

The triangle closure is a polyhedron

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Abstract Recently, cutting planes derived from maximal lattice-free convex sets have been studied intensively by the integer programming community. An important question in this research area has been to decide whether the closures associated with certain families of lattice-free sets are polyhedra. For a long time, the only result known was the celebrated theorem of Cook, Kannan and Schrijver who showed that the split closure is a polyhedron. Although some fairly general results were obtained by Andersen et al. (Math Oper Res 35(1):233–256, 2010) and Averkov (Discret Optimiz 9(4):209–215, 2012), some basic questions have remained unresolved. For example, maximal lattice-free triangles are the natural family to study beyond the family of splits and it has been a standing open problem to decide whether the triangle closure is a polyhedron. In this paper, we show that when the number of integer variables $m = 2$ the triangle closure is indeed a polyhedron and its number of facets can be bounded by a polynomial in the size of the input data. The techniques of this proof are also used to give a refinement of necessary conditions for valid inequalities being facet-defining due to Cornuéjols and Margot (Math Program 120:429–456, 2009) and obtain polynomial complexity results about the mixed integer hull.

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1 Introduction

We study the following system, introduced by Andersen et al. [2]:

$$\begin{aligned} x &= f + \sum_{j=1}^k r^j s_j \\ x &\in \mathbb{Z}^m \\ s_j &\geq 0 \quad \text{for all } j = 1, \dots, k. \end{aligned} \quad (1)$$

This model has been studied extensively with the purpose of providing a unifying theory for cutting planes and exploring new families of cutting planes [2, 6–8, 12–14]. In this theory, an interesting connection is explored between valid inequalities for the convex hull of solutions to (1) (the *mixed integer hull*) and maximal lattice-free convex sets in \mathbb{R}^m . A *lattice-free* convex set is a convex set that does not contain any integer point in its interior. A *maximal* lattice-free convex set is a lattice-free convex set that is maximal with respect to set inclusion. Since the x variables are uniquely determined by the s_j variables, only the values of the s_j variables need to be recorded for the system (1), as done with the following notation:

$$R_f = \left\{ s \in \mathbb{R}_+^k \mid f + \sum_{j=1}^k r^j s_j \in \mathbb{Z}^m \right\} \quad (2)$$

where \mathbb{R}_+^k denotes the nonnegative orthant in \mathbb{R}^k . The mixed integer hull is then denoted by $\text{conv}(R_f)$ and can be obtained by intersecting all valid inequalities derived using the Minkowski functional of maximal lattice-free convex sets containing f in their interior [2, 5, 9, 20]. We explain this more precisely after introducing some notation.

Let $B \in \mathbb{R}^{n \times m}$ be a matrix with n rows $b^1, \dots, b^n \in \mathbb{R}^m$. We write $B = (b^1; \dots; b^n)$. Let

$$M(B) = \{ x \in \mathbb{R}^m \mid B \cdot (x - f) \leq e \}, \quad (3)$$

where e is the vector of all ones. This is a polyhedron with f in its interior. We will denote the set of its vertices by $\text{vert}(B)$. In fact, any polyhedron with f in its interior can be given such a description. We will mostly deal with matrices B such that $M(B)$ is a maximal lattice-free convex set in \mathbb{R}^m .

The Minkowski functional for the set $M(B)$ can be defined as

$$\psi_B(r) = \max_{i \in \{1, \dots, n\}} b^i \cdot r \quad \text{for } r \in \mathbb{R}^m.$$

Proposition 1.1 *If $B \in \mathbb{R}^{n \times m}$ is a matrix such that $M(B)$ is a lattice-free convex set in \mathbb{R}^m with f in its interior, then the inequality $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ is a valid inequality for (1).*

Proof Let $s \in R_f$. Then $x = f + \sum_{j=1}^k r^j s_j \in \mathbb{Z}^m$. Consider $\psi_B(x - f)$ and let $b \in \{b^1, \dots, b^n\}$ such that $\psi_B(x - f) = b \cdot (x - f)$. Since $x \in \mathbb{Z}^m$ and $M(B)$ is lattice-free, by definition of $M(B)$, we have $b \cdot (x - f) \geq 1$. Thus

$$1 \leq b \cdot (x - f) = b \cdot \sum_{j=1}^k r^j s_j = \sum_{j=1}^k (b \cdot r^j) s_j \leq \sum_{j=1}^k \psi_B(r^j) s_j.$$

Therefore, the inequality $\sum_{j=1}^k \psi_B(r^j)s_j \geq 1$ holds for s . Since $s \in R_f$ was chosen arbitrarily, it is a valid inequality for (1). \square

We define the vector of coefficients as

$$\gamma(B) = \left(\psi_B(r^j) \right)_{j=1}^k$$

and therefore can write the mixed integer hull as

$$\text{conv}(R_f) = \left\{ s \in \mathbb{R}_+^k \mid \begin{array}{l} \gamma(B) \cdot s \geq 1 \text{ for all } B \in \mathbb{R}^{n \times m} \text{ such that} \\ M(B) \text{ is a maximal lattice-free convex set} \end{array} \right\}. \tag{4}$$

Motivation All maximal lattice-free convex sets are polyhedra [7, 16]. The most primitive type of maximal lattice-free convex set in \mathbb{R}^m is the *split*, which is of the form $\pi_0 \leq \pi \cdot x \leq \pi_0 + 1$ for some $\pi \in \mathbb{Z}^m$ and $\pi_0 \in \mathbb{Z}$. A famous theorem due to Cook et al. [11] implies that the intersection of all valid inequalities for (1) derived from splits, known as the *split closure*, is a polyhedron. The split closure result has been used repeatedly as a theoretical as well as practical tool in many diverse settings within the integer programming community. This motivates the following question: For which families of lattice-free convex sets is the associated closure a polyhedron? Not much was known about this question until very recently when some elegant results of a more general nature were obtained in [1] and [3]. Even so, some basic questions have remained open. Consider the case $m = 2$. For this case, the different types of maximal lattice-free convex sets have been classified quite satisfactorily. Lovász characterized the maximal lattice-free convex sets in \mathbb{R}^2 as follows.

Theorem 1.2 (Lovász [16]) *In the plane, a maximal lattice-free convex set with non-empty interior is one of the following:*

- (a) *A split $c \leq ax_1 + bx_2 \leq c + 1$ where a and b are co-prime integers and c is an integer;*
- (b) *A triangle with an integral point in the interior of each of its edges;*
- (c) *A quadrilateral containing exactly four integral points, with exactly one of them in the interior of each of its edges. Moreover, these four integral points are vertices of a parallelogram of area 1.*

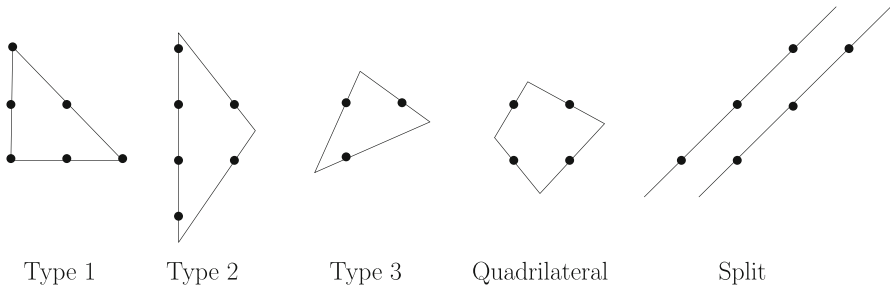


Fig. 1 Types of maximal lattice-free convex sets in \mathbb{R}^2

Following Dey and Wolsey [14], the maximal lattice-free triangles can be further partitioned into three canonical types:

- *Type 1 triangles*: triangles with integral vertices and exactly one integral point in the relative interior of each edge;
- *Type 2 triangles*: triangles with at least one fractional vertex v , exactly one integral point in the relative interior of the two edges incident to v and at least two integral points on the third edge;
- *Type 3 triangles*: triangles with exactly three integral points on the boundary, one in the relative interior of each edge.

Figure 1 shows these three types of triangles as well as a maximal lattice-free quadrilateral and a split satisfying the properties of Theorem 1.2.

For this simple case of $m = 2$, it was not even known whether the triangle closure (the convex set formed by the intersection of all inequalities derived from maximal lattice-free triangles) is a polyhedron. The results from [1] and [3] cannot be used as they use an assumption of the so-called bounded lattice-width, which is not applicable here. In this paper, we settle this question in the affirmative under the assumption of rationality of all the data. The techniques used are substantially different from those in [1] and [3].

Statement of results Given a matrix $B \in \mathbb{R}^{3 \times 2}$, if $M(B)$ is a lattice-free set, then it will be either a triangle or a split in \mathbb{R}^2 (not necessarily maximal); the latter case occurs when one row of B is a scaling of another row.

We define the *split closure* as

$$S = \{s \in \mathbb{R}_+^k \mid \gamma(B) \cdot s \geq 1 \text{ for all } B \in \mathbb{R}^{3 \times 2} \text{ such that } M(B) \text{ is a lattice-free split}\}.$$

Note that we are using a redundant description of convex sets that are splits, i.e., using 3 inequalities to describe it, instead of the standard 2 inequalities. It follows from the result of Cook et al. [11] that the split closure is a polyhedron. We are interested in the closure using all inequalities derived from lattice-free triangles.

We define the *triangle closure*, first defined in [6], as

$$T = \left\{ s \in \mathbb{R}_+^k \mid \begin{array}{l} \gamma(B) \cdot s \geq 1 \text{ for all } B \in \mathbb{R}^{3 \times 2} \text{ such that} \\ M(B) \text{ is a lattice-free triangle} \end{array} \right\}.$$

It is proved in [6] that $T \subseteq S$, and therefore, $T = T \cap S$. This is because we can write a sequence of triangles whose limit is a split, and therefore all split inequalities are limits of triangle inequalities. Hence, using the fact that $T = T \cap S$, we can write the triangle closure as

$$T = \{ s \in \mathbb{R}_+^k \mid \gamma(B) \cdot s \geq 1 \text{ for all } B \in \mathbb{R}^{3 \times 2} \text{ such that } M(B) \text{ is lattice-free} \}. \tag{5}$$

The reason we describe split sets using 3 inequalities is to write the triangle closure in a uniform manner using 3×2 matrices as in (5). We note here that in the definition of T , we do not insist that the lattice-free set $M(B)$ is maximal.

We will prove the following theorem.

Theorem 1.3 *Let $m = 2$. Suppose that the data in (1) is rational, i.e., $f \in \mathbb{Q}^2$ and $r^j \in \mathbb{Q}^2$ for all $j = 1, \dots, k$. Then the triangle closure T is a polyhedron with only a polynomial number of facets with respect to the binary encoding sizes of f, r^1, \dots, r^k .*

We will first use convex analysis in Sect. 2 to illuminate the convex geometry of T by studying a set obtained from the defining inequalities of T . We will then demonstrate in Lemma 2.4 that it suffices to show that an associated convex set has finitely many extreme points. In Sect. 3, we prove that there are indeed only finitely many such extreme points, and in Sect. 4, we complete the proof of Theorem 1.3.

The tools developed in Sect. 3 for proving Theorem 1.3 will then be used to prove the following result about the mixed integer hull in Sect. 5.

Theorem 1.4 *For $m = 2$, the number of facets of the mixed integer hull $\text{conv}(R_f)$ is polynomial in the binary encoding sizes of f, r^1, \dots, r^k .*

We prove the following result in Sect. 5 as a direct consequence of our proof for Theorem 1.4.

Theorem 1.5 *There exists a polynomial time algorithm to enumerate all the facets of $\text{conv}(R_f)$ when $m = 2$.*

Apart from being the main machinery behind Theorems 1.3, 1.4 and 1.5, the results in Sect. 3 also shed light on the classification results of Cornuéjols and Margot for the facets of the mixed integer hull [12]. In Sect. 3, we provide a more detailed set of necessary conditions for a maximal lattice-free convex set to give a facet-defining inequality. This avoids the use of an algorithm for a statement of such necessary conditions (as was done in [12] via the *Reduction Algorithm*) and also provides a completely different proof technique for such classifications. This might also help towards obtaining such results in dimensions higher than two, i.e., $m \geq 3$. On the other hand, we do not provide sufficient conditions, as was done in [12].

We make a remark about the proof structure of Theorem 1.3 here. In this context, the most important result from Sect. 3 is Theorem 3.2. Theorem 3.2 can be viewed as the bridge between Sects. 2 and 4. The reader can follow the proof of Theorem 1.3 by reading only Sects. 2 and 4, if Theorem 3.2 is assumed true. One can then return to Sect. 3 to see the proof of Theorem 3.2, which is rather technical.

2 Preliminaries: convex analysis and the geometry of T

We will prove several preliminary convex analysis lemmas relating to the geometry of T . We show that we can write the triangle closure T using a smaller set of inequalities. We begin by defining the set of vectors which give the inequalities defining T ,

$$\Delta = \{ \gamma(B) \mid B \in \mathbb{R}^{3 \times 2} \text{ such that } M(B) \text{ is lattice-free (not necessarily maximal)} \}.$$

It is easily verified that for any matrix $B \in \mathbb{R}^{3 \times 2}$, if $M(B)$ is lattice-free, then $\psi_B(r) \geq 0$ for all $r \in \mathbb{R}^2$ and therefore $\Delta \subseteq \mathbb{R}_+^k$.

Let $\Delta' = \text{cl}(\text{conv}(\Delta)) + \mathbb{R}_+^k$ where $\text{cl}(\text{conv}(\Delta))$ denotes the closed convex hull of Δ and $+$ denotes the Minkowski sum. Then $\Delta \subseteq \Delta'$ and Δ' is convex as it is the Minkowski sum of two convex sets (see Theorem 3.1 in [18]). In general the Minkowski sum of two closed sets is not closed (for example, $X = \{ (x, y) \mid y \geq 1/x, x > 0 \}$, $Y = \{ (x, y) \mid x = 0 \}$, $X + Y = \{ (x, y) \mid x > 0 \}$). However, in this particular case, we show now that Δ' is closed. We will use the well-known fact that the Minkowski sum of two compact sets is indeed closed. We prove the following more general result.

