, affine independence follows since $a_{i'} > W_{j_{i'}}^{i'}$ implies that $(0, 0)$, $(1, 1)$, and $\left(1, \frac{W_{j_{i'}}^{i'}}{a_{i'}}\right)$ are affinely independent. For $j_{i'} = q_{i'+1}$, affine independence follows from the affine independence of $(0, 0)$, $(1, 1)$, and $(0, 1)$. □

Inequalities (64) can be facet-defining depending on the value of the coefficients a_i and the choice of lifting coefficients (α_i, β_i) for $i \in M$. As mentioned before, $\widehat{M}$ may be chosen to be any subset of $\overline{M}$ that satisfies $\sum_{i \in \widehat{M}} a_i < \mu'$. In this case, (64) will be facet-defining if $\max\{a_i \mid i \in M, j_i = 0\} \geq \bar{\mu}$ but it may not be facet-defining otherwise. The next example illustrates the use of (64) in deriving facets of PB.

Example 7 Consider the clique inequality (62) of Example 6 and its corresponding approximate lifting function. We have $q_1 = 2$ and $q_2 = 1$ with $W_0^1 = 0$, $W_1^1 = 3$, $W_2^1 = 20$, $W_3^1 = 21$, and $W_0^2 = 0$, $W_1^2 = 3$, $W_2^2 = 19$. Applying Theorem 3, we obtain the following nine inequalities

$$\left\{\begin{array}{c} \frac{21}{3}y_1 \\ \frac{14}{17}x_1 + \frac{21}{17}y_1 \\ 2x_1 \end{array}\right\} + \left\{\begin{array}{c} \frac{19}{3}y_2 \\ \frac{21}{24}x_2 + \frac{19}{24}y_2 \\ \frac{5}{3}x_2 \end{array}\right\} + x_3 + x_4 + x_5 \geq 2,$$

which define faces of PB of dimension at least 8 since $I(a_1) = 0$ and $I(a_2) = 1$. It follows from the first condition of Theorem 3 that the following three inequalities

$$\left\{\begin{array}{c} \frac{21}{3}y_1 \\ \frac{14}{17}x_1 + \frac{21}{17}y_1 \\ 2x_1 \end{array}\right\} + \frac{19}{3}y_2 + x_3 + x_4 + x_5 \geq 2$$

are facet-defining for PB since $j_2 < q_2$. The following two inequalities also define facets of PB

$$\frac{21}{3}y_1 + \left\{\begin{array}{c} \frac{21}{24}x_2 + \frac{19}{24}y_2 \\ \frac{5}{3}x_2 \end{array}\right\} + x_3 + x_4 + x_5 \geq 2,$$

since they satisfy the second condition for facet-defining inequalities in Theorem 3 as $j_1 = 0$.

4 Relations to fixed-charge single-node flow model without inflows

In Sect. 3, we derived strong valid inequalities for the bilinear set B using lifting. In this section, we show that many of these lifted inequalities are also facet-defining for the convex hull of the fixed-charge single-node flow model without inflows F defined in Sect. 1. It is easy to see that $F \subseteq B$; see [9] for a proof.

Lemma 3 *The bilinear covering set B is a relaxation of the flow set F.* □

Fixed-charge single-node flow sets are important in practice since they can be used as a source of cutting planes for 0−1 mixed-integer programs. Further, they naturally arise in the formulation of fixed-charge network problems; see [2,12,18,19,22]. The fixed-charge single-node flow set F without inflows was first studied by [22] under the assumptions that (i) $a_i \leq d$ and (ii) $\sum_{j=1}^{n} a_j > d + a_i$ for all $i \in N$. In the following, we relate the facets of PF to those of PB without assuming that the sets are full-dimensional.

Lemma 4 (Adapted from Proposition 8 in [22]) *Every facet-defining inequality of PF that is not a multiple of $y_i \leq x_i$ can be expressed as $\alpha x + \beta y \geq \delta$, where $\beta \geq 0$.*

Proof If for some i, $\beta_i < 0$ then the only points tight on this inequality are such that $y_i = x_i$. If F satisfies this equality then we may rewrite the facet-defining inequality as $\alpha x + \beta_i x_i + \beta y - \beta_i y_i \geq \delta$. □

In the following, we refer to the facet-defining inequalities of PF that are not multiples of $y_i \leq x_i$ as *non-trivial* facet-defining inequalities.

Lemma 5 *aff*$(F) =$ *aff*(B).

Proof Clearly, $\text{aff}(F) \subseteq \text{aff}(B)$ since $F \subseteq B$ by Lemma 3. It therefore remains to prove that $\text{aff}(B) \subseteq \text{aff}(F)$. Consider any point $(x, y) \in B$. If $(x, y) \in F$, then clearly $(x, y) \in \text{aff}(F)$. We may therefore assume that $(x, y) \in B \backslash F$. Define $p = (x', y')$ where $(x'_i, y'_i) = (x_i, x_i y_i)$ for $i \in N$. It is easy to see that $\sum_{i \in N} a_i y'_i = \sum_{i \in N} a_i x_i y_i \geq d$ and $y'_i \leq x'_i$ for $i \in N$ and so $p \in F$. Let $I' = \{i \in N \mid y_i > x_i\}$. We show next that for each $i \in I'$, $p^i = p + (0, e_i) \in \text{aff}(F)$. To this end, observe that $x'_i = 0$ for each $i \in I'$. It follows easily that $q^i = p + (e_i, 0)$ and $r^i = p + (e_i, e_i)$ belong to F. Therefore, $p^i = p + (r^i - q^i) \in \text{aff}(F)$. Now, observe that $(x, y) = p + \sum_{i \in I'} y_i (p^i - p) \in \text{aff}(F)$. It follows that $B \subseteq \text{aff}(F)$ and therefore $\text{aff}(B) \subseteq \text{aff}(F)$. □

Proposition 23 *Assume that*

$$\alpha x + \beta y \geq \delta \tag{70}$$

is valid for PF and, for each $i \in N$, either $\alpha_i \leq 0$ or $\beta_i \geq 0$. Then, (70) is valid for PB. Further, for every non-trivial facet (70) of PF with $\beta \geq 0$, (70) is facet-defining inequality for PB.