Lemma 2.1 *Let $X, Y \subseteq \mathbb{R}_+^k$ be closed subsets of \mathbb{R}_+^k . Then $X + Y$ is closed.*

Proof Let $Z = X + Y$ and let $(z^n) \in Z$ such that $z^n \rightarrow z \in \mathbb{R}^k$. We want to show that $z \in Z$. Let $A = \{ x \in \mathbb{R}_+^k \mid \|x\|_\infty \leq \|z\|_\infty + 1 \}$. Since $\|z^n - z\|_\infty \rightarrow 0$ as $n \rightarrow \infty$, for some $N \in \mathbb{N}$, we must have that $\|z^n\|_\infty \leq \|z\|_\infty + 1$, that is, $z^n \in Z \cap A$, for all $n \geq N$. Since $X, Y \subseteq \mathbb{R}_+^k$, we see that $Z \cap A \subseteq (X \cap A) + (Y \cap A)$. Since $(X \cap A) + (Y \cap A)$ is a Minkowski sum of two closed and bounded subsets of \mathbb{R}^k , i.e., compact, $(X \cap A) + (Y \cap A)$ is closed. Therefore, the tail of (z^n) is contained in a closed set, so it must converge to a point in the set, that is, $z \in (X \cap A) + (Y \cap A)$. Since $(X \cap A) + (Y \cap A) \subseteq X + Y = Z$, we have that $z \in Z$. Therefore, Z is closed. □

Lemma 2.2 $T = \{ s \in \mathbb{R}_+^k \mid \gamma \cdot s \geq 1 \text{ for all } \gamma \in \Delta' \}$.

Proof Since $\Delta \subseteq \Delta'$, we have that

$$\{ s \in \mathbb{R}_+^k \mid \gamma \cdot s \geq 1 \text{ for all } \gamma \in \Delta' \} \subseteq \{ s \in \mathbb{R}_+^k \mid \gamma \cdot s \geq 1 \text{ for all } \gamma \in \Delta \} = T.$$

We now show the reverse inclusion. Consider any $s \in T$ and $\gamma \in \Delta'$. We show that $\gamma \cdot s \geq 1$.

Since $\Delta' = \text{cl}(\text{conv}(\Delta)) + \mathbb{R}_+^k$, there exists $r \in \mathbb{R}_+^k$ and $a \in \text{cl}(\text{conv}(\Delta))$ such that $\gamma = a + r$. Moreover, there exists a sequence (a^n) such that (a^n) converges to a and (a^n) is in the convex hull of points $q^j \in \Delta$, $j \in J$. Since $q^j \cdot s \geq 1$ for all $j \in J$,

we have that $a^n \cdot s \geq 1$ for all $n \in \mathbb{N}$. Therefore $a \cdot s = \lim_{n \rightarrow \infty} a^n \cdot s \geq 1$. Since $r \in \mathbb{R}_+^k, r \cdot s \geq 0$ and so $\gamma \cdot s = (a + r) \cdot s \geq a \cdot s \geq 1$. \square

We say that $a \in \Delta'$ is a *minimal* point if there does not exist $x \in \Delta'$ such that $a - x \in \mathbb{R}_+^k \setminus \{0\}$. If such an x exists then we say that a is *dominated* by x . We introduce some standard terminology from convex analysis. Given a convex set $C \subseteq \mathbb{R}^k$, a *supporting hyperplane* for C is a hyperplane $H = \{x \in \mathbb{R}^k \mid h \cdot x = d\}$ such that $h \cdot c \leq d$ for all $c \in C$ and $H \cap C \neq \emptyset$. A point $x \in C$ is called *extreme* if there do not exist y^1 and y^2 in C different from x such that $x = \frac{1}{2}(y^1 + y^2)$. If such $y^1 \neq y^2$ exist, we say that x is a *strict convex combination* of y^1 and y^2 . A point x is called *exposed* if there exists a supporting hyperplane H for C such that $H \cap C = \{x\}$. We will denote the closed ball of radius r around a point y as $\mathfrak{B}(y, r)$. We denote the boundary of this ball by $\partial\mathfrak{B}(y, r)$.

Lemma 2.3 Δ' is a closed convex set with \mathbb{R}_+^k as its recession cone.

Proof Recall that $\Delta \subseteq \mathbb{R}_+^k$. Since \mathbb{R}_+^k is closed and convex, $\text{cl}(\text{conv}(\Delta)) \subseteq \mathbb{R}_+^k$ and so $\Delta' = \text{cl}(\text{conv}(\Delta)) + \mathbb{R}_+^k$ is closed by Lemma 2.1. Since the Minkowski sum of two convex sets is convex, Δ' is convex. Moreover since $\Delta' \subseteq \mathbb{R}_+^k$, the recession cone of Δ' is \mathbb{R}_+^k . \square

Lemma 2.4 Let C be the set of extreme points of Δ' . Then

$$T = \{s \in \mathbb{R}_+^k \mid a \cdot s \geq 1 \text{ for all } a \in C\}.$$

Proof Let $\hat{T} = \{s \in \mathbb{R}_+^k \mid a \cdot s \geq 1 \text{ for all } a \in C\}$. Since $C \subseteq \Delta'$, we have that $T \subseteq \hat{T}$. We show the reverse inclusion. Consider any $s \in \hat{T}$.

By Lemma 2.3, Δ' is a closed convex set with \mathbb{R}_+^k as its the recession cone. Therefore, Δ' contains no lines. This implies that any point $a \in \Delta'$ can be represented as $a = z + \sum_j \lambda_j v^j$ where z is a recession direction of Δ' , v^j 's are extreme points of Δ' , $\lambda_j \geq 0$ and $\sum_j \lambda_j = 1$ (see Theorem 18.5 in [18]). Moreover, since the v^j 's are extreme points, $v^j \in C$ and therefore $v^j \cdot s \geq 1$ for all j because $s \in \hat{T}$. Since $z \in \mathbb{R}_+^k, s \in \mathbb{R}_+^k, \lambda_j \geq 0$ for all j and $\sum_j \lambda_j = 1, a \cdot s = z \cdot s + \sum_j \lambda_j (v^j \cdot s) \geq 1$. Therefore, for all $a \in \Delta', a \cdot s \geq 1$. By Lemma 2.2, $s \in T$. \square

Observation 2.5 Since the recession cone of Δ' is \mathbb{R}_+^k by Lemma 2.3, every extreme point of Δ' is minimal.

We end this section with the following technical lemma.

Lemma 2.6 Let A be any subset of \mathbb{R}^k and let $A' = \text{cl}(\text{conv}(A))$. Then for any extreme point x of A' , there exists a sequence of points $(a^n) \in A$ converging to x .

Proof We first show the following claim.

Claim α For any exposed point a of A' , there exists a sequence of points $(a^n) \in A$ converging to a .

Proof Let $H = \{x \in \mathbb{R}^k \mid h \cdot x = d\}$ be a supporting hyperplane for A' such that $H \cap A' = \{a\}$. Suppose to the contrary that there does not exist such a sequence in A . This implies that there exists $\varepsilon > 0$ such that $\mathfrak{B}(a, \varepsilon) \cap A = \emptyset$. Let $D = \partial\mathfrak{B}(a, \varepsilon) \cap H$. Since $H \cap A' = \{a\}$, for any point $c \in D$, $\text{dist}(c, A') > 0$. Since D is a compact set and the distance function is a Lipschitz continuous function, there exists $\delta > 0$ such that $\text{dist}(c, A') > \delta$ for all $c \in D$. We choose δ' such that for any $y \in \partial\mathfrak{B}(a, \varepsilon)$ satisfying $d \geq h \cdot y > d - \delta'$, there exists $c \in D$ with $\text{dist}(c, y) < \delta$.

Since $a \in \text{cl}(\text{conv}(A))$, there exists a sequence of points $(b^n) \in \text{conv}(A)$ converging to a . This implies that $(h \cdot b^n)$ converges to $h \cdot a = d$. Therefore, we can choose b in this sequence such that $h \cdot b > d - \delta'$ and $b \in \mathfrak{B}(a, \varepsilon)$. Since $b \in \text{conv}(A)$ there exist $v^j \in A$, $j = 0, \dots, k$ such that $b = \text{conv}(\{v_0, \dots, v_k\})$. Therefore, for some j , $h \cdot v^j > d - \delta'$. Moreover, since $v^j \in A$ and $\mathfrak{B}(a, \varepsilon) \cap A = \emptyset$, $v^j \notin \mathfrak{B}(a, \varepsilon)$. Since $b \in \mathfrak{B}(a, \varepsilon)$ and $v^j \notin \mathfrak{B}(a, \varepsilon)$, there exists a point $p \in \partial\mathfrak{B}(a, \varepsilon)$ such that p is a convex combination of b and v^j . Since $h \cdot b > d - \delta'$ and $h \cdot v^j > d - \delta'$, we have that $h \cdot p > d - \delta'$. Moreover $b \in \text{conv}(A)$ implying $b \in A'$ and $v^j \in A'$, so we have $p \in A'$ and so $d \geq h \cdot p$ since H is a supporting hyperplane for A' . So by the choice of δ' , we have that there exists $c \in D$ with $\text{dist}(c, p) < \delta$. However, $\text{dist}(c, A') > \delta$ for all $c \in D$ which is a contradiction because $p \in A'$. \square

By Straszewicz's theorem (see for example Theorem 18.6 in [18]), for any extreme point x of A' , there exists a sequence of exposed points converging to x . So for any $n \in \mathbb{N}$, there exists an exposed point e^n such that $\text{dist}(e^n, x) < \frac{1}{2n}$ and using Claim α , there exists $a^n \in A$ such that $\text{dist}(e^n, a^n) < \frac{1}{2n}$. Now the sequence (a^n) converges to x since $\text{dist}(a^n, x) < \frac{1}{n}$. \square

3 Polynomially many extreme points in Δ'

In this section, we will introduce certain tools and use them to prove the following proposition.

Proposition 3.1 *There exists a finite set $\mathcal{E} \subseteq \Delta$, such that if $\gamma \in \Delta \setminus \mathcal{E}$, then γ is dominated by some $\gamma' \in \Delta$, or γ is the strict convex combination of γ^1 and $\gamma^2 \in \Delta$. Furthermore, the cardinality of \mathcal{E} is bounded polynomially in the binary encoding sizes of f, r^1, \dots, r^k .*

This proposition will be proved by carefully counting the triangles and splits $M(B)$ such that $\gamma(B)$ is not dominated by a point in Δ and is not a strict convex combination of points in Δ . To this end, we define the following subsets of \mathbb{R}_+^k :

$$\Delta_i = \{ \gamma(B) \mid B \in \mathbb{R}^{3 \times 2}, M(B) \text{ is a Type } i \text{ triangle} \} \quad \text{for } i = 1, 2, 3$$

and

$$\Pi = \{ \gamma(B) \mid B \in \mathbb{R}^{3 \times 2}, M(B) \text{ is a maximal lattice-free split} \}.$$

Note that these sets are not disjoint, as the same vector γ can be realized by maximal lattice-free convex sets of different kinds.

We first develop some important concepts in Sect. 3.1, followed by the main counting arguments in Sect. 3.2. In Sect.3.3, we prove Proposition 3.1. It has the following theorem as a consequence, which is the most important ingredient for the triangle closure result.

Theorem 3.2 *There exists a finite set $\mathcal{E} \subset \Delta$ such that if γ is an extreme point of Δ' , then $\gamma \notin \Delta \setminus \mathcal{E}$. Furthermore, the cardinality $\#\mathcal{E}$ is bounded polynomially in the binary encoding sizes of f, r^1, \dots, r^k .*

Proof Let \mathcal{E} be the set from Proposition 3.1. Since $\Delta \subseteq \Delta'$, together with Observation 2.5 and the definition of extreme point, this implies that $\Delta \setminus \mathcal{E}$ does not contain any extreme points of Δ' . □

3.1 Tools

For fixed f , given a set of rays $R \subseteq \mathbb{R}^m$ and a matrix $B = (b^1; \dots; b^n) \in \mathbb{R}^{n \times m}$, we refer to the set of ray intersections

$$P(B, R) = \{ p(B, r) \in \mathbb{R}^m \mid r \in R, \psi_B(r) > 0 \}$$

where $p(B, r) = f + \frac{1}{\psi_B(r)}r$ is the point where r meets the boundary of the set $M(B)$. Therefore, $P(B, R) \subset \partial M(B)$. Furthermore, if $P(B^1, \{r^1, \dots, r^k\}) = P(B^2, \{r^1, \dots, r^k\})$, then $\gamma(B^1) = \gamma(B^2)$. If $p(B, r)$ is a vertex of $M(B)$, then we call r a *corner ray* of $M(B)$.

Whenever $\psi_B(r) > 0$, the set $I_B(r) = \operatorname{argmax}_{i=1, \dots, n} b^i \cdot r$ is the index set of all defining inequalities of the polyhedron $M(B)$ that the ray intersection $p(B, r)$ satisfies with equality. In particular, for $m = 2$, when all the inequalities corresponding to the rows of B are different facets of $M(B)$, we have $\#I_B(r) = 1$ when r points to the relative interior of a facet, and $\#I_B(r) = 2$ when r points to a vertex of $M(B)$, where $\#X$ denotes the cardinality of the set X .

Let $F_i(B) = M(B) \cap \{x \in \mathbb{R}^m \mid b^i \cdot (x - f) = 1\}$ for each $i = 1, \dots, n$ (in what follows, $F_i(B)$ will usually be a facet of $M(B)$). Let $Y(B)$ be the set of integer points contained in $M(B)$. Recall that if $M(B)$ is a maximal lattice-free convex set, then each facet $F_i(B)$ contains at least one integer point in its relative interior. In our proofs, it is convenient to choose, for every $i = 1, \dots, n$, a certain subset $Y_i \subseteq Y(B) \cap F_i(B)$ of the integer points on $F_i(B)$.

Definition 3.3 Let \mathcal{Y} denote the tuple (Y_1, Y_2, \dots, Y_n) . The *tilting space* $\mathcal{T}(B, \mathcal{Y}, R) \subseteq \mathbb{R}^{n \times m}$ is defined as the set of matrices $A = (a^1; a^2; \dots; a^n) \in \mathbb{R}^{n \times m}$ that satisfy the following conditions:

$$a^i \cdot (y - f) = 1 \quad \text{for } y \in Y_i, i = 1, \dots, n, \tag{6a}$$

$$a^i \cdot r = a^{i'} \cdot r \quad \text{for } i, i' \in I_B(r), \text{ and all } r \in R \tag{6b}$$

$$a^i \cdot r > a^{i'} \cdot r \quad \text{for } i \in I_B(r), i' \notin I_B(r), \text{ and all } r \in R. \tag{6c}$$

Note that if $R' \supseteq R$, then $\mathcal{T}(B, \mathcal{Y}, R') \subseteq \mathcal{T}(B, \mathcal{Y}, R)$. The tilting space $\mathcal{T}(B, \mathcal{Y}, R)$ is defined for studying perturbations of the lattice-free set $M(B)$. This is done by changing or tilting the facets of $M(B)$ subject to certain constraints to construct a new lattice-free set $M(A)$. Constraint (6a) requires that $F_i(A)$ must contain subset Y_i of integer points. Constraints (6b) and (6c) together imply that for any $r \in R$, the ray intersection $p(A, r) = f + \frac{1}{\psi_A(r)}r$ lies on $F_i(A)$ of $M(A)$ if and only if the ray intersection $p(B, r) = f + \frac{1}{\psi_B(r)}r$ for $M(B)$ lies on the corresponding $F_i(B)$ of $M(B)$. Thus we have $I_A(r) = I_B(r)$. In particular, this means that if $r \in R$ is a corner ray of $M(B)$, then r must also be a corner ray for $M(A)$ if $A \in \mathcal{T}(B, \mathcal{Y}, R)$.

Note that $\mathcal{T}(B, \mathcal{Y}, R)$ is defined by linear equations and strict linear inequalities and, since $B \in \mathcal{T}(B, \mathcal{Y}, R)$, it is non-empty. Thus it is a convex set whose dimension is the same as that of the affine space given by the Eqs. (6a) and (6b) only. This motivates the following definition.

Definition 3.4 $\mathcal{N}(B, \mathcal{Y}, R)$ will denote the nullspace of the Eqs. (6a) and (6b), i.e.,

$$\mathcal{N}(B, \mathcal{Y}, R) = \left\{ A = (a^1; \dots; a^n) \in \mathbb{R}^{n \times m} \mid \begin{array}{l} a^i \cdot (y - f) = 0 \quad \forall y \in Y_i, i = 1, \dots, n, \\ a^i \cdot r = a^{i'} \cdot r \quad \forall i, i' \in I_B(r) \text{ and } r \in R \end{array} \right\}$$

If $R' \supseteq R$, then $\mathcal{N}(B, \mathcal{Y}, R') \subseteq \mathcal{N}(B, \mathcal{Y}, R)$. For many cases when $\gamma(B)$ is not extreme, we will find a matrix $\bar{A} \in \mathcal{N}(B, \mathcal{Y}, R)$ such that $\gamma(B)$ can be expressed as the convex combination of $\gamma(B + \varepsilon\bar{A})$ and $\gamma(B - \varepsilon\bar{A})$ and $M(B + \varepsilon\bar{A}), M(B - \varepsilon\bar{A})$ are lattice-free polytopes. If $\gamma(B) \in \Delta$, then $\gamma(B + \varepsilon\bar{A}), \gamma(B - \varepsilon\bar{A})$ will also be in Δ .

For convenience, we shorten notation in the following way: for the rest of the paper we fix the input data f and the set of rays $R = \{r^1, \dots, r^k\}$. Furthermore, whenever the matrix B is clear from context, we write $\mathcal{N}(\mathcal{Y}) = \mathcal{N}(B, \mathcal{Y}, R)$, $\mathcal{T}(\mathcal{Y}) = \mathcal{T}(B, \mathcal{Y}, R)$, $P = P(B, R)$, $p^j = p(B, r^j)$ for $j = 1, \dots, k$, and $F_i = F_i(B)$.

Next we introduce a tool that helps to ensure that sets $M(B \pm \varepsilon\bar{A})$ are lattice-free. This will be done by utilizing now-classic results in the theory of parametric linear programming. Specifically, consider a parametric linear program,

$$\sup\{c(t) \cdot x : A(t)x \leq b(t)\} \in \mathbb{R} \cup \{\pm\infty\}, \tag{7}$$

where all coefficients depend continuously on a parameter vector t within some parameter region $\mathcal{R} \subseteq \mathbb{R}^q$.

Theorem 3.5 (D. H. Martin [17], Lemma 3.1) *Suppose that the solution set of (7) for $t = t_0$ is non-empty and bounded. Then, in parameter space, there is an open neighborhood \mathcal{O} of t_0 such that the union of all solution sets for $t \in \mathcal{O}$ is bounded.*

We use this theorem of parametric linear programming to prove the following lemma.

Lemma 3.6 *Let $B \in \mathbb{R}^{n \times m}$ be such that $M(B)$ is a bounded maximal lattice-free set. Then for every $\bar{A} \in \mathbb{R}^{n \times m}$, there exists $\delta > 0$ such that for all $0 < \varepsilon < \delta$, the set $Y(B + \varepsilon\bar{A})$ of integer points contained in $M(B + \varepsilon\bar{A})$ is a subset of $Y(B)$.*

Proof Consider the parametric linear program

$$\max\{0 \mid a^i \cdot (x - f) \leq 1, i = 1, \dots, n\}$$

with parameters $t = A = (a^1; \dots; a^n) \in \mathbb{R}^{n \times m}$. By the assumption of the lemma, the solution set for $t_0 = B = (b^1; \dots; b^n)$ is bounded. Let \mathcal{O} be the open neighborhood of t_0 from Theorem 3.5, and let \hat{S} be the union of all solution sets for $t \in \mathcal{O}$, which is by the theorem a bounded set.

For each of the finitely many lattice points $y \in \hat{S} \setminus M(B)$, let $i(y) \in \{1, \dots, n\}$ be an index of an inequality that cuts off y , that is, $b^{i(y)} \cdot (y - f) > 1$. Then, given any $\bar{A} \in \mathbb{R}^{n \times m}$, there exists $\delta > 0$ such that for all $0 < \varepsilon < \delta$ such that $(b^{i(y)} + \varepsilon \bar{a}^{i(y)}) \cdot (y - f) > 1$ for all $y \in \hat{S} \setminus M(B)$. Therefore, for all such ε , we have $Y(B + \varepsilon \bar{A}) \subseteq Y(B)$. \square

Lemma 3.7 (General tilting lemma) *Let $B \in \mathbb{R}^{n \times m}$ be such that $M(B)$ is a bounded lattice-free set. Suppose $\mathcal{Y} = (Y_1, \dots, Y_n)$ is a covering of $Y(B)$, i.e., $Y(B) \subseteq Y_1 \cup \dots \cup Y_n$. For any $\bar{A} \in \mathcal{N}(\mathcal{Y}) \setminus \{0\}$, there exists $\delta > 0$ such that for all $0 < \varepsilon < \delta$ the following statements hold:*

- (i) $I_B(r^j) = I_{B+\varepsilon\bar{A}}(r^j) = I_{B-\varepsilon\bar{A}}(r^j)$ for all $j = 1, \dots, k$.
- (ii) $\gamma(B) = \frac{1}{2}\gamma(B + \varepsilon\bar{A}) + \frac{1}{2}\gamma(B - \varepsilon\bar{A})$.
- (iii) Both $M(B \pm \varepsilon\bar{A})$ are lattice-free.
- (iv) Suppose $m = 2$ and let $\bar{a}^1, \dots, \bar{a}^n \in \mathbb{R}^2$ denote the rows of \bar{A} . Suppose there exists an index $i \in \{1, \dots, n\}$ such that $\bar{a}^i \neq 0, \#Y_i = 1$, and $(F_i \cap P) \setminus \mathbb{Z}^2 \neq \emptyset$. Then $\gamma(B)$ is a strict convex combination of $\gamma(B + \varepsilon\bar{A})$ and $\gamma(B - \varepsilon\bar{A})$.

Proof Since $\bar{A} \in \mathcal{N}(\mathcal{Y}) \setminus \{0\}$, $B \pm \varepsilon\bar{A}$ satisfies the Eqs. (6a) and (6b) for any ε , and there exists $\delta_1 > 0$ such that $B \pm \varepsilon\bar{A}$ satisfies the strict inequalities (6c) for $0 \leq \varepsilon < \delta_1$. Thus $B \pm \varepsilon\bar{A} \in \mathcal{T}(\mathcal{Y})$ for $0 \leq \varepsilon < \delta_1$. Let $\delta_2 > 0$ be obtained by applying Lemma 3.6. Choose $\delta = \min\{\delta_1, \delta_2\}$.

Part (i). This follows from the fact that $B \pm \varepsilon\bar{A} \in \mathcal{T}(\mathcal{Y})$, and thus (6b) and (6c) hold, for all $0 < \varepsilon < \delta$.

Part (ii). This is a consequence of (i), since $\gamma(B)_j = \psi_B(r^j) = b^i \cdot r^j$ for any $i \in I_B(r^j)$, and $\gamma(B \pm \varepsilon\bar{A})_j = \psi_{B \pm \varepsilon\bar{A}}(r^j) = (b^i \pm \varepsilon \bar{a}^i) \cdot r^j$ for any $i \in I_{B \pm \varepsilon\bar{A}}(r^j)$.

Part (iii). This follows from Lemma 3.6 and the fact that $Y(B)$ remains on the boundary of $M(B \pm \varepsilon\bar{A})$, due to constraint (6a).

Part (iv). Since $(F_i \cap P) \setminus \mathbb{Z}^2 \neq \emptyset$, there exists a ray $r^j \in R$ such that $p^j \in (F_i \cap P) \setminus \mathbb{Z}^2$. We will show that $\bar{a}^i \cdot r^j \neq 0$. Suppose for the sake of contradiction that $\bar{a}^i \cdot r^j = 0$. Let $y \in Y_i$. By definition of $\mathcal{N}(\mathcal{Y})$, $\bar{a}^i \cdot (y - f) = 0$. Since r^j does not point to an integer point from f , the vectors r^j and $y - f$ are not parallel. Therefore, the system $\bar{a}^i \cdot r^j = 0, \bar{a}^i \cdot (y - f) = 0$ has the unique solution $\bar{a}^i = 0$, which is a contradiction since we assumed $\bar{a}^i \neq 0$. Hence, $\bar{a}^i \cdot r^j \neq 0$ and therefore $\psi_{B+\varepsilon\bar{A}}(r^j) \neq \psi_{B-\varepsilon\bar{A}}(r^j)$. Therefore, $\gamma(B)$ is a strict convex combination of $\gamma(B + \varepsilon\bar{A})$ and $\gamma(B - \varepsilon\bar{A})$. \square

We will now apply this principle to obtain two lemmas for the case $m = 2$. The first simple application is to tilt a single facet of a polytope to show that the corresponding

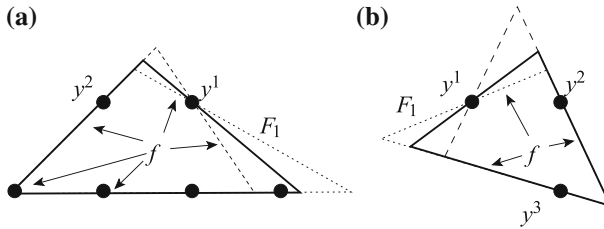


Fig. 2 Single facet tilts: tilting one facet of a polytope to generate new inequalities. In both examples, there is a ray pointing to a non-integer point on the interior of the facet being tilted. This ensures that the inequalities from the tilted sets are distinct, and therefore we see that $\gamma(B)$ is the strict convex combination of other points in Δ . This is the assertion of Lemma 3.8

inequality is a strict convex combination of other inequalities, as shown in Fig. 2. This is summarized in the following lemma.

Lemma 3.8 (Single facet tilt lemma) *Let $M(B)$ be a lattice-free polytope for some matrix $B \in \mathbb{R}^{n \times 2}$. Suppose that $F_1(B) \cap \mathbb{Z}^2 = \{y^1\}$, $y^1 \in \text{rel int}(F_1(B))$, and $P \cap F_1(B) \subset \text{rel int}(F_1(B))$, i.e., there are no ray intersections on the lower-dimensional faces of $F_1(B)$, and $P \cap F_1(B) \setminus \mathbb{Z}^2 \neq \emptyset$. Then there exist $\bar{a}^1 \neq 0$ and $\varepsilon > 0$ such that $\bar{A} = (\bar{a}^1; 0; \dots; 0)$ has the property that $\gamma(B)$ is a strict convex combination of $\gamma(B + \varepsilon \bar{A})$ and $\gamma(B - \varepsilon \bar{A})$ and $M(B + \varepsilon \bar{A})$ and $M(B - \varepsilon \bar{A})$ are lattice-free.*

Proof Let $Y_1 = \{y^1\}$ and $Y_i = Y(B) \cap F_i, i = 2, \dots, n$, so that $\mathcal{Y} = (Y_1, \dots, Y_n)$ is a covering of the set $Y(B)$ of integer points in $M(B)$.

Let $\bar{a}^1 \neq 0$ such that $\bar{a}^1 \cdot (y^1 - f) = 0$ and observe that $\bar{A} = (\bar{a}^1; 0; \dots; 0) \in \mathcal{N}(\mathcal{Y})$ because there are no constraints (6b) involving \bar{a}^1 , as there are no ray intersections on the lower-dimensional faces of $F_1(B)$. By Lemma 3.7, for some $\varepsilon > 0$, $\gamma(B)$ is a strict convex combination of $\gamma(B + \varepsilon \bar{A})$ and $\gamma(B - \varepsilon \bar{A})$ and $M(B + \varepsilon \bar{A})$ and $M(B - \varepsilon \bar{A})$ are lattice-free. \square

We give another application of the perturbation arguments to Type 3 triangles and quadrilaterals, before moving on to more specific counting arguments in the next subsection.