Proof We first show that (70) is valid for B. Consider $(x, y) \in B$. Let $I = \{i \in N \mid \alpha_i \leq 0\}$. Define (x', y') such that $(x'_i, y'_i) = (1, y_i)$ for $i \in I$ and $(x'_i, y'_i) = (x_i, x_i y_i)$ for $i \in N \backslash I$. Then,

$$\sum_{i \in N} a_i y'_i = \sum_{i \in I} a_i y_i + \sum_{i \in N \backslash I} a_i x_i y_i \geq \sum_{i \in N} a_i x_i y_i \geq d,$$

where the last inequality holds because $(x, y) \in B$. Further, since $y'_i \leq x'_i$, it follows that $(x', y') \in F$. Then,

$$\delta \leq \alpha x' + \beta y' \leq \alpha x + \beta y,$$

where the first inequality holds because $(x', y') \in F$ and the second inequality is satisfied since, by construction, $\alpha(x' - x) + \beta(y' - y) \leq 0$. It follows that (70) is valid for PB.

Consider a non-trivial facet-defining inequality $\alpha' x + \beta' y \geq \delta'$ of PF with $\beta' \geq 0$. Then, by the first part of this result, it follows that $\alpha' x + \beta' y \geq \delta'$ is valid for PB. Since, by Lemmas 3 and 5 respectively, $B \supseteq F$ and $\dim(B) = \dim(F)$, it follows that $\alpha' x + \beta' y \geq \delta$ defines a facet of PB. □

In Proposition 23, the assumption that $\beta \geq 0$ for a facet-defining inequality is without loss of generality because of Lemma 4. As a consequence of Proposition 23, it can be shown that lifting functions associated with inequalities $\alpha x + \beta y \geq \delta$, where for each i either $\alpha_i \leq 0$ or $\beta_i \geq 0$, are identical when computed over B or over F. Since the seed and lifted inequalities we derived for PB satisfy these assumptions, our results in Sect. 3 extend to the study of F. We record this observation in the following corollary.

Corollary 4 *Let $(\alpha, \beta) \in \mathbb{R}^{2n}$ and, for each $i \in N$, assume that either $\alpha_i \leq 0$ or $\beta_i \geq 0$. Let $B(w) = \{(x, y) \in \{0,1\}^n \times [0,1]^n \mid \sum_{i \in N} a_i x_i y_i \geq d - w\}$, and $F(w) = \{(x, y) \in \{0,1\}^n \times [0,1]^n \mid \sum_{i \in N} a_i y_i \geq d - w$ and $y_i \leq x_i$ for all $i \in N\}$, where $a_i \geq 0$ for all $i \in N$. Let $z_B(w) = \min\{\alpha x + \beta y \mid (x, y) \in B(w)\}$ and $z_F(w) = \min\{\alpha x + \beta y \mid (x, y) \in F(w)\}$. Then, $z_B(w) = z_F(w)$.*

Proof By Lemma 3, $B(w) \supseteq F(w)$. It follows that $z_B(w) \leq z_F(w)$. We now argue that $z_B(w) \geq z_F(w)$. Because $z_F(w)$ is defined as the minimum value that $\alpha x + \beta y$ takes over $F(w)$, $\alpha x + \beta y \geq z_F(w)$ is valid for $F(w)$, which is a flow-set. Let $(x', y') \in \operatorname{argmin}\{\alpha x + \beta y \mid (x, y) \in B(w)\}$. Then, $z_B(w) = \alpha x' + \beta y' \geq z_F(w)$, where the inequality follows from Proposition 23. We conclude that $z_B(w) = z_F(w)$. □

Now, we illustrate Proposition 23 via an example.

Example 8 Consider the fixed-charge single-node flow set without inflows

$$F = \left\{ (x, y) \in \{0,1\}^4 \times [0,1]^4 \;\middle|\; 19y_1 + 17y_2 + 15y_3 + 10y_4 \geq 20, \quad x_j \geq y_j,\ \forall j = 1, \ldots, 4 \right\},$$

corresponding to the bilinear covering set B discussed in Example 1. We obtained a complete linear description of PF using PORTA; see [7]. We observe that inequalities (5), (6), (12), and (13) are facets for both PB and PF. However, it can be verified that inequalities (7), (8), (10), and (11) are facet-defining for PB but not for PF.

Proposition 23 is surprising in light of Lemma 3 because on the one hand $F \subseteq B$ and on the other hand, the nontrivial facets of PF are facets of PB. In other words, a polyhedral description of PF can be derived from that of PB by adding the trivial facets of PF. The converse, however, is not true. As an illustration, inequality

$$14x_1 + 10x_3 + 5x_4 + 17y_2 \geq 15 \tag{71}$$

is a non-trivial facet-defining inequality of PB that is not facet-defining of PF. Surprisingly, a partial converse to Proposition 23 does hold.

We will show that an inequality description of PB can be obtained using the facet-defining inequalities for PF. The key to this construction is the result of Lemma 6 which shows that F and B can be viewed as projections of the same set onto different subspaces. Let

$$S = \left\{ (x, y, z) \in \{0,1\}^n \times [0,1]^n \times \mathbb{R}^n \;\middle|\; \sum_{j=1}^{n} a_j z_j \geq d, z_j = x_j y_j, \forall j \in N \right\}.$$

Lemma 6 *The projection of S onto the (x, z) space is F while the projection of S onto (x, y) space is B. Consequently, $\text{proj}_{(x,z)} \text{conv}(S) = PF$ and $\text{proj}_{(x,y)} \text{conv}(S) = PB$.*