Lemma 3.9 *Let $B \in \mathbb{R}^{n \times 2}$ be such that $M(B)$ is a Type 3 triangle ($n = 3$) or a maximal lattice-free quadrilateral ($n = 4$). Let $\text{rel int}(F_i) \cap \mathbb{Z}^2 = \{y^i\}$ and set $Y_i = \{y^i\}$; then (Y_1, \dots, Y_n) form a covering of $Y(B)$. If $P \not\subset \mathbb{Z}^2$ and $M(B)$ has fewer than n corner rays, then $\gamma(B)$ is a strict convex combination of $\gamma(B^1)$ and $\gamma(B^2)$, where $B^1, B^2 \in \mathbb{R}^{n \times 2}$ are matrices such that $M(B^1), M(B^2)$ are both lattice-free.*

Proof Let $Y_i = \{y^i\}$ for $i = 1, \dots, n$. Let $R' \supseteq R = \{r^1, \dots, r^k\}$ such that R' contains exactly $n - 1$ corner rays pointing to different vertices of $M(B)$ from f . We examine the null space $\mathcal{N}(B, \mathcal{Y}, R')$. With $n - 1$ corner rays, $\mathcal{N}(B, \mathcal{Y}, R')$ is the set of matrices $A = (a^1; \dots; a^n)$ satisfying the following system of equations, where, for convenience, we define $\bar{y}^i := y^i - f$:

$$a^i \cdot \bar{y}^i = 0 \quad \text{for } i = 1, \dots, n \quad \text{and} \quad a^i \cdot r^i = a^{i+1} \cdot r^i \quad \text{for } i = 1, \dots, n - 1.$$

We have assumed, without loss of generality, that the corner rays and facets are numbered such that we have corner rays $r^i \in F_i \cap F_{i+1}$ for $i = 1, \dots, n - 1$. Note that \bar{y}^i is linearly independent from r^i for $i = 1, \dots, n - 1$ and linearly independent from r^{i-1} for $i = 2, \dots, n$, because y^i lies in the relative interior of F_i and the rays point to the vertices.

There are $2n - 1$ equations and $2n$ variables, so $\dim \mathcal{N}(B, \mathcal{Y}, R') \geq 1$. Choose $\bar{A} = (\bar{a}^1; \dots; \bar{a}^n) \in \mathcal{N}(B, \mathcal{Y}, R') \setminus \{0\}$. Notice that for $i = 1, \dots, n - 1$, if $\bar{a}^i = 0$, then \bar{a}^{i+1} must satisfy $\bar{a}^{i+1} \cdot r^i = 0$ and $\bar{a}^{i+1} \cdot \bar{y}^{i+1} = 0$, which implies that $\bar{a}^{i+1} = 0$, since \bar{y}^{i+1} and r^i are linearly independent. Similarly, for $i = 2, \dots, n$, if $\bar{a}^i = 0$, then \bar{a}^{i-1} must satisfy $\bar{a}^{i-1} \cdot r^{i-1} = 0$ and $\bar{a}^{i-1} \cdot \bar{y}^{i-1} = 0$, which implies that $\bar{a}^{i-1} = 0$. By induction, this shows that if $\bar{a}^i = 0$ for any $i = 1, \dots, n$, then $\bar{A} = 0$, which contradicts our assumption. Hence, $\bar{a}^i \neq 0$ for all $i = 1, \dots, n$.

Since $R' \supseteq R$, we have that $\mathcal{N}(B, \mathcal{Y}, R') \subseteq \mathcal{N}(B, \mathcal{Y}, R)$; thus, $\bar{A} \in \mathcal{N}(B, \mathcal{Y}, R) \setminus \{0\}$. Since $P \not\subseteq \mathbb{Z}^2$, there exists $r \in R$ that does not point to an integer point. Let $i \in I_B(r)$. Since $\bar{a}_i \neq 0$, by applying Lemma 3.7 with \bar{A} we obtain $\varepsilon > 0$ such that $\gamma(B)$ is a strict convex combination of $\gamma(B + \varepsilon\bar{A})$ and $\gamma(B - \varepsilon\bar{A})$ and $M(B + \varepsilon\bar{A}), M(B - \varepsilon\bar{A})$ are lattice-free. \square

3.2 Counting arguments

An important ingredient in the proof of Proposition 3.1 is the following consequence of the Cook et al. [10] theorem on the polynomial-size description of the integer hulls of polyhedra in fixed dimension combined with an algorithm by Hartmann [15] for enumerating all the vertices, which runs in polynomial time in fixed dimension.

Lemma 3.10 *Given two rays r^1 and r^2 in \mathbb{R}^2 , we define the affine cone*

$$C(r^1, r^2) = \{x \in \mathbb{R}^2 \mid x = f + s_1r^1 + s_2r^2 \text{ for } s_1, s_2 \geq 0\}.$$

The number of facets and vertices of the integer hull

$$(C(r^1, r^2))_{\mathbb{I}} = \text{conv}(C(r^1, r^2) \cap \mathbb{Z}^2)$$

is bounded by a polynomial in the binary encoding sizes of f, r^1, r^2 . Furthermore, the facets and vertices of the integer hull can be enumerated in polynomial time in the binary encoding sizes of f, r^1, r^2 .

In the following, the closed line segment between two points x^1 and x^2 will be denoted by $[x^1, x^2]$, and the open line segment will be denoted by (x^1, x^2) .

Lemma 3.11 *Consider any lattice-free convex set $M(B)$ for $B \in \mathbb{R}^{n \times 2}$. Suppose there exist two rays r^{j_1}, r^{j_2} such that the corresponding ray intersections p^{j_1}, p^{j_2} are distinct and lie on a facet F of $M(B)$.*

- (i) *If $[p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2 = \{y\}$ and $y \in (p^{j_1}, p^{j_2})$, then y is a vertex of the integer hull $(C(r^{j_1}, r^{j_2}))_{\mathbb{I}}$. Moreover, the line $\text{aff}(F)$ is a supporting hyperplane for $(C(r^{j_1}, r^{j_2}))_{\mathbb{I}}$, i.e., $(C(r^{j_1}, r^{j_2}))_{\mathbb{I}}$ lies on one side of this line.*

(ii) If $[p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2$ contains at least two points, then the line $\text{aff}(F)$ contains a facet of the integer hull $(C(r^{j_1}, r^{j_2}))_{\mathbb{I}}$.

Proof Suppose H is the halfspace corresponding to F that contains f . Then $H \cap C(r^{j_1}, r^{j_2}) \subset M(B)$ and since $M(B)$ does not contain any integer points in its interior, neither does $H \cap C(r^{j_1}, r^{j_2})$. Since we assume $[p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2$ is non-empty and p^{j_1}, p^{j_2} lie on the line defining H (and also F), this line is a supporting hyperplane for $(C(r^{j_1}, r^{j_2}))_{\mathbb{I}}$.

If $[p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2$ contains the single point y and $y \in (p^{j_1}, p^{j_2})$, then clearly y is an extreme point of $(C(r^{j_1}, r^{j_2}))_{\mathbb{I}}$.

If $[p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2$ contains two (or more) points, then the line defining H (and also F) defines a facet of $(C(r^{j_1}, r^{j_2}))_{\mathbb{I}}$. □

Observation 3.12 (Integral ray intersections) *Let $R = \{r^1, \dots, r^k\}$. Then there is a unique $\gamma \in \mathbb{R}^k$ such that $\gamma = \gamma(B)$, $M(B)$ is a lattice-free convex set and $P(B, R) \subset \mathbb{Z}^2$. (Note that there may be multiple matrices $B \in \mathbb{R}^{n \times 2}$ yielding γ .)*

Proof Let \mathcal{B} be the family of matrices $B \in \mathbb{R}^{n \times 2}$ such that $P(B, R) \subset \mathbb{Z}^2$ and $M(B)$ is a lattice-free set. Then $P(B_1, R) = P(B_2, R)$ for all $B_1, B_2 \in \mathcal{B}$ because $M(B_1)$ and $M(B_2)$ are lattice-free. Hence, $\gamma(B)$ is the same vector for all $B \in \mathcal{B}$. □

Proposition 3.13 (Counting Type 3 triangles) *There exists a finite subset $\mathcal{E}_3 \subseteq \Delta_3$ such that for any $\gamma \in \Delta_3 \setminus \mathcal{E}_3$, there exist $\gamma^1, \gamma^2 \in \Delta$ such that γ is a strict convex combination of γ^1 and γ^2 . Moreover, the cardinality of \mathcal{E}_3 is bounded polynomially in the binary encoding sizes of f, r^1, \dots, r^k . Specifically, \mathcal{E}_3 can be chosen as the set of all $\gamma(B)$ such that $M(B)$ is a Type 3 triangle and one of the following holds, where $R = \{r^1, \dots, r^k\}$:*

Case a. $P(B, R) \subset \mathbb{Z}^2$.

Case b. $M(B)$ has three corner rays, that is, $\text{vert}(B) \subseteq P(B, R)$.

Proof Step 1. Let $\gamma = \gamma(B)$ for some fixed $B \in \mathbb{R}^{3 \times 2}$ such that $M(B)$ is a Type 3 triangle. By Lemma 3.9, if $\gamma \in \Delta_3 \setminus \mathcal{E}_3$, then there exist $\gamma^1, \gamma^2 \in \Delta$ such that γ is a strict convex combination of γ^1 and γ^2 . Therefore, we are left to determine the cardinality of \mathcal{E}_3 .

Step 2. We now bound the cardinality of \mathcal{E}_3 by considering each case.

Case a. Observation 3.12 shows that there is a unique $\gamma \in \mathcal{E}_3$ corresponding to this case.

Case b. $M(B)$ has three corner rays. First we pick any triplet of pairwise distinct rays, say $r^{j_1}, r^{j_2}, r^{j_3} \in R$, as the corner rays; there are $O(k^3)$ such triplets. Because we are constructing a Type 3 triangle, there needs to be an integer point $y_3 \in (p^{j_1}, p^{j_2})$. By Lemma 3.11 (i), y_3 is a vertex of $(C(r^{j_1}, r^{j_2}))_{\mathbb{I}}$. By the same argument, there exist integer points $y_2 \in (p^{j_1}, p^{j_3})$ and $y_1 \in (p^{j_2}, p^{j_3})$ that are vertices of $(C(r^{j_1}, r^{j_3}))_{\mathbb{I}}$ and $(C(r^{j_2}, r^{j_3}))_{\mathbb{I}}$, respectively. By Proposition 7.1 in ‘‘Appendix 7’’, a triangle is uniquely determined by three corner rays and one point on the relative interior of each facet. Thus, we can use a triplet of rays and a vertex from each integer hull of the three cones spanned by consecutive rays to uniquely define the triangle. These are polynomially bounded in number by Lemma 3.10.

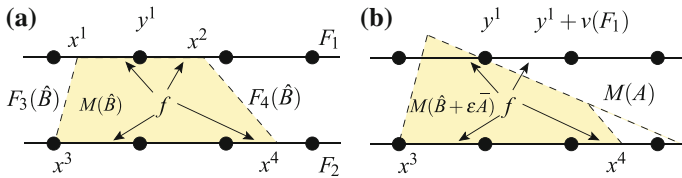


Fig. 3 Geometry of necessary conditions for splits in proof of Proposition 3.14. **a** Lattice-free quadrilateral $M(\hat{B})$ that is created such that $\gamma(B) = \gamma(\hat{B})$. **b** Lattice-free quadrilateral $M(\hat{B} + \varepsilon\bar{A})$ and the lattice-free triangle $M(A)$. We see that $P(A, R) = P(\hat{B} + \varepsilon\bar{A}, R)$ and hence $\gamma(A) = \gamma(\hat{B} + \varepsilon\bar{A})$

Thus the number of elements of \mathcal{E}_3 corresponding to each case has a polynomial bound, and the result is proved. \square

Proposition 3.14 (Counting splits) *There exists a finite subset $\mathcal{E}_0 \subseteq \Pi$ such that for any $\gamma \in \Pi \setminus \mathcal{E}_0$, there exist $\gamma^1, \gamma^2 \in \Delta$ such that γ is a strict convex combination of γ^1 and γ^2 . Moreover, the cardinality of \mathcal{E}_0 is bounded polynomially in the binary encoding sizes of f, r^1, \dots, r^k . Specifically, \mathcal{E}_0 can be chosen as the set of all $\gamma(B)$ such that $M(B)$ is a maximal lattice-free split and one of the following holds, where $R = \{r^1, \dots, r^k\}$:*

Case a. $P(B, R) \subset \mathbb{Z}^2$.

Case b. There exists $j \in \{1, \dots, k\}$ such that r^j lies in the recession cone of the split.

Case c. $\#(\text{conv}(P(B, R) \cap F_1) \cap \mathbb{Z}^2) \geq 2$.

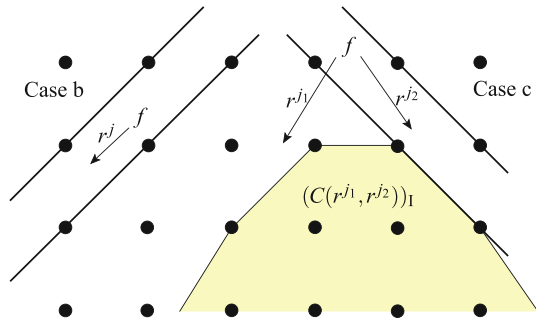
Proof Step 1. Consider $\gamma(B) \in \Pi \setminus \mathcal{E}_0$, and so $M(B)$ is a maximal lattice-free split such that none of Case a, Case b, or Case c hold. So we suppose that, possibly by exchanging the rows of B , no ray in R lies in the recession cone of the split, $P(B, R) \cap F_1 \setminus \mathbb{Z}^2 \neq \emptyset$, and $\#(\text{conv}(P(B, R) \cap F_1) \cap \mathbb{Z}^2) \leq 1$.

We will first construct a lattice-free quadrilateral $M(\hat{B})$ such that $\gamma(B) = \gamma(\hat{B})$. We will consider the sub-lattice of \mathbb{Z}^2 contained in the linear space parallel to F_1 . We use the notation $v(F_1)$ to denote a primitive lattice vector which generates this one-dimensional lattice. Choose $y^1 \in \mathbb{Z}^2$ such that $P(B, R) \cap F_1 \subset (y^1 - v(F_1), y^1 + v(F_1))$. Pick any $x^1, x^2 \in F_1$ such that

$$(P(B, R) \cap F_1) \cup \{y^1\} \subsetneq (x^1, x^2) \subsetneq (y^1 - v(F_1), y^1 + v(F_1)).$$

We can assume that F_2 corresponds to the facet opposite F_1 , with another exchange of the rows of B if necessary. Next, choose distinct integer points $x^3, x^4 \in F_2 \cap \mathbb{Z}^2$ such that $P(B, R) \cap F_2 \subset (x^3, x^4)$ and $f \in \text{int}(\text{conv}(\{x^1, x^2, x^3, x^4\}))$. Now, let $\hat{B} = (\hat{b}^1; \hat{b}^2; \hat{b}^3; \hat{b}^4) \in \mathbb{R}^{4 \times 2}$ such that $M(\hat{B}) = \text{conv}(\{x^1, x^2, x^3, x^4\})$ and let $F_1(\hat{B}) = [x^1, x^2]$, $F_2(\hat{B}) = [x^3, x^4]$, $F_3(\hat{B}) = [x^1, x^3]$ and $F_4(\hat{B}) = [x^2, x^4]$ (see Fig. 3a). The set $M(\hat{B})$ is a lattice-free quadrilateral with no corner rays. By construction, the ray intersections are the same for $M(\hat{B})$ and $M(B)$, i.e., $P(\hat{B}, R) = P(B, R)$, and therefore $\gamma(\hat{B}) = \gamma(B)$. Furthermore, $F_1(\hat{B}) \cap P(B, R) \subset \text{rel int}(F_1(\hat{B}))$, that is, all ray intersections on $F_1(\hat{B})$ are contained in its relative interior. Since $P(B, R) \cap F_1(\hat{B}) \not\subset \mathbb{Z}^2$ and is non-empty, by Lemma 3.8, $\gamma(\hat{B}) = \gamma(B)$ is a strict convex combination of $\gamma(\hat{B} + \varepsilon\bar{A})$ and $\gamma(\hat{B} - \varepsilon\bar{A})$, where $\bar{A} = (\bar{a}^1; 0; 0; 0)$ and $\bar{a}^1 \neq 0$.

Fig. 4 Step 3 of the proof of Proposition 3.14: uniquely determining the split $M(B)$ for Cases (b) and (c). Case (b) requires only the ray r^j which is in the recession cone of the split, whereas Case (c) is determined by a facet of $(C(r^{j_1}, r^{j_2}))_I$



Lastly, we need to show that there exist $A, A' \in \mathbb{R}^{3 \times 2}$ such that $\gamma(A) = \gamma(\hat{B} + \varepsilon \bar{A})$ and $\gamma(A') = \gamma(\hat{B} - \varepsilon \bar{A})$, and $M(A), M(A')$ are lattice-free triangles. Since the cases are similar, we will just show that such a matrix A exists. More concretely, we want to exhibit a matrix A such that $P(A, R) = P(\hat{B} + \varepsilon \bar{A}, R)$ and that $M(A)$ is lattice-free. As \bar{A} comes from Lemma 3.8, $\bar{a}^1 \cdot v(F_1) \neq 0$. Suppose, without loss of generality, that $\bar{a}^1 \cdot v(F_1) > 0$. Let $a^1 = \hat{b}^1 + \varepsilon \bar{a}^1, a^2 = \hat{b}^2$, and $a^3 = \hat{b}^3$ and $A = (a^1; a^2; a^3)$ (see Fig. 3b). Let $\alpha > 0$. Then

$$\begin{aligned}
 (\hat{b}^1 + \bar{a}^1) \cdot (y^1 + \alpha v(F_1) - f) &= (\hat{b}^1 + \bar{a}^1) \cdot (y^1 - f) + \alpha \hat{b}^1 \cdot v(F_1) + \alpha \bar{a}^1 \cdot v(F_1) \\
 &= 1 + 0 + \alpha \bar{a}^1 \cdot v(F_1) > 1,
 \end{aligned}$$

and therefore, $y^1 + \alpha v(F_1) \notin M(A)$ for all $\alpha > 0$. Recalling that $M(B)$ is a split, it follows that $\text{int}(M(A)) \setminus \text{int}(M(B)) \subseteq \text{int}(M(\hat{B} + \varepsilon \bar{A}))$ and $\text{int}(M(A)) \cap \text{int}(M(B)) \subseteq \text{int}(M(B))$. Therefore, $\text{int}(M(A)) \subseteq \text{int}(M(\hat{B} + \varepsilon \bar{A})) \cup \text{int}(M(B))$. Because of this inclusion, $M(A)$ is lattice-free.

Step 2. We now bound the cardinality of \mathcal{E}_0 by considering each case.

Case a. $P(B, R) \subset \mathbb{Z}^2$. Observation 3.12 shows that there is a unique $\gamma \in \mathcal{E}_0$ corresponding to this case.

The two remaining cases are illustrated in Fig. 4.

Case b. A ray direction r^j is parallel to the split. There are at most k such ray directions, and thus at most k splits in this case.

Case c. There exist $p^{j_1}, p^{j_2} \in P(B, R)$ such that $\#([p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2) \geq 2$, and therefore, the split must run parallel to a facet of $(C(r^{j_1}, r^{j_2}))_I$ by Lemma 3.11 (ii), of which there are only polynomially many. There are only $\binom{k}{2}$ ways to choose two rays for this possibility.

Since each case has a polynomial bound, we conclude that $\#\mathcal{E}_0$ is polynomially bounded as well. □

Proposition 3.15 (Counting Type 1 triangles) *There exists a finite subset $\mathcal{E}_1 \subseteq \Delta_1$, such that for any $\gamma \in \Delta_1 \setminus (\mathcal{E}_1 \cup \Pi \cup \Delta_2)$, there exist $\gamma^1, \gamma^2 \in \Delta$ such that γ is a strict convex combination of γ^1 and γ^2 or there exists $\gamma' \in \Delta$ such that γ is dominated by γ' . Moreover, the cardinality of \mathcal{E}_1 is bounded polynomially in the binary encoding sizes of f, r^1, \dots, r^k . Specifically, \mathcal{E}_1 is chosen as the set of all $\gamma(B)$ such that $M(B)$*

Fig. 5 Proposition 3.15, Step 2: Uniquely determining Type 1 triangles that are in Cases a and b. In both cases, the triangle is uniquely determined by y^1, y^2 and the ray intersections $p^{j_1}, p^{j_2} \in \text{vert}(B) \cap F_3$

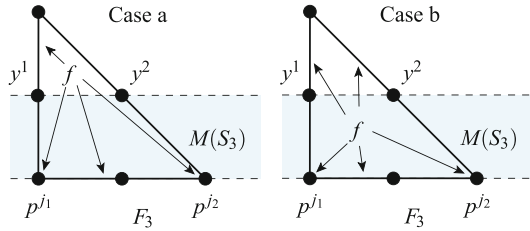
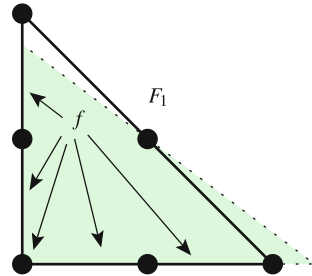


Fig. 6 In the proof of Proposition 3.15, Step 1, a Type 1 triangle can be replaced by a Type 2 triangle (dotted) that gives the same inequality



is a Type 1 triangle, there exist distinct points $p^{j_1}, p^{j_2} \in \text{vert}(B) \cap F_3 \cap P(B, R)$, i.e., F_3 has two corner rays, and one of the following holds:

Case a. $f \notin M(S_3)$.

Case b. $f \in M(S_3)$, and $P(B, R) \not\subset M(S_3)$.

Here $R = \{r^1, \dots, r^k\}$ and $S_3 \in \mathbb{R}^{3 \times 2}$ is a matrix such that $M(S_3)$ is a maximal lattice-free split with the property that one facet of $M(S_3)$ contains F_3 and $M(S_3) \cap \text{int}(M(B)) \neq \emptyset$.

Figure 5 illustrates these two cases.

Proof Consider any $\gamma \in \Delta_1 \setminus (\mathcal{E}_1 \cup \Pi \cup \Delta_2)$ and let $\gamma = \gamma(B)$ for some $B \in \mathbb{R}^{3 \times 2}$ such that $M(B)$ is a Type 1 triangle. For the sake of brevity, we use P to denote $P(B, R)$ in the remainder of this proof.

Step 1. Suppose $\#(\text{vert}(B) \cap P) \leq 1$. This implies that some facet has no corner rays; without loss of generality, let this be F_1 . Thus $F_1 \cap \text{vert}(B) \cap P = \emptyset$. Let y^1 be the integer point in $\text{rel int}(F_1)$. If $P \cap F_1 \setminus \mathbb{Z}^2 = \emptyset$, then we can tilt F_1 slightly in either direction without making new ray intersections on F_1 . This creates a Type 2 triangle that realizes γ (see Fig. 6), which contradicts the hypothesis that $\gamma \notin \Delta_2$. Therefore, we can assume $P \cap \text{rel int}(F_1) \setminus \mathbb{Z}^2 \neq \emptyset$. Under this assumption, Lemma 3.8 shows that there exist $\gamma^1, \gamma^2 \in \Delta$ such that γ is a strict convex combination of γ^1 and γ^2 .

Step 2. Suppose $\#(\text{vert}(B) \cap P) \geq 2$. Thus there exist $p^{j_1}, p^{j_2} \in \text{vert}(B) \cap P$, and we assume, by possibly exchanging rows of B , that F_3 is a facet containing p^{j_1}, p^{j_2} . Since $\gamma \notin \mathcal{E}_1$, neither Case a nor Case b holds. Thus $P \cup \{f\} \subset M(S_3)$. But then γ is dominated by or equal to $\gamma(S_3)$. Since $\gamma \notin \Pi$, γ must be dominated by $\gamma(S_3)$.

Thus we have shown that either γ is a strict convex combination of γ^1 and γ^2 with $\gamma^1, \gamma^2 \in \Delta$ or there exists $\gamma' \in \Delta$ such that γ is dominated by γ' .

Step 3. We next bound the cardinality of Ξ_3 , i.e., the number of Type 1 triangles with two corner rays r^{j_1}, r^{j_2} on F_3 such that Case a or Case b holds. There are $O(k^2)$ ways to choose the two corner rays on F_3 , which uniquely determine F_3 .

Case a. Since f does not lie in the split S_3 , the integer points y^1, y^2 are uniquely determined.

Case b. In this case f lies in the split S_3 and there exists a ray intersection p^{j_3} outside the split. After choosing the ray $r^{j_3} \in R$, the integer points y^1, y^2 are uniquely determined; there are at most k choices for r^{j_3} .

Since the triangle is uniquely determined from the points y^1, y^2 and the two corner rays r^{j_1}, r^{j_2} on F_3 , there are only polynomially many Type 1 triangles which give vectors in Ξ_1 . □

We next consider Type 2 triangles, which are the most complicated to handle. For this, we first establish some notation and an intermediate lemma.

Consider a matrix $B = (b^1; b^2; b^3) \in \mathbb{R}^{3 \times 2}$ such that $M(B)$ is a Type 2 triangle. For $i = 1, 2, 3$, we denote $F_i = F_i(B)$. Without loss of generality, we assume that the facet containing multiple integer points is F_3 . We label the unique integer points in the relative interiors of F_1 and F_2 as y^1 and y^2 , respectively. Within the case analysis of some of the proofs, we will refer to certain points lying within splits. For convenience, for $i = 1, 2, 3$, we define $S_i \in \mathbb{R}^{3 \times 2}$ such that $M(S_i)$ is the maximal lattice-free split with the properties that one facet of $M(S_i)$ contains F_i and $M(S_i) \cap \text{int}(M(B)) \neq \emptyset$.

Lemma 3.16 (Type 3 dominating Type 2 lemma) *Let $R = \{r^1, \dots, r^k\}$. Consider any $B \in \mathbb{R}^{3 \times 2}$ such that $M(B)$ is a Type 2 triangle. Denote the vertex $F_1 \cap F_3$ by v and let $y^3 \in F_3$ be the integer point in $\text{rel int}(F_3)$ closest to v . Suppose $P(B, R) \cap F_3$ is a subset of the line segment connecting v and y^3 . Then there exists a matrix $B' \in \mathbb{R}^{3 \times 2}$ such that $M(B')$ is a Type 3 triangle and either $\gamma(B)$ is dominated by $\gamma(B')$, or $\gamma(B) = \gamma(B')$.*

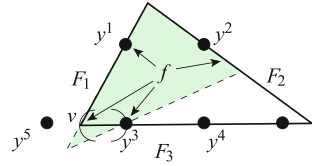
Proof Choose \bar{a}^3 such that $\bar{a}^3 \cdot (y^3 - f) = 0$ and $\bar{a}^3 \cdot (y^3 - v) > 0$. Consider tilting F_3 by adding $\varepsilon \bar{A} = \varepsilon(0; 0; \bar{a}^3)$ to B for some small enough $\varepsilon > 0$ so that the following two conditions are met. First, ε is chosen small enough such that the set of integer points contained in $M(B + \varepsilon \bar{A})$ is a subset of $Y(B)$; this can be done by Lemma 3.6. Second, since $P(B, R) \cap F_3 \subset [y^3, v]$, we know that there is no corner ray pointing to $F_2 \cap F_3$, and therefore we can choose ε small enough such that for all rays r^j such that $2 \in I_B(r^j), I_{B+\varepsilon\bar{A}}(r^j) = I_B(r^j)$. This means that $p^j = p(B, r^j) \in F_2$ if and only if $p(B + \varepsilon \bar{A}, r^j) \in F_2(B + \varepsilon \bar{A})$.

Now suppose r^j is a ray pointing from f to F_3 . Since $P(B, R) \cap F_3 \subset [y^3, v]$, we can describe r^j as the linear combination $r^j = \alpha_1(y^3 - f) - \alpha_2(y^3 - v)$ for some $\alpha_1, \alpha_2 \geq 0$. Observe that $\psi_{B+\varepsilon\bar{A}}(r^j) = \max\{b^1 \cdot r^j, (b^3 + \varepsilon \bar{a}^3) \cdot r^j\}$ and

$$(b^3 + \varepsilon \bar{a}^3) \cdot r^j = b^3 \cdot r^j + \varepsilon \bar{a}^3 \cdot (\alpha_1(y^3 - f) - \alpha_2(y^3 - v)) \leq b^3 \cdot r^j = \psi_B(r^j). \tag{8}$$

By definition of $\psi_B(r^j)$, we also have $b^1 \cdot r^j \leq \psi_B(r^j)$. Therefore, $\psi_{B+\varepsilon\bar{A}}(r^j) \leq \psi_B(r^j)$.

Fig. 7 We depict the geometry of Lemma 3.16 and show how we can change a facet of $M(B)$ to find a new matrix $B' \in \mathbb{R}^{3 \times 2}$ such that $M(B')$ is a Type 3 triangle and $\gamma(B')$ either dominates $\gamma(B)$, or $\gamma(B) = \gamma(B')$



Finally suppose r^j is such that $p^j \in P(B, R) \cap (F_1 \cup F_2) \setminus F_3$. Then $\psi_{B+\varepsilon\bar{A}}(r^j) = \psi_B(r^j)$ since by construction $I_B(r^j) = I_{B+\varepsilon\bar{A}}(r^j)$ for all such rays. Also, note that for any $y \in F_3 \cap \mathbb{Z}^2$, $y = y^3 + \beta(y^3 - v)$ for some $\beta \geq 0$. Therefore,

$$(b^3 + \varepsilon\bar{a}^3) \cdot (y - f) \geq b^3 \cdot (y - f) = 1,$$

meaning that none of these integer points are contained in the interior of $M(B + \varepsilon\bar{A})$. Since the set of integer points contained in $M(B + \varepsilon\bar{A})$ is a subset of $Y(B)$ and facets F_1 and F_2 were not tilted, $M(B + \varepsilon\bar{A})$ is lattice-free; in fact, it is a Type 3 triangle (see Fig. 7).