Proof First, we show that $\text{proj}_{(x,z)}S = F$. If $(x, y, z) \in S$, it is clear that $(x, z) \in F$ since $0 \leq z_j \leq x_j$ and $\sum_{j=1}^{n} a_j z_j \geq d$. If $(x, z) \in F$, then $0 \leq z_j \leq x_j$ and $x_j \in \{0, 1\}$ imply that $z_j = x_j z_j$. Therefore, $(x, z, z) \in S$. Second, we show that $\text{proj}_{(x,y)}S = B$. This follows by substituting $x_j y_j$ for z_j in $\sum_{j=1}^{n} a_j z_j \geq d$. The last statement follows since $\text{proj}_{(x,y)}\text{conv}(S) = \text{conv}(\text{proj}_{(x,y)}S) = \text{conv}(B) = PB$ and $\text{proj}_{(x,z)}\text{conv}(S) = \text{conv}(\text{proj}_{(x,z)}S) = \text{conv}(F) = PF$ as $\text{conv}(AS) = A\text{conv}(S)$ for any linear transformation A. $\square$

Surprisingly, $\text{conv}(S)$ can be described using the facet-defining inequalities of PF. We write that $(\alpha, \beta, \gamma) \in \mathcal{F}(PF)$ if $\alpha x + \beta y \geq \gamma$ is a facet-defining inequality of PF that is not a multiple of $y_j \leq x_j$. Define

$$G = \left\{ (x, y, z) \in \mathbb{R}^{3n} \;\middle|\; \alpha x + \beta z \geq \gamma \; \forall (\alpha, \beta, \gamma) \in \mathcal{F}(PF), \, z \leq \min\{x, y\}, \, y \leq z + \mathbf{1} - x \right\}.$$

Theorem 4 $G = \mathit{conv}(S)$.

Proof ($\supseteq$) To show that $\text{conv}(S) \subseteq G$, it suffices to show that $S \subseteq G$ because G is convex. Consider $(x, y, z) \in S$. Then, by Lemma 6, $(x, z) \in F$ and, therefore, $\alpha x + \beta z \geq \gamma$ for all (α, β, γ) in $\mathcal{F}(PF)$. Further, McCormick envelopes of $x_j y_j$ yield, $x_j + y_j - 1 \leq x_j y_j \leq \min\{x_j, y_j\}$. Therefore, (x, y, z) satisfies the defining inequalities of G.

($\subseteq$) Now, we show that $G \subseteq \text{conv}(S)$. If $(x, y, z) \in G$, then $(x, z) \in PF$ and $z \leq y \leq z + \mathbf{1} - x$. Therefore, there exists a set of points $(x^i, z^i) \in F$ indexed by I, such that $(x, z) = \sum_{i \in I} \lambda_i (x^i, z^i)$ where $\lambda_i \geq 0$ for $i \in I$, and $\sum_{i \in I} \lambda_i = 1$. We define $f_j = \frac{y_j - z_j}{1 - x_j}$ if $x_j < 1$ and 0 otherwise. Let $I_j^1 = \{i \in I \mid x_j^i = 1\}$. Now, consider (x^i, y^i, z^i) where $y_j^i = z_j^i$ if $i \in I_j^1$ and $y_j^i = f_j$ if $i \in I \backslash I_j^1$. Then, $z_j^i \leq x_j^i$ and $x_j^i \in \{0, 1\}$ imply that $z_j^i = x_j^i y_j^i$. Further,

$$\sum_{i \in I} \lambda_i y_j^i = \sum_{i \in I_j^1} \lambda_i z_j^i + \sum_{i \in I \backslash I_j^1} \lambda_i f_j = z_j + (1 - x_j) f_j = y_j,$$

where the second equality follows since $z_j = \sum_{i \in I} \lambda_i z_j^i = \sum_{i \in I_j^1} \lambda_i z_j^i$, $\sum_{i \in I_j^1} \lambda_i = x_j$, and $\sum_{i \in I} \lambda_i = 1$, and the last equality since $x_j = 1$ implies that $z_j = y_j$. Therefore, $(x, y, z) = \sum_{i \in I} \lambda_i (x^i, y^i, z^i) \in \text{conv}(S)$. $\square$

Finally, we show that the projections of G to the (x, z) and (x, y) spaces are not altered even if G is relaxed in a certain way. Let

$$R = \left\{ (x, y, z) \in \mathbb{R}^{3n} \;\middle|\; \alpha x + \beta z \geq \gamma \; \forall (\alpha, \beta, \gamma) \in \mathcal{F}(PF), \, z \leq \min\{x, y\}, \, y \leq \mathbf{1} \right\}.$$

Corollary 5 $PF = \mathit{proj}_{(x,z)}R$ *and* $PB = \mathit{proj}_{(x,y)}R$.

Proof We will show that $\text{proj}_{(x,z)}R = \text{proj}_{(x,z)}G$ and $\text{proj}_{(x,y)}R = \text{proj}_{(x,y)}G$. The result then follows from Lemma 6 and Theorem 4. Since $z + \mathbf{1} - x \leq \mathbf{1}$, we know that $R \supseteq G$. It follows that $\text{proj}_{(x,z)} R \supseteq \text{proj}_{(x,z)} G$ and $\text{proj}_{(x,y)} R \supseteq \text{proj}_{(x,y)} G$. To complete the proof, we first show that $\text{proj}_{(x,z)} R \subseteq \text{proj}_{(x,z)} G$. Assume that $(x, y, z) \in R$. Then, define $y' = z + \mathbf{1} - x$. Since $z + \mathbf{1} - x \geq z$ it follows that $(x, y', z) \in G$. Second, we show that $\text{proj}_{(x,y)} R \subseteq \text{proj}_{(x,y)} G$. Assume that $(x, y, z) \in R$. Then, let $z' = \max\{z, x + y - \mathbf{1}\}$. By Lemma 4, for all $(\alpha, \beta, \gamma) \in \mathcal{F}(PF)$, $\beta \geq 0$. Therefore, $\alpha x + \beta z' \geq \alpha x + \beta z \geq \gamma$. Further, $z' = \max\{z, x + y - \mathbf{1}\} \leq \min\{x, y\}$ since $z \leq \min\{x, y\}$ and $x, y \in [0, 1]^2$. Finally, by construction, $y \leq z' + \mathbf{1} - x$. Therefore, $(x, y, z') \in G$. □

Corollary 5 implies every non-trivial facet of *PB* arises as a conic combination of a single non-trivial facet of *PF* and (possibly multiple) trivial facet-defining inequalities $y_j \leq x_j$.

Corollary 6 *Let $\alpha x + \beta y \geq \gamma$ be a facet-defining inequality for PB where $\beta \geq 0$. Then, $\alpha x + \beta y \geq \gamma$ defines a non-empty face of PF. Further, there exists (α', β') and $\lambda \geq 0$ such that $(\alpha, \beta) = (\alpha' + \lambda, \beta' - \lambda)$, where $\alpha' x + \beta' y \geq \gamma$ is facet-defining for PF and $\lambda_j \beta_j = 0$ for $j = 1, \ldots, n$.*

Proof Let $\delta = \min\{\alpha x + \beta y \mid (x, y) \in PF\}$. Since, by Lemma 3, $F \subseteq B$, it follows that $\delta \geq \gamma$. By Proposition 23, $\alpha x + \beta y \geq \delta$ is valid for *PB*. Therefore, $\delta \leq \gamma$. In other words, $\delta = \gamma$ and $\alpha x + \beta y \geq \gamma$ defines a non-empty face of *PF*. By Corollary 5 and Fourier-Motzkin elimination of z from R, it follows that,

$$PB = \left\{ (x, y) \;\middle|\; \alpha' x + \beta'^J x + \beta'^{N \setminus J} y \geq \gamma' \,\forall (\alpha', \beta', \gamma') \in \mathcal{F}(PF) \text{and } J \subseteq N, y \leq 1 \right\},$$

where $\beta'^J_j = \beta'_j$ if $j \in J$ and $\beta'^J_j = 0$ otherwise. Since (α, β, γ) is not a multiple of $y_j \leq 1$, it follows that there exists $J \subseteq N$ and $(\alpha', \beta', \gamma') \in \mathcal{F}(PF)$ such that $(\alpha, \beta) = (\alpha' + \beta'^J, \beta' - \beta'^J)$. □

Example 9 Consider the inequality $126x_1 + 90x_3 + 45x_4 + 153y_2 \geq 135$*, which is (71) scaled by a factor of* 9*. This inequality is facet-defining for the bilinear covering set of Example 1 but not facet-defining for the corresponding flow set presented in Example 8. Then, as Corollary 6 proves, this inequality can be expressed as the sum of* $50x_1 + 90x_3 + 45x_4 + 76y_1 + 153y_2 \geq 135$ *and* $76x_1 - 76y_1 \geq 0$*, which are facet-defining inequalities for the corresponding flow set.*

Proposition 23 and Corollary 5 show that a polyhedral description of either *PF* or *PB* can be derived explicitly given the facet-defining inequalities of the other. In fact, Proposition 23 also shows that an affine function over either B or F can be optimized if we have an oracle for optimizing an affine function over the other set. We discuss the reduction below. Let $l(x, y) = \alpha x + \beta y - \gamma$ and define $I = \{i \in N \mid \alpha_i > 0, \beta_i < 0\}$. Let $z_B(l) = \min\{l(x, y) \mid (x, y) \in B\}$ and $z_F(l) = \min\{l(x, y) \mid (x, y) \in F\}$. First, consider minimizing $l(x, y)$ over B using an oracle for minimizing an affine function over F. Define $l'(x, y) = \alpha x + \sum_{i \in N \setminus I} \beta_i y_i + \sum_{i \in I} \beta_i - \gamma$. While minimizing $l(x, y)$ over B, y_i can be set to 1 whenever $\beta_i \leq 0$. Therefore, it follows that $z_B(l) = z_B(l')$. However, by Proposition 23, $z_F(l') = z_B(l')$. Therefore, $z_B(l) = z_F(l')$. If (x, y) is an optimal solution to $z_F(l')$, then (x, y') where $y'_i = 1$ if $i \in I$ and

$y'_i = y_i$ otherwise, is an optimal solution to $z_B(l)$. Now, consider minimizing $l(x, y)$ over F using an oracle for minimizing an affine function over B. Define $l''(x, y) = \alpha x + \sum_{i \in I} \beta_i x_i + \sum_{i \in N \setminus I} \beta_i y_i - \gamma$. While minimizing $l(x, y)$ over F, y_i can be set to x_i whenever $\beta_i \leq 0$. Therefore, $z_F(l) = z_F(l'')$. However, by Proposition 23, $z_F(l'') = z_B(l'')$. Therefore, $z_F(l) = z_B(l'')$. If (x, y) is an optimal solution to $z_B(l'')$, then (x', y'), where $(x'_i, y'_i) = (x_i, x_i)$ for $i \in I$, $(x'_i, y'_i) = (1, y_i)$ if $\alpha_i \leq 0$, and $(x'_i, y'_i) = (x_i, x_i y_i)$ otherwise is an optimal solution to $z_F(l)$.

Given the relationships between the polyhedra *PB* and *PF*, it is reasonable to expect that the inequalities we developed in Sect. 3 reveal facets of *PF*. We now provide a detailed discussion of which inequalities are facet-defining for *PF*. For the remainder of this section, we assume, as we did for *PB*, that

Assumption 3 $\sum_{j=1}^{n} a_j \geq d + a_i$ *for all* $i \in N$.

Under Assumption 3, it follows from Lemma 5 that *PF* is a full-dimensional polytope.

Theorem 5 *A lifted bilinear cover inequality (45) is facet-defining for PF if and only if*

$$(\alpha_i, \beta_i) \in \bigcup_{j=0}^{q_i} \left\{ \left(P^C\left(Q^i_j\right) - \frac{P^C\left(Q^i_{j+1}\right) - P^C\left(Q^i_j\right)}{\Delta^i_j} Q^i_j, \frac{P^C\left(Q^i_{j+1}\right) - P^C\left(Q^i_j\right)}{\Delta^i_j} a_i \right) \right\} \tag{72}$$

for all $i \in M$.

Proof The proof of Proposition 9 already shows that (33) is facet-defining for $PF(M, C', M, C')$ since all the points considered are feasible to the flow set.