Thus, we can choose $B' = B + \varepsilon\bar{A}$. The vector $\gamma(B)$ is dominated by $\gamma(B')$ when the inequality (8) is strict for some r^j ; otherwise, $\gamma(B) = \gamma(B')$. □

Proposition 3.17 (Counting Type 2 triangles) *There exists a finite subset $\mathcal{E}_2 \subseteq \Delta_2$ such that if $\gamma \in \Delta_2 \setminus (\mathcal{E}_2 \cup \Delta_3 \cup \Pi)$, then there exist $\gamma^1, \gamma^2 \in \Delta$ such that γ is a strict convex combination of γ^1 and γ^2 or there exists $\gamma' \in \Delta$ such that γ is dominated by γ' . Moreover, the cardinality of \mathcal{E}_2 is bounded polynomially in the binary encoding sizes of f, r^1, \dots, r^k . Specifically, \mathcal{E}_2 can be chosen as the set of all $\gamma(B)$ such that $M(B)$ is a Type 2 triangle satisfying one of the following, where $P = P(B, R)$ and $F_i = F_i(B)$ such that F_3 is the facet of $M(B)$ containing multiple integer points:*

Case a. $P \subset \mathbb{Z}^2$.

Case b. $P \not\subset \mathbb{Z}^2$ and there exist $p^{j_1} \in P \cap F_1 \cap F_3$ (i.e., there is a corner ray pointing from f to $F_1 \cap F_3$) and $p^{j_2} \in P \cap F_3$ with $\#([p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2) \geq 2$. Moreover, if $P \cap \text{rel int}(F_2) \setminus \mathbb{Z}^2 \neq \emptyset$, then there is a corner ray of $M(B)$ pointing to a vertex different from $F_1 \cap F_3$. Also, one of the following holds:

Case b1. $f \notin M(S_3)$.

Case b2. $f \in M(S_3)$ and $P \not\subset M(S_3)$.

Case c. $P \not\subset \mathbb{Z}^2$ and there exist $p^{j_1} \in P \cap F_1 \cap F_3 \cap \mathbb{Z}^2$ (i.e., there is a corner ray pointing from f to $F_1 \cap F_3 \subset \mathbb{Z}^2$) and $p^{j_2} \in P \cap F_1$ with $\#([p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2) \geq 2$. Moreover, if $P \cap \text{rel int}(F_2) \setminus \mathbb{Z}^2 \neq \emptyset$, then p^{j_2} can be chosen such that $p^{j_2} \in F_1 \cap F_2$ (i.e., there is a corner ray pointing from f to $F_1 \cap F_2$). Also, one of the following holds:

Case c1. $f \notin M(S_1)$.

Case c2. $f \in M(S_1)$ and $P \not\subset M(S_1)$.

Case d. $P \not\subset \mathbb{Z}^2$, for all $i \in \{1, 2, 3\}$ and all $p^{j_1}, p^{j_2} \in P \cap F_i$ we have $\#([p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2) \leq 1$, there exists a corner ray pointing from f to $F_1 \cap F_3$, and $F_1 \cap F_3 \not\subset \mathbb{Z}^2$. Let $y^3, y^4 \in F_3$ such that y^3 is the closest integer point in $\text{rel int}(F_3)$ to $F_1 \cap F_3$, and y^4 is the next closest integer point. Let $H_{2,4}$ be the half-space adjacent to $[y^2, y^4]$ and containing y^1 .

Then, we further have $P \cap (y^3, y^4) \neq \emptyset$. Moreover, one of the following holds:

Case d₁. $f \notin H_{2,4}$, there exists a corner ray pointing from f to $F_1 \cap F_2$.

Case d₂. $f \notin H_{2,4}$, there exists a ray pointing from f through (y^1, y^2) to F_1 and there are no rays pointing from f to $\text{rel int}(F_2) \setminus \mathbb{Z}^2$.

Case d₃. $f \in H_{2,4}$, $P \not\subset H_{2,4}$, and there exists a corner ray pointing from f to $F_1 \cap F_2$.

Proof Consider any $\gamma \in \Delta_2$. By definition of Δ_2 , there exists a matrix $B \in \mathbb{R}^{3 \times 2}$ such that $\gamma(B) = \gamma$ and $M(B)$ is a Type 2 triangle. Recall the labeling of the facets of $M(B)$ as F_1, F_2, F_3 with corresponding labels for the rows of B . For the sake of brevity, let P denote the set $P(B, R)$ of the ray intersections in $M(B)$ for the rest of this proof. If $P \subset \mathbb{Z}^2$, we are in Case a. Therefore, in the remainder of the proof, we always assume $P \not\subset \mathbb{Z}^2$. □

Proof steps 1 and 2: dominated, convex combination, or Case d Suppose $P \not\subset \mathbb{Z}^2$ and for all $i \in \{1, 2, 3\}$ and all $p^{j_1}, p^{j_2} \in P \cap F_i$, we have $\#([p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2) \leq 1$. We will show that at least one of the following occurs:

- (i) $\gamma(B)$ is dominated by some $\gamma' \in \Delta$, or is a strict convex combination of some $\gamma^1, \gamma^2 \in \Delta$, or there exists a maximal lattice-free split or Type 3 triangle $M(B')$ such that $\gamma(B') = \gamma(B)$.
- (ii) Either Case d₁, Case d₂, or Case d₃ occurs.

First note that if there exist distinct $p^{j_1}, p^{j_2} \in P \cap \text{vert}(B) \cap F_3$, then $[p^{j_1}, p^{j_2}] = F_3$ and $\#([p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2) \geq 2$ since F_3 contains multiple integer points, and thus violating the assumptions. Therefore $\#(P \cap \text{vert}(B) \cap F_3) \leq 1$.

Recall that F_3 is the facet of $M(B)$ that contains at least 2 integer points and consider the sub-lattice of \mathbb{Z}^2 contained in the linear space parallel to F_3 . We use the notation $v(F_3)$ to denote the primitive lattice vector which generates this one-dimensional lattice and lies in the same direction as the vector pointing from $F_1 \cap F_3$ to $F_2 \cap F_3$. Since $\#([p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2) \leq 1$ for all $p^{j_1}, p^{j_2} \in P \cap F_3$, there exists $y^3 \in F_3 \cap \mathbb{Z}^2$ such that $P \cap F_3 \subset (y^3 - v(F_3), y^3 + v(F_3))$. Let $y^4 = y^3 + v(F_3)$ and let $y^5 = y^3 - v(F_3)$ and so $P \cap F_3$ is a subset of the open segment (y^5, y^4) . Note that y^4, y^5 are not necessarily contained in F_3 . In Step 1 we will analyze the case with no corner rays on F_3 and see that we always arrive in conclusion (i), whereas in Step 1 we will analyze the case with a corner ray on F_3 and see that we will also arrive in conclusion (i), except for the last step, Step 2d, where we arrive in conclusion (ii).

Step 1. Suppose that F_3 has no corner rays, i.e., $\text{vert}(B) \cap P \cap F_3 = \emptyset$.

Step 1a. Suppose $P \cap (F_1 \cup F_2) \setminus \mathbb{Z}^2 \neq \emptyset$, i.e., there exists an index $j \in \{1, \dots, k\}$ such that $p^j \in (F_1 \cup F_2) \setminus (F_3 \cup \mathbb{Z}^2)$. We will use the tilting space to show that $\gamma(B)$ is a strict convex combination of points in Δ . Let $Y_1 = \{y^1\}, Y_2 = \{y^2\}, Y_3 = F_3 \cap \mathbb{Z}^2, \mathcal{Y} = \{Y_1, Y_2, Y_3\}$. Let r' be a corner ray that points from f to the vertex $F_1 \cap F_2$ and let $R' = \{r^1, \dots, r^k, r'\}$. Since $R' \supseteq R$, we have $\mathcal{N}(B, \mathcal{Y}, R') \subseteq \mathcal{N}(B, \mathcal{Y}, R)$.

We first count the equations that define $\mathcal{N}(B, \mathcal{Y}, R')$. The equation $a^3 = b^3$ is implicit in $\mathcal{S}(B, \mathcal{Y}, R')$ since there are multiple integer points on F_3 . There are two other equations for integer points on F_1 and F_2 . The null space $\mathcal{N}(B, \mathcal{Y}, R')$ is given by the equations

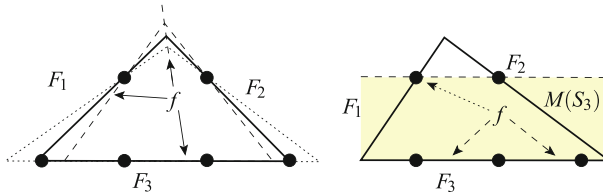


Fig. 8 Steps 1a and 1b₁. The *left figure* depicts Step 1a where $P \cap (F_1 \cup F_2) \neq \emptyset$ and there are no corner rays on F_3 , and shows that $\gamma(B)$ is a strict convex combination of other points in Δ by finding two lattice-free triangles through tilting the facets F_1 and F_2 . The *right figure* depicts Step 1b₁, where we find the split $M(S_3)$ such that $\gamma(S_3)$ dominates $\gamma(B)$

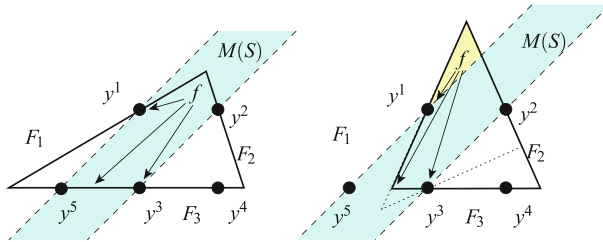


Fig. 9 Step 1b₂. In this step we consider $f \notin M(S_3)$. On the *left* we see that $P \cap F_3 \subset [y^5, y^3]$ and both $y^5, y^3 \in F_3$, which allows $\gamma(B)$ to be dominated by $\gamma(S)$. This split satisfies $\gamma(S) \in \Delta$ because $f \notin M(S_3)$, meaning that f is located somewhere on the top of the triangle, which is completely contained by $M(S)$. On the *right*, $y^5 \notin F_3$, which means that the split S cuts off the top corner of the triangle, potentially leaving f outside the split. This is problematic, so instead, we use Lemma 3.16 to create a new Type 3 triangle $M(B')$ such that $\gamma(B')$ dominates $\gamma(B)$

$$a^1 \cdot (y^1 - f) = 0, \quad a^2 \cdot (y^2 - f) = 0, \quad a^1 \cdot r' = a^2 \cdot r', \quad a^3 = 0.$$

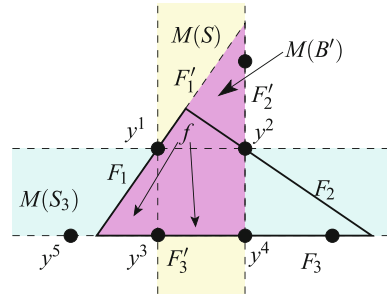
Since $\mathcal{N}(B, \mathcal{Y}, R') \subset \mathbb{R}^{3 \times 2}$ and there are 5 equations (note that $a_3 = 0$ is actually two equations), we see $\dim(\mathcal{N}(B, \mathcal{Y}, R)) \geq \dim(\mathcal{N}(B, \mathcal{Y}, R')) \geq 1$. Let $\bar{A} = (\bar{a}^1; \bar{a}^2; \bar{a}^3) \in \mathcal{N}(B, \mathcal{Y}, R) \setminus \{0\}$. Since \mathcal{Y} is a covering of the lattice points in $M(B)$, by Lemma 3.7, there exists $\varepsilon > 0$ such that $\gamma(B) = \frac{1}{2}\gamma(B + \varepsilon\bar{A}) + \frac{1}{2}\gamma(B - \varepsilon\bar{A})$ and $M(B \pm \varepsilon\bar{A})$ are both lattice-free. See Fig. 8 for these possible triangles.

We next show that $\gamma(B - \varepsilon\bar{A}) \neq \gamma(B + \varepsilon\bar{A})$. Observe that $\bar{a}^3 = 0$ since we are restricted by the equation $a^3 = b^3$. If $\bar{a}^1 = 0$, then \bar{a}^2 must satisfy $\bar{a}^2 \cdot r' = 0$ and $\bar{a}^2 \cdot (y^2 - f) = 0$, which implies that $\bar{a}^2 = 0$ since r' and $y^2 - f$ are linearly independent (since $y^2 \in \text{rel int}(F_2)$ and r' points to a corner of F_2). Similarly, if $\bar{a}^2 = 0$, then $\bar{a}^1 = 0$. Since $\bar{A} \neq 0$, we must have both $\bar{a}^1, \bar{a}^2 \neq 0$. Then, since $\#Y_1 = \#Y_2 = 1$ and $1, 2 \in I_B(r')$ and $p^j \in (F_1 \cup F_2) \setminus (F_3 \cup \mathbb{Z}^2)$, by Lemma 3.7 $\gamma(B)$ is a strict convex combination of $\gamma(B + \varepsilon\bar{A})$ and $\gamma(B - \varepsilon\bar{A})$ and $M(B \pm \varepsilon\bar{A})$ are lattice-free triangles.

Step 1b. Suppose $P \cap (F_1 \cup F_2) \setminus \mathbb{Z}^2 = \emptyset$, i.e., there only exist rays pointing from f to F_3, y^1, y^2 . Therefore, $P \subset M(S_3)$. We now analyze further subcases.

Step 1b₁. Suppose $f \in M(S_3)$. Then $\gamma(B)$ is either dominated by or equal to $\gamma(S_3)$. If $P \cap F_3 = \emptyset$, then $P \subset \{y^1, y^2\} \subset \mathbb{Z}^2$, which is a contradiction with the assumption of Step 1 that $P \not\subset \mathbb{Z}^2$ (see Fig. 8).