Now, it suffices to show that sufficiently many of the tight points added when lifting variables (x_i, y_i) for $i \in M \cup C'$ also belong to *PF*. When we lifted variables (x_i, y_i) for $i \in C'$ in the proof of Proposition 16, we added the two affinely independent points $(0, 0)$ and $\left(1, \frac{(a_i - \mu)^+}{a_i}\right)$ that both correspond to feasible solutions of F; see (33) and Corollary 4. When lifting the variables (x_i, y_i) for $i \in M$ in Theorem 1, we added the two points $\left(1, \frac{Q^i_j}{a_i}\right)$ and $\left(1, \frac{Q^i_{j+1}}{a_i}\right)$ that both correspond to feasible solutions of F; see (36) and Corollary 4.

Next, we show that if (45) is facet-defining for *PF*, then (α_i, β_i) must be chosen as in (72). It suffices to show that if $(\alpha_i, \beta_i) = (P^C(a_i), 0)$ for some $i \in M$ and if at least one of the coefficients pair $(P^C(a_i), 0)$ does not reduce to coefficients studied before (which happens when $P^C(a_v) \neq P^C(Q^v_{q_v})$ for some v), then (45) is not facet-defining for *PF*. We will do so by showing that in such a case, (45) can be obtained by combining a different (facet-defining) inequality of the form (45) for *PF* with trivial facets $y_i \leq x_i$ of *PF*. Let $V \subseteq M$ be the set of lifting coefficients (α_v, β_v) chosen to be $(P^C(a_v), 0)$. Inequality (45) then reduces to

$$\sum_{v\in V} P^C(a_v)x_v+\sum_{i\in C}(a_i-\mu)^+x_i+\sum_{j\in T} a_j y_j+\sum_{i\in M\setminus V}\alpha_i x_i+\sum_{i\in M\setminus V}\beta_i y_i\geq\sum_{i\in C}(a_i-\mu)^+. \tag{73}$$

Using the first part of this proof, we know that choosing lifting coefficients

$$\left(\left(P^C\left(Q^v_{q_v}\right)-\frac{P^C\left(Q^v_{q_v+1}\right)-P^C\left(Q^v_{q_v}\right)}{\Delta^v_{q_v}}Q^v_{q_v}\right),\left(\frac{P^C\left(Q^v_{q_v+1}\right)-P^C\left(Q^v_{q_v}\right)}{\Delta^v_{q_v}}a_v\right)\right)$$

for $v \in V$ yields the following facet-defining inequality

$$\begin{aligned}&\sum_{v\in V}\left(P^C\left(Q^v_{q_v}\right)-\frac{P^C\left(Q^v_{q_v+1}\right)-P^C\left(Q^v_{q_v}\right)}{\Delta^v_{q_v}}Q^v_{q_v}\right)x_v\\&\quad+\left(\frac{P^C\left(Q^v_{q_v+1}\right)-P^C\left(Q^v_{q_v}\right)}{\Delta^v_{q_v}}a_v\right)y_v\\&\quad+\sum_{i\in C}(a_i-\mu)^+x_i+\sum_{j\in T}a_jy_j+\sum_{i\in M\setminus V}\alpha_ix_i+\sum_{i\in M\setminus V}\beta_iy_i\geq\sum_{i\in C}(a_i-\mu)^+\end{aligned} \tag{74}$$

for PF. Summing (74) with

$$\left(\frac{P^C\left(Q^v_{q_v+1}\right)-P^C\left(Q^v_{q_v}\right)}{\Delta^v_{q_v}}a_v\right)(x_v-y_v)\geq 0,\forall v\in V \tag{75}$$

we obtain (73) since $Q^v_{q_v+1} = a_v$ and $\Delta^v_{q_v} = a_v - Q^v_{q_v}$. Since we assumed that $P^C(a_v) - P^C(Q^v_{q_v}) > 0$ for some $v \in V$ and because it is easy to see that (75) does not define the same face of PF that (74) defines, we conclude that (73) is not facet-defining for PF. □

We remark that in the proof of Theorem 5, we proved that a few inequalities of the type (45) are facet-defining for PB but not for PF. This was shown by expressing these inequalities using another non-trivial facet of PF and the inequalities $y_j \leq x_j$. We have already shown in Corollary 6 that such construction can be used to describe all facet-defining inequalities of PB that are not facet-defining for PF. We will use similar constructions later in the section. As a consequence of Theorem 5, we obtain the following result initially obtained in [22].

Corollary 7 (Adapted from Proposition 12 in [22]) *Assume that (i) $\mathcal{C}$ is a cover with excess $\bar{\mu} = \sum_{j\in\mathcal{C}} a_j - d$ such that $\bar{a} = \max_{j\in\mathcal{C}} a_j > \bar{\mu}$ and (ii) $\mathcal{L} \subseteq N\setminus\mathcal{C}$ is chosen so that $0 < \bar{a} - \bar{\mu} < a_k \leq \bar{a}$ for all $k \in \mathcal{L}$ and $\sum_{j\in N\setminus\mathcal{L}} a_j > d + \bar{a}$. Then*

$$\sum_{j\in\mathcal{C}}(a_j-\bar{\mu})^+x_j+\sum_{j\in\mathcal{L}}(\bar{a}-\bar{\mu})x_j+\sum_{j\in N\setminus(\mathcal{C}\cup\mathcal{L})}a_jy_j\geq\sum_{j\in\mathcal{C}}(a_j-\bar{\mu})^+ \tag{76}$$

is facet-defining for PF.

Proof Let $\mathcal{C}$ and $\mathcal{L}\subseteq N\setminus\mathcal{C}$ be given that satisfy conditions (i) and (ii) of Corollary 7. Select $l\in\text{argmax}\{a_j \mid j\in\mathcal{C}\}$. Define $C'=\mathcal{C}\setminus\{l\}$, $M=\mathcal{L}$, and $T=N\setminus(\mathcal{C}\cup\mathcal{L})$. Clearly, $\mu=\bar{\mu}$. Observe further that $a_l=\bar{a}>\mu$ and that $\sum_{j\in T}a_j>a_l-\bar{\mu}$ since $\sum_{j\in N\setminus\mathcal{L}}a_j>d+\bar{a}$. It follows that $(C',\{l\},M,T)$ is a partition of N that satisfies Conditions (A1), (A2), (A3), and (A4) of Theorem 1. We obtain from Assumption (ii) that $A_1-\mu<a_i\leq A_1<A_2-\mu$ for $i\in M$, which implies that $q_i=1$ for all $i\in M$ in Lemma 1. Further, since $Q_1^i=A_1-\mu$ and $Q_2^i=a_i$ for $i\in M$, we can select (α_i,β_i) as $(A_1-\mu,0)$ in (45), yielding

$$\sum_{j\in C}(a_j-\mu)^+x_j+\sum_{j\in T}a_jy_j+\sum_{j\in M}(A_1-\mu)x_j\geq\sum_{j\in C}(a_j-\mu)^+,$$

which is exactly (76) after substituting $C=\mathcal{C}$, $T=N\setminus(\mathcal{C}\cup\mathcal{L})$, $M=\mathcal{L}$, $A_1=\bar{a}$ and $\mu=\bar{\mu}$. □

Observe that in (76), for each $j\in N$, either the coefficient of x_j or that of y_j is zero, whereas this is not the case for (45). Therefore, the family of facet-defining inequalities obtained via (76) is strictly contained in the family of facet-defining inequalities obtained via (45). In [22], the authors did not explicitly impose the condition $\sum_{j\in N\setminus\mathcal{L}}a_j>d+\bar{a}$. However, in its absence, the inequalities are not necessarily facet-defining as we show in Example 10. The authors' proof implicitly used this assumption during an induction step.

Example 10 Consider the flow set defined by

$$F=\left\{(x,y)\in\{0,1\}^4\times[0,1]^4\;\middle|\;7y_1+6y_2+5y_3+4y_4\geq 10,\,x_j\geq y_j\;\forall j=1,\ldots,4\right\}.$$

Define $\mathcal{C}=\{1,3\}$ and $\mathcal{L}=\{2\}$ where $\bar{a}=7$ and $\bar{\mu}=2$. Clearly $\bar{a}-\bar{\mu}<a_2\leq\bar{a}$. However, the assumption that $\sum_{j\in N\setminus\mathcal{L}}a_j>d+\bar{a}$ does not hold. Inequality (76) takes the form

$$5x_1+5x_2+3x_3+4y_4\geq 8. \tag{77}$$

Observe that whenever (77) is satisfied at equality by a point of F, the inequality $x_1+x_2\geq 1$ is also tight. Since $x_1+x_2\geq 1$ is clearly valid for F, it follows that (77) is not facet-defining for PF.

We next show that the family of lifted bilinear cover inequalities that are proven to be facet-defining for PF in Theorem 5 is larger than the family given by (76).

Example 11 As established in Example 8, (5) and (6) are facet-defining lifted bilinear cover inequalities (45) for both PB and PF. They are obtained by choosing

$(C', l, M, T) = (\{4\}, \{3\}, \{1\}, \{2\})$ *and* $(C', l, M, T) = (\{4\}, \{2\}, \{1\}, \{3\})$ *respectively in Theorems* 1 and 5. *However, as mentioned above, (5) and (6) cannot be obtained using Corollary 7.*

Next, we describe when the lifted reverse bilinear cover inequalities (54) that define facets of PB also define facets of PF.

Theorem 6 *Lifted bilinear reverse cover inequalities (54) are facet-defining for PF if and only if* $a_i > a_l - \mu$ *for all* $i \in M$. □

The proof is similar to that of Theorem 5, constructing tight points when $a_i > a_l - \mu$ for $i \in M$ and showing that (54) can be obtained as a conic combination of the lifted reverse bilinear cover inequality based on partition $(C', \{l\}, M_1, T \cup M_2)$ where $M_2 = \{i \in M \,|\, a_i \le a_l - \mu\} \neq \emptyset$ and $M_1 = M \backslash M_2$ and the inequalities $a_j(x_j - y_j) \ge 0$ for $j \in M_2$ when $a_i \le a_l - \mu$ for some $i \in M$; see [9] for details.

The inequalities of Theorem 6 are known to be valid for PF, as first shown in [12].

Corollary 8 (Adapted from Theorem 12 in [12]) *Assume that (i)* $\mathcal{C} \subseteq N$ *is a generalized cover for* F *such that* $\sum_{j \in \mathcal{C}} a_j = d - \lambda$ *with* $\lambda > 0$ *and (ii)* $\mathcal{L} \neq \emptyset$ *and* $\sum_{j \in N \backslash \mathcal{L}} a_j > d$ *where* $\mathcal{L} = \{j \in N \backslash \mathcal{C} \mid a_j > \lambda\}$. *Assume also that* $\mathcal{L} = \{j_1, j_2, \ldots, j_r\}$ *with* $a_{j_i} \ge a_{j_{i+1}}$ *for* $i = 1, \ldots, r-1$. *Let* $r = |\mathcal{L}|$, $A_0 = 0$, *and* $A_i = \sum_{k=1}^{i} a_{j_k}$ *for* $i = 1, \ldots, r$. *Further, let* $d' = \sum_{j \in N \backslash \mathcal{C}} a_j - \lambda$. *Define*

$$f(z) = \begin{cases} i\lambda & \text{if } A_i \le z \le A_{i+1} - \lambda, \quad i = 0, \ldots, r-1, \\ z - A_i + i\lambda & \text{if } A_i - \lambda \le z \le A_i, \qquad i = 1, \ldots, r-1, \\ z - A_r + r\lambda & \text{if } A_r - \lambda \le z \le d'. \end{cases} \tag{78}$$

Then, the lifted simple generalized flow cover inequality (LSGFCI)

$$\sum_{j \in \mathcal{L}} \lambda x_j + \sum_{j \in \mathcal{C}} f(a_j) x_j + \sum_{j \in N \backslash (\mathcal{C} \cup \mathcal{L})} a_j y_j \ge \lambda + \sum_{j \in \mathcal{C}} f(a_j) \tag{79}$$

is facet-defining for PF.