Fig. 10 Step 1b₃. This figure demonstrates a new triangle $M(B')$ that yields the same inequality as $M(B)$



Step 1b₂. Suppose that $f \notin M(S_3)$ and $P \cap F_3 \neq \emptyset$. Suppose further that either $P \cap F_3 \subset [y^3, y^4]$ or $P \cap F_3 \subset [y^5, y^3]$. These two cases are very similar, so we will just show the case with $P \subset [y^5, y^3]$, which is illustrated in Fig. 9.

If both $y^5, y^3 \in F_3$, then $\gamma(B)$ is dominated by $\gamma(S)$ where S is the maximal lattice-free split with its two facets along $[y^3, y^2]$ and $[y^5, y^1]$. This is because $P \not\subset \mathbb{Z}^2$ and so there exists a ray intersection lying in the open segment (y^5, y^3) .

Otherwise, suppose $y^5 \notin F_3$. Note that $y^3 \notin \text{vert}(B)$, because otherwise since $P \cap F_3 \neq \emptyset$, we find that $P \cap F_3 \subset [y^5, y^3] \cap F_3 = \{y^3\}$, and therefore, $y^3 \in P$, contradicting the fact that there are no corner rays on F_3 . Thus y^3 is the integer point in $\text{rel int}(F_3)$ closest to $F_1 \cap F_3$. This implies that $M(B)$ satisfies the hypotheses of Lemma 3.16. Hence there exists B' such that $M(B')$ is a Type 3 triangle and either $\gamma(B)$ is dominated by $\gamma(B')$ or $\gamma(B) = \gamma(B')$.

Step 1b₃. Suppose that $f \notin M(S_3)$, $P \cap F_3 \neq \emptyset$, and $P \not\subset [y^3, y^4]$, $P \not\subset [y^5, y^3]$, i.e., $\text{conv}(P \cap F_3)$ contains the integer point y^3 in its relative interior.

We define a new triangle $M(B')$ by choosing its facets $F'_i = F_i(B')$, and thus, uniquely defining the matrix $B' \in \mathbb{R}^{3 \times 2}$. Let $F'_3 = F_3 \cap [y^5, y^4]$. Next, let F'_1 and F'_2 be given by lines from the endpoints of F'_3 through y^1 and y^2 , respectively (see Fig. 10). We will show $\gamma(B') = \gamma(B)$ and then that $\gamma(B')$ is a strict convex combination of other points in Δ .

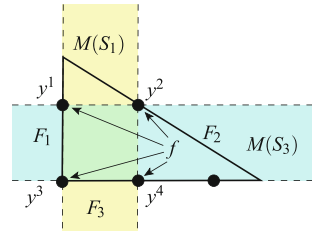
Claim β The set $M(B')$ is lattice-free.

Proof First note that $\#(F_3 \cap [y^5, y^4] \cap \mathbb{Z}^2) \geq 2$ since y^3 is in the relative interior of F_3 and F_3 contains multiple integer points. Without loss of generality, suppose $y^4 \in F_3$. Let $S \in \mathbb{R}^{3 \times 2}$ such that $M(S)$ is the maximal lattice-free split with facets on $[y^1, y^3]$ and $[y^2, y^4]$. Then $M(B')$ is lattice-free since $M(B') \subset M(S) \cup M(B)$, which are both lattice-free sets. □

Claim γ We have $f \in M(B')$ and $\gamma(B') = \gamma(B)$.

Proof Since $f \in M(B) \cap M(S)$ and $M(B) \cap M(S) \subset M(B') \cap M(S)$, it follows that $f \in M(B')$. Recall that we are under the assumption that all rays point to y^1, y^2 or F_3 . Moreover, all ray intersections on F_3 are contained in (y^5, y^4) and hence the ray intersections are contained in F'_3 . Therefore, the set $P(B', R)$ of ray intersections with respect to $M(B')$ is the same as P , and therefore $\gamma(B) = \gamma(B')$. □

Fig. 11 Step 2a. Depending on where f is located in the triangle, at least one of $\gamma(S_3)$ or $\gamma(S_1)$ either dominates or realizes $\gamma(B)$. In this picture, $\gamma(S_3)$ realizes $\gamma(B)$, while $\gamma(S_1) \notin \Delta$ since $f \notin \text{int}(M(S_1))$



Since $\text{conv}(P \cap F_3)$ contains y^3 in its relative interior and $P \cap F_3$ is contained in the open segment (y^5, y^4) , we must have $P \cap \text{rel int}(F_3) \setminus \mathbb{Z}^2 \neq \emptyset$. Furthermore, let $P' = P(B', R)$ be the set of ray intersections for $M(B')$, then $P' \cap F'_3 = P \cap F_3$ by definition of F'_3 . Therefore, $P' \cap \text{rel int}(F'_3) \setminus \mathbb{Z}^2 \neq \emptyset$. Furthermore, $P' \cap \text{vert}(B') \cap F'_3 = \emptyset$ since $M(B)$ has no corner rays pointing to F_3 and there cannot exist rays pointing to y^4 or y^5 since $P \cap F_3$ is contained in the open segment (y^5, y^4) . Moreover, $\text{rel int}(F'_3) \cap \mathbb{Z}^2 = \{y^3\}$. Therefore, Lemma 3.8 can be applied to $M(B')$ with $F = F'_3$, which shows that $\gamma(B') = \gamma(B)$ is a strict convex combination of other points in Δ .

Step 2. Suppose there is a corner ray on F_3 and, if necessary, relabel the facets of $M(B)$ (and the rows of B) such that this corner ray points from f to the intersection $F_1 \cap F_3$. Recall that we label the integer points $y^1 \in F_1, y^2 \in F_2$. Since $F_1 \cap F_3 \subseteq P$, observe that $y^3 \in F_3$ (as defined in the paragraph before Step 1) is the closest integer point in F_3 to $F_1 \cap F_3$, and since $M(B)$ is a Type 2 triangle, we have $y^4 \in F_3$. Let $H_{2,4}$ be the half-space with boundary containing the segment $[y^2, y^4]$ and with interior containing y^1 (see Fig. 12).

Step 2a. Suppose $y^3 \in F_1 \cap F_3$ and recall that there is a corner ray pointing from f to $F_1 \cap F_3$. Note that this implies that $P \cap F_2 \cap \text{vert}(B) = \emptyset$, because $\#([p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2) \leq 1$ for all $p^{j_1}, p^{j_2} \in P \cap F_i$ for any $i \in \{1, 2, 3\}$ and including any corner ray pointing from f to $F_1 \cap F_2$ or $F_2 \cap F_3$ would contradict this. Therefore, $P \cap F_2 \subset \text{rel int}(F_2)$.

If $P \cap F_2 \setminus \mathbb{Z}^2 = P \cap \text{rel int}(F_2) \setminus \mathbb{Z}^2 \neq \emptyset$, then $M(B)$ satisfies the hypotheses of Lemma 3.8 with $F = F_2$ and $\gamma(B)$ is a strict convex combination of points in Δ .

If instead $P \cap F_2 \setminus \mathbb{Z}^2 = \emptyset$, then $P \cap F_2 \subset \{y^2\}$, and since $F_1 \cap F_3 \subseteq P$ and no two ray intersections within a facet can contain two integer points between them, we must have $P \cap F_1 \subseteq [y^3, y^1]$ and $P \cap F_3 \subseteq [y^3, y^4]$. Therefore, $P \subset \text{conv}(\{y^1, y^2, y^3, y^4\})$. Since $M(S_1) \cup M(S_3) \supset M(B)$, we must have $P \cup \{f\} \subset M(S_i)$ for $i = 1$ or 3 , and hence $\gamma(B)$ is either dominated by or equal to $\gamma(S_i)$ (see Fig. 11).

Step 2b. Suppose $y^3 \notin F_1 \cap F_3$ and $P \cup \{f\} \subset H_{2,4}$. Let $B' \in \mathbb{R}^{3 \times 2}$ such that $M(B')$ is the lattice-free Type 2 triangle with base F'_3 along $[y^2, y^4]$, the facet F'_1 given by the line defining F_1 for $M(B)$ and the facet F'_2 given by the line defining F_3 for $M(B)$. Let P' be the set of ray intersections for $M(B')$ (see Fig. 12).

If $P \cap \text{rel int}(F_2) \setminus \{y^2\} \neq \emptyset$, then $\gamma(B')$ dominates $\gamma(B)$ because $P \cup \{f\} \subset H_{2,4}$.

Otherwise, $\gamma(B) = \gamma(B')$ and $P \cap \text{rel int}(F_2) \setminus \{y^2\} = \emptyset$. This implies that no ray points from f to the corner $F'_1 \cap F'_3$ of $M(B')$. Recall that $P \cap F_3$ is a subset of the open segment (y_5, y_4) , therefore, $y^4 \notin P$. Hence, $M(B')$ has no corner rays on F'_3 . Also, since there exists a corner ray pointing from f to $F_1 \cap F_3 = F'_1 \cap F'_2$, we see that $P' \cap (F'_1 \cup F'_2) \setminus \mathbb{Z}^2 \neq \emptyset$. Hence, $M(B')$ is a Type 2 triangle satisfying the conditions

Fig. 12 Step 2b. We show that if $P \cup \{f\} \subset H_{2,4}$, then we can create a different Type 2 triangle $M(B')$ such that $\gamma(B')$ dominates $\gamma(B)$. If $\gamma(B) = \gamma(B')$, i.e., the ray pointing from f to the facet F_2 does not exist in the above picture, then the new triangle is a Type 2 triangle that was considered in Step 1a

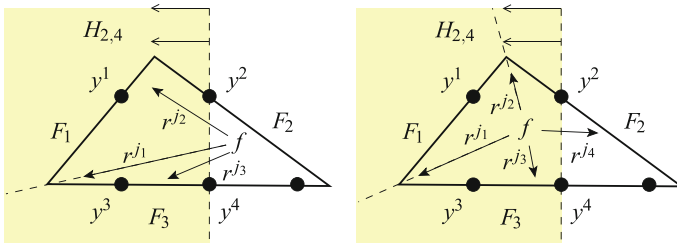
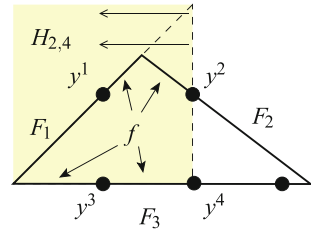


Fig. 13 Steps 2d₁ and 2d₂. These steps arrive at Cases d₁, d₂, and d₃, which are depicted here. Other rays may also exist. Cases d₁ and d₂, because of their commonalities, are represented in one picture on the left, and Case d₃ is on the right

considered in Step 1a, and using the same reasoning from that step, $\gamma(B') = \gamma(B)$ can be shown to be a strict convex combination of points in Δ .

Step 2c. Suppose $P \cap F_3 \subset [F_1 \cap F_3, y^3]$, $y^3 \notin F_1 \cap F_3$. Recall that y^3 is the closest integer point in F_3 to the corner $F_1 \cap F_3$. Then $M(B)$ satisfies the hypotheses of Lemma 3.16 and we can find a Type 3 triangle $M(B')$ such that $\gamma(B)$ is dominated by $\gamma(B')$ or $\gamma(B) = \gamma(B')$.

Step 2d. We can now assume that $P \not\subset \mathbb{Z}^2$ (the assumption for Steps 1 and 2), there is a corner ray pointing from f to $F_1 \cap F_3$ (assumption in beginning of Step 2), $y^3 \notin F_1 \cap F_3$ (negation of the assumption in Step 2a), $P \cup \{f\} \not\subset H_{2,4}$ (negation of the second assumption in Step 2b), and $(y^3, y^4) \cap P \neq \emptyset$ (negation of the assumption in Step 2c), which implies $y^3 \in \text{int}(\text{conv}(P \cap F_3))$. Furthermore, we may be in one of the following subcases.

Step 2d₁. $f \notin H_{2,4}$. This implies $f \in M(S_3)$ since $M(B) \setminus H_{2,4} \subset M(S_3)$. If P is also contained in $M(S_3)$, then either $\gamma(B)$ is dominated by $\gamma(S_3)$, or $\gamma(B) = \gamma(S_3)$. Therefore, we assume $P \not\subset M(S_3)$, and so there must exist a ray r pointing from f through (y^1, y^2) .

Suppose there is a ray that points from f to $\text{rel int}(F_2)$. If there is no corner ray pointing from f to $F_1 \cap F_2$, then $M(B)$ would satisfy the hypotheses of Lemma 3.8 with $F = F_2$ since no ray points to $F_2 \cap F_3$. Therefore, $\gamma(B)$ can be expressed as the strict convex combination of points from Δ . On the other hand, if there is a corner ray pointing to $F_1 \cap F_2$, then we satisfy the statement of Case d₁ (see Fig. 13).

Suppose now that no ray points from f to $\text{rel int}(F_2)$. This implies that the ray r points from f to F_1 through (y^1, y^2) and $P \cap \text{rel int}(F_2) \setminus \mathbb{Z}^2 = \emptyset$. This is Case d₂.

Step 2d₂. $f \in H_{2,4}$ and $P \not\subset H_{2,4}$. Because also $P \cap F_3 \subseteq H_{2,4}$, this implies that $P \cap \text{rel int}(F_2) \setminus \mathbb{Z}^2 \neq \emptyset$.

If there is no corner ray pointing from f to $F_1 \cap F_2$, then $M(B)$ satisfies the hypotheses of Lemma 3.8 with $F = F_2$ because there is no ray intersection in $F_2 \cap F_3$. Then $\gamma(B)$ can be expressed as the strict convex combination of points from Δ .

On the other hand, if there is a corner ray pointing from f to $F_1 \cap F_2$, then we satisfy the statement of Case d₃.

From the analysis of Steps 1 and 2, we conclude that if $P \not\subset \mathbb{Z}^2$, γ is not dominated by any $\gamma' \in \Delta$, is not a strict convex combination of any $\gamma^1, \gamma^2 \in \Delta$, and there does not exist a maximal lattice-free split or Type 3 triangle $M(B')$ such that $\gamma(B') = \gamma$, then one of the following holds:

- (i) There exist ray intersections $p^{j_1}, p^{j_2} \in P \cap F_i$ with $\#([p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2) \geq 2$ for some $i \in \{1, 2, 3\}$.
- (ii) We are in Case d.