Proof For a given *generalized cover* $\mathcal{C}$ of F, we define $C = \mathcal{C} \cup \{l\}$ where $l \in \mathcal{L} \neq \emptyset$. Set C is a cover since $a_j > \lambda$ for all $j \in \mathcal{L}$. Further, $\sum_{j \in C} a_j = d + a_l - \lambda > d$ and so $\mu = a_l - \lambda > 0$, *i.e.*, C satisfies Conditions (A1) and (A3) in Theorem 2. Now set $M = \mathcal{L} \backslash \{l\}$ in (54). Condition (A4) in Theorem 2 also holds since $\sum_{j \in N \backslash (\mathcal{L} \backslash \{l\})} a_j - d = \sum_{j \in N \backslash \mathcal{L}} a_j + a_l - d > a_l$. Next, we observe that $C \cup M = \mathcal{C} \cup \mathcal{L}$ and that $\min\{a_i, a_l - \mu\} = \min\{a_i, \lambda\} = \lambda = a_l - \mu$ for all $i \in M$. Substituting $a_l - \mu = \lambda$ in Proposition 19, we obtain that $f(w) = -P^M(-w)$ since $M \cup \{l\} = \mathcal{L}$. Therefore, we conclude that (79) is a lifted reverse bilinear cover inequality (54). □

Because in [12] the fixed-charge single-node flow set studied is more general than F, the authors focused mainly on the derivation of valid inequalities and discussed only indirectly whether the resulting inequalities are facet-defining. The result of Corollary 8 is therefore different from that of Theorem 12 in [12] in two ways. First we added the condition $\sum_{j \in N \backslash \mathcal{L}} a_j > d$. This condition guarantees that the simple generalized flow cover inequality (SGFCI) that is used as seed inequality for lifting procedures in [12] is facet-defining for the problem restriction. Second, we replaced the statement that inequality (79) is valid for PF with the stronger statement that it is facet-defining for PF.

We conclude this section by presenting conditions under which the lifted clique inequalities (64) are facet-defining for the flow set PF.

Theorem 7 *A lifted clique inequality (64) is facet-defining for PF if (i) $\sum_{j \in K} a_j - a_k > d$ for all $k \in K$, (ii) lifting coefficients are chosen according to (65) and (iii) one of the following conditions holds:*

1. $L = \emptyset$.
2. $\exists \bar{\imath} \in M$ *such that* $j_{\bar{\imath}} = 0$ *and, for all* $i \in L \backslash \{\bar{\imath}\}$, $j_i = q_i$.

Proof Using a proof technique similar to that used in Theorems 5 and 6, we show that seed inequality (59) is facet-defining for $PF(\overline{M} \backslash \widehat{M}, \emptyset, \overline{M} \backslash \widehat{M}, \emptyset)$ and that lifting (x_i, y_i) for $i \in M$ adds two tight independent points in (64) that belong to F. Let $K = \{1, \ldots, l\}$ and $\widehat{M} = \{l+1, \ldots, h\}$. Define the vector $\chi \in \mathbb{R}^{|N|}$ such that $\chi_j = 1$ for $j \leq l$ and $\chi_j = 0$ for $l+1 \leq j \leq |N|$. Consider $p^i = (\chi - e_i, \chi - e_i)$ for $i = 1, \ldots, l$, $q^i = (\chi - e_i, \chi - e_i - \epsilon e_{i+1})$ for $i = 1, \ldots, l-1$, $q^l = (\chi - e_l, \chi - e_l - \epsilon e_1)$ where ϵ is positive, and, for $j = l+1, \ldots, h$, $r^j = \big(\chi - e_1 + e_j, \chi - e_1\big)$ and $s^j = (\chi - e_1 + e_j, \chi - e_1 + e_j)$. These points satisfy (59) at equality, are affinely independent and, because of Assumption (i), belong to F when ϵ is sufficiently small. This shows that (59) is facet-defining for $PF(M, \emptyset, M, \emptyset)$. Assume first that $L = \emptyset$ and consider now the lifting of variables (x_i, y_i) for $i \in M$ in the proof of Theorem 3. For $j_i \in \{0, \ldots, q_i\}$, lifting adds the two independent points $\left(1, \frac{W_j^i}{a_i}\right)$ and $\left(1, \frac{W_{j+1}^i}{a_i}\right)$ that both correspond to feasible solutions of F because of (59) and Corollary 4, proving the result. Then it follows from the first part of this proof that the inequality obtained after lifting the variables in $M \backslash L$ is facet-defining for $PF(L \backslash \{\bar{\imath}\}, \emptyset, L \backslash \{\bar{\imath}\}, \emptyset)$. Consider now the lifting of variables (x_i, y_i) for $i \in L \backslash \{\bar{\imath}\}$. When $j_i = q_i$, we derived in the proof of Theorem 3 that lifting adds the two independent points $(1, 1)$ and $\left(1, \frac{W_{j_i}^i}{a_i}\right)$ that both correspond to feasible solutions of F because the first point sets $(x_{\bar{\imath}}, y_{\bar{\imath}}) = \left(1, \frac{B_{q_i} + \bar{\mu} - a_i}{a_{\bar{\imath}}}\right)$, and the structure of (59) satisfies the assumptions of Corollary 4. □

To the best of our knowledge, Theorem 7 presents a new family of facet-defining inequalities for fixed-charge single-node flow models without inflows. We remark that the facet-defining inequalities hitherto known in the literature are such that, for all $j \in N$, only one of x_j or y_j has a non-zero coefficient in the inequality; see (76) and (79). However, the families of lifted bilinear cover inequalities (72) and of lifted clique inequalities (64) each contain exponentially many facet-defining inequalities in which there exists an index $j \in N$ such that both x_j and y_j variables have non-zero coefficients. Therefore, the inequalities we have obtained exhibit a fundamentally more general structure. In particular, it follows from Proposition 10 and the ensuing discussion that these inequalities do not arise as facet-defining inequalities of mixed 0-1 knapsack sets obtained by fixing some x and y variables to one.

5 Discussion and conclusion

Many of the results presented in this paper extend to $0-1$ mixed integer nonlinear sets defined by constraints of the form $\sum_{i \in N}(a_i x_i y_i + b_i x_i + c_i y_i) \geq d$ where a linear term has been added to the left-hand-side, provided that $a_i + b_i \geq 0$ and $a_i + c_i \geq 0$ for all $i \in N$. The primary reason that the techniques extend to this case is that the lifting functions are equal to those derived in Sect. 3 if the seed inequalities contain no more than one of the variables (x_i, y_i) for each $i \in N$. We give a proof of this assertion in the next proposition.

Proposition 24 *Let $a_i \in \mathbb{R}$ and $(b_i, c_i) \in \mathbb{R}^2_+$ be such that $a_i + \min\{b_i, c_i\} \geq 0$ for each $i \in N$. Let $(\alpha_i, \beta_i) \in \mathbb{R}^2$ be such that $\alpha_i \beta_i = 0$ for each $i \in N$. Define $I = \{i \in N \mid \alpha_i = 0\}$ and $I^c = N \backslash I$. Also define*

$$z_{A(w)} = \min \left\{ \sum_{i \in N} (\alpha_i x_i + \beta_i y_i) \,\middle|\, (x, y) \in A(w) \right\}$$

where $A(w) = \{(x, y) \in \{0, 1\}^n \times [0, 1]^n \mid \sum_{i=1}^n (a_i x_i y_i + b_i x_i + c_i y_i) \geq d - w\}$, and

$$z_{B(w)} = \min \left\{ \sum_{i \in N} (\alpha_i x_i + \beta_i y_i) \,\middle|\, (x, y) \in B(w) \right\}$$

where $B(w) = \{(x, y) \in \{0, 1\}^n \times [0, 1]^n \mid \sum_{i \in I}(a_i + c_i) x_i y_i + \sum_{i \in I^c}(a_i + b_i) x_i y_i \geq d - \sum_{i \in I} b_i - \sum_{i \in I^c} c_i - w\}$. Then $z_{A(w)} = z_{B(w)}$. Further, for $H^+ = \{(x, y) \in \mathbb{R}^{2n} \mid \sum_{i \in N}(\alpha_i x_i + \beta_i y_i) \geq \gamma\}$, $H^+ \supseteq A(w)$ if and only if $H^+ \supseteq B(w)$.

Proof We first claim that, for any point $(x', y') \in \{0, 1\}^n \times [0, 1]^n$ with $x'_i = 1$ for each $i \in I$ and $y'_i = 1$ for each $i \in I^c$, $(x', y') \in A(w)$ if and only if $(x', y') \in B(w)$. Consider such a point (x', y'). Then, for $i \in I$, $a_i x'_i y'_i + b_i x'_i + c_i y'_i = (a_i + c_i) x'_i y'_i + b_i$. Similarly, for $i \in I^c$, $a_i x'_i y'_i + b_i x'_i + c_i y'_i = (a_i + b_i) x'_i y'_i + c_i$. In other words, $\sum_{i \in I}(a_i + c_i) x'_i y'_i + \sum_{i \in I^c}(a_i + b_i) x'_i y'_i + \sum_{i \in I} b_i + \sum_{i \in I^c} c_i = \sum_{i=1}^n (a_i x'_i y'_i + b_i x'_i + c_i y'_i)$; proving the claim.

Defining α and β to be vectors with components α_i and β_i respectively, we see that

$$\begin{aligned} z_{A(w)} &= \min\{\alpha x + \beta y \mid (x, y) \in A(w)\} \\ &= \min\{\alpha x + \beta y \mid (x, y) \in A(w), x_i = 1 \forall i \in I, y_i = 1 \forall i \in I^c\} \\ &= \min\{\alpha x + \beta y \mid (x, y) \in B(w), x_i = 1 \forall i \in I, y_i = 1 \forall i \in I^c\} \\ &= \min\{\alpha x + \beta y \mid (x, y) \in B(w)\} = z_{B(w)}, \end{aligned}$$

where the second and the second last equalities follow from the assumptions which imply that $a_i x_i y_i + b_i x_i + c_i y_i \leq \min\{a_i x_i + b_i x_i + c_i, a_i y_i + b_i + c_i y_i\}$, $(a_i + c_i) x_i y_i \leq (a_i + c_i) y_i$, and $(a_i + b_i) x_i y_i \leq (a_i + b_i) x_i$ for each $i \in N$. Since $z_{A(w)} = z_{B(w)}$ and $H^+ \supseteq A(w)$ (resp. $H^+ \supseteq B(w)$) if and only if $z_{A(w)} \geq \gamma$ (resp. $z_{B(w)} \geq \gamma$), it follows that $H^+ \supseteq A(w)$ if and only if $H^+ \supseteq B(w)$. $\square$

In Proposition 24, $z_A(w)$ can be thought of as the lifting function of an inequality of the form $\alpha x + \beta y \geq \delta$ for some δ in a 0-1 mixed integer bilinear covering set with linear term while $z_B(w)$ can be thought of as the lifting function of the same inequality in PB. Because the seed inequalities we use for lifting satisfy the condition $\alpha_i \beta_i = 0$, Proposition 24 essentially shows that the lifting functions derived for the problem with only bilinear terms on the left-hand-side carry over to problems containing linear terms. For example, it is shown in [8] that the two families of lifted bilinear cover inequalities and lifted reverse bilinear cover inequalities have natural extensions for bilinear covering sets with linear terms where, for all $i \in N$, $a_i \geq 0$, $b_i c_i = 0$, $b_i \geq 0$ and $c_i = a_i$ whenever $c_i \neq 0$. Such sets occur naturally after 0-1 and continuous branching is performed in PB.

In this paper, we study the polyhedral structure of the 0−1 mixed-integer bilinear covering set. We give a complete linear description of its convex hull when $n = 2$. We then derive three families of strong inequalities for PB that can be obtained using sequence-independent lifting. Among them, two families have an exponential number of members. We study relations between 0−1 mixed-integer bilinear covering sets and fixed-charge single-node flow sets without inflows. We show that valid inequalities for bilinear sets are also valid for flow sets and prove that all nontrivial facets of PF can be obtained through the study of facets of PB. We then show that the inequalities we derive generalize two classical families of lifted flow cover inequalities for PF and provide a new family for PF. Future research will focus on evaluating the computational benefits of using these lifted cuts in branch-and-bound frameworks for both linear and nonlinear mixed integer programming.

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