Proof steps 3 and 4: remaining cases We now assume that $\gamma = \gamma(B)$ is not dominated by any $\gamma' \in \Delta$, is not a strict convex combination of any $\gamma^1, \gamma^2 \in \Delta$, and there does not exist a maximal lattice-free split or Type 3 triangle $M(B')$ such that $\gamma(B') = \gamma(B)$, and we are not in Case d, and we are not in Case a (so $P \not\subset \mathbb{Z}^2$). Therefore, from our previous analysis, there exist ray intersections $p^{j_1}, p^{j_2} \in P \cap F_i$ with $\#([p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2) \geq 2$ for some $i \in \{1, 2, 3\}$. We will show that either Case b₁, b₂, c₁, or c₂ occurs. In Step 3 below, we analyze the case when $i = 3$, and in Step 4, we analyze the case when $i = 1$ or $i = 2$.

Step 3. Suppose $P \not\subset \mathbb{Z}^2$ and there exist $p^{j_1}, p^{j_2} \in P \cap F_3$ with $\#([p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2) \geq 2$. Let p^{j_1}, p^{j_2} be such that $P \cap F_3 \subseteq [p^{j_1}, p^{j_2}]$.

Step 3a. We first show that there exists a matrix B' such that $M(B')$ is a lattice-free Type 2 triangle that has a corner ray on F_3 , and $\gamma(B) = \gamma(B')$.

If either p^{j_1} or p^{j_2} is a vertex of $M(B)$, then we let $B' = B$ and move to Step 3b.

We now deal with the case that $p^{j_1}, p^{j_2} \notin \text{vert}(B)$, i.e., there are no corner rays pointing from f to F_3 .

Claim δ If there exists a ray $r^j \in R$ such that $p^j \in F_1 \cup F_2 \setminus (F_3 \cup \mathbb{Z}^2)$, then $\gamma(B)$ is a strict convex combination of other points in Δ .

Proof We define $\mathcal{Y} = (Y_1, Y_2, Y_3)$ as $Y_1 = \{y^1\}$, $Y_2 = \{y^2\}$ and $Y_3 = F_3 \cap \mathbb{Z}^2$. Hence, \mathcal{Y} is a covering of $Y(B)$. Define a new ray r' to be a corner ray pointing from f to the intersection $F_1 \cap F_2$ and let $R' = \{r^1, \dots, r^k, r'\}$. Then $\mathcal{N}(B, \mathcal{Y}, R') \subseteq \mathcal{N}(\mathcal{Y})$.

Since there are no corner rays pointing to F_3 , there is only one independent equation coming from a corner ray condition in the system defining $\mathcal{N}(B, \mathcal{Y}, R')$. The integer point sets Y_1 and Y_2 each contribute one equation. Since Y_3 contains two integer points, it contributes a system of equalities involving a^3 with rank 2. Therefore, $\dim \mathcal{N}(B, \mathcal{Y}, R') = 6 - 5 = 1$.

Note that the equations from Y_3 impose that $\bar{a}^3 = 0$. Therefore, either $\bar{a}^1 \neq 0$ or $\bar{a}^2 \neq 0$. Since $p^j \in \text{rel int}(F_i) \setminus \mathbb{Z}^2$ for either $i = 1, 2$, we have that $i \in I_B(r^j)$ for either $i = 1, 2$. Then, since $\#Y_1 = \#Y_2 = 1$, by Lemma 3.7, there exists $\varepsilon > 0$ such

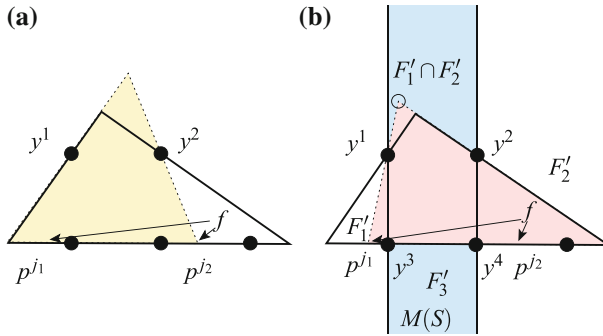


Fig. 14 Step 3a. Either F_1 or F_2 is tilted to give a new triangle $M(B')$ (dotted). **a** Here F_2 cannot be used because tilting would remove f from the interior. **b** Instead, F_1 needs to be used

that $\gamma(B)$ is a strict convex combination of $\gamma(B + \varepsilon \bar{A})$ and $\gamma(B - \varepsilon \bar{A})$ and $M(B + \varepsilon \bar{A})$ and $M(B - \varepsilon \bar{A})$ are lattice-free. □

We now show that either F_1 or F_2 can be tilted to create a new Type 2 triangle $M(B')$ that has a corner ray pointing to the facet that contains multiple integer points, $F_3(B')$, and $\gamma(B') = \gamma(B)$. Let y^1, y^2 be the integer points in $\text{rel int}(F_1), \text{rel int}(F_2)$, respectively. Since, by Claim δ , there are no rays pointing to $F_1 \cup F_2 \setminus \mathbb{Z}^2$, tilting F_i with fulcrum y^i does not change $\gamma(B)$ unless f is no longer on the interior of the perturbed set. For $i = 1, 2$, consider changing F_i to now lie on the line through p^{j_i} and y^i . At most one of these facet tilts puts f outside the perturbed set, and therefore, at least one of these changes is possible. This is illustrated in Fig. 14. We can assume that the tilt of facet F_1 is possible (by exchanging the rows of B , if necessary). Let $B' \in \mathbb{R}^{3 \times 2}$ such that $M(B')$ is the new set after this tilting operation. The facets $F'_i = F_i(B')$ of $M(B')$ are chosen such that F'_1 corresponds to the new tilted F_1 and F'_2, F'_3 correspond to F_2, F_3 respectively.

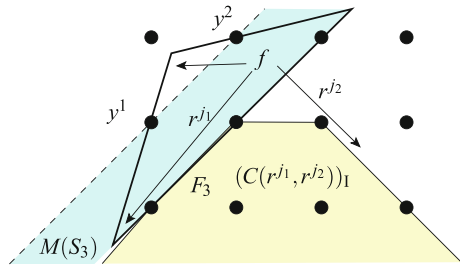
We claim that $M(B')$ is lattice-free. To see this, we will show that $\text{int}(M(B'))$ is a subset of the union of interiors of two lattice-free sets. Let $y^3, y^4 \in [p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2$ be distinct integer points adjacent to each other. Then consider the maximal lattice-free split $M(S)$, where $S \in \mathbb{R}^{2 \times 2}$, whose facets contain the segments $[y^3, y^1]$ and $[y^4, y^2]$, respectively. Since $[y^3, y^4] \subset (p^{j_1}, F_2 \cap F_3)$ is a strict subset, the former and new intersections $F_1 \cap F_2$ and $F'_1 \cap F'_2$ are contained in the split $M(S)$. Observe that $\text{int}(M(B')) \cap M(S_3) \subseteq \text{int}(M(B))$ and $\text{int}(M(B')) \setminus M(S_3) \subseteq \text{int}(M(S))$. Hence $\text{int}(M(B')) \subset \text{int}(M(B)) \cup \text{int}(M(S))$. Therefore $M(B')$ is lattice-free since $M(S)$ and $M(B)$ are both lattice-free.

Therefore, we have shown that $\gamma(B) = \gamma(B')$ where $M(B')$ is a Type 2 triangle that has a corner ray pointing to the facet F'_3 . Note that since $\gamma(B) = \gamma(B')$, the sets of ray intersections coincide, that is, $P(B', R) = P$.

Step 3b. After Step 3a, we now focus on the new triangle $M(B')$ with facets $F'_i = F_i(B')$ for $i = 1, 2, 3$ that has a corner ray pointing from f to the vertex at $F'_1 \cap F'_3$. We will show that the conditions of Case b are satisfied.

If $P(B', R) \cap \text{rel int}(F'_2) \setminus \mathbb{Z}^2 \neq \emptyset$ and there are no corner rays on F'_2 , then $M(B')$ satisfies the hypotheses of Lemma 3.8 and $\gamma(B) = \gamma(B')$ could be expressed as a

Fig. 15 Proposition 3.17, Step 5: Uniquely determining Type 2 triangles that are in Case b using a facet of $(C(r^{j_1}, r^{j_2}))_1$ and the integer points y^1 and y^2



strict convex combination of points in Δ . Therefore, if $P(B', R) \cap \text{rel int}(F'_2) \setminus \mathbb{Z}^2 \neq \emptyset$, then there must be a corner ray pointing from f to F'_2 (and thus pointing to a vertex different from $F'_1 \cap F'_3$).

Hence, the conditions of Case b are met for B' (instead of B). Furthermore, if $P \cup \{f\} \subset M(S_3)$, then $\gamma(B) = \gamma(B')$ is dominated by or equal to $\gamma(S_3)$, a contradiction to our assumption; hence, either Case b_1 or Case b_2 occurs.

Thus, from the analysis of Step 3, when there exist points $p^{j_1}, p^{j_2} \in P \cap F_3$ with $\#([p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2) \geq 2$, then $M(B')$ is a Type 2 triangle satisfying the statement of Case b.

Step 4. Suppose $P \not\subset \mathbb{Z}^2$ and there exist $p^{j_1}, p^{j_2} \in P \cap F_i$ with $\#([p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2) \geq 2$, for $i = 1$ or $i = 2$. After a relabeling of the facets of $M(B)$ and the rows of B , we can assume $i = 1$. Since $M(B)$ is a Type 2 triangle and F_1 is a facet with at most one integer point in its relative interior, we must have $\#(F_1 \cap \mathbb{Z}^2) \leq 2$. In order for $\#([p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2) \geq 2$, it has to equal exactly two, and one of the points, say p^{j_1} , must lie in $F_1 \cap F_3 \cap \mathbb{Z}^2$. Thus, p^{j_1} corresponds to a corner ray.

If $P \cap \text{rel int}(F_2) \setminus \mathbb{Z}^2 \neq \emptyset$, then again, there must be a corner ray on F_2 ; otherwise, Lemma 3.8 shows that $\gamma(B)$ is a strict convex combination of points in Δ . We can assume that this corner ray points from f to $F_1 \cap F_2$, otherwise we are back to the assumptions in Step 3 and $M(B)$ will satisfy the conditions of Case b. Thus p^{j_2} can be chosen such that $p^{j_2} \in F_1 \cap F_2$.

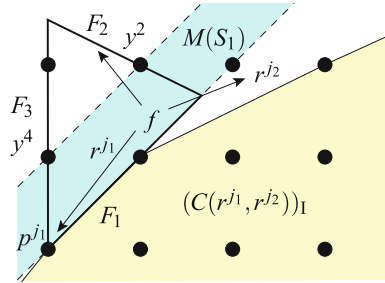
As in Case b, if $P \cup \{f\} \subset M(S_1)$, then $\gamma(B)$ is dominated by or equal to $\gamma(S_1)$. Hence, we are either in Case c_1 or Case c_2 .

Proof step 5: Bounding the cardinality of Ξ_2 by a polynomial Recall that we have a set of k rays $\{r^1, \dots, r^k\}$ and P is the set of ray intersections. Given this set of rays, we count how many distinct vectors $\gamma(B)$ can arise when $M(B)$ satisfies the conditions in Cases a, b, c and d. We will apply Lemma 3.10 to show that there are only polynomially many possibilities for $\gamma(B)$ in each case.

Case a. $P(B, R) \subset \mathbb{Z}^2$. Observation 3.12 shows that there is a unique $\gamma \in \Xi_2$ corresponding to this case.

Case b. We now count the vectors $\gamma(B)$ such that $M(B)$ satisfies the conditions of Case b with respect to our set of rays $\{r^1, \dots, r^k\}$. Consider any such $M(B)$. From the conditions stated in Case b, we can assume that $M(B)$ has a corner ray on F_3 .

Fig. 16 Proposition 3.17, Step 5: Uniquely determining Type 2 triangles that are in Case c using a facet of $(C(r^{j_1}, r^{j_2}))_1$ and the integer points y^2 and y^4



We label as r^{j_1}, r^{j_2} the two rays whose corresponding ray intersections are on F_3 , so that r^{j_1} points to $F_1 \cap F_3$ and the ray intersection of r^{j_2} is closest to $F_2 \cap F_3$; and so r^{j_1} is a corner ray by the statement of Case b. There are $2 \times \binom{k}{2}$ ways to choose r^{j_1}, r^{j_2} from the set $\{r^1, \dots, r^k\}$ with one of them as the corner ray (see Fig. 15). By Lemma 3.11, $\text{aff}(F_3)$ contains a facet of $(C(r^{j_1}, r^{j_2}))_1$. By Lemma 3.10, we have polynomially many choices for $\text{aff}(F_3)$. Once we choose $\text{aff}(F_3)$, we consider the possible choices for y^1, y^2 , which are the integer points on F_1, F_2 , respectively.

In Case b_1 , where $f \notin M(S_3)$, y^1, y^2 are given uniquely by where f is. To see this, we observe a few things. Let y^3 and y^4 be the integer points on F_3 that are closest to $F_1 \cap F_3$. The split with one side going through y^1, y^3 and the other side going through y^2, y^4 contains f . Now consider the family of maximal lattice-free splits with one side going through y^3 and the other side going through y^4 . Observe that since $f \notin M(S_3)$, only one member of this family of splits contains f . This uniquely determines y^1 and y^2 .

In Case b_2 , $P \not\subset M(S_3)$, which implies that there exists a ray r^{j_3} such that r^{j_3} points between y^1 and y^2 . Moreover, since y^1, y^2 have to lie on the lattice plane adjacent to F_3 , we have a unique choice for y^1, y^2 once we choose r^{j_3} from our set of k rays. Now r^{j_3} can be chosen in $O(k)$ ways and so there are $O(k)$ ways to pick y^1, y^2 .

We already know there is a corner ray pointing to $F_1 \cap F_3$. By the statement of Case b, either $P \cap \text{rel int}(F_2) \setminus \mathbb{Z}^2 \neq \emptyset$, in which case we have a corner ray of $M(B)$ pointing to a different vertex, or $P \cap \text{rel int}(F_2) \setminus \mathbb{Z}^2 = \emptyset$. If $M(B)$ has a corner ray pointing to a vertex different from $F_1 \cap F_3$, then we can choose it in $O(k)$ ways, and the triangle is uniquely determined by these two corner rays, $\text{aff}(F_3)$, y^1 , and y^2 .

On the other hand, if $M(B)$ has corner rays pointing only to $F_1 \cap F_3$ (one of which is r^{j_1}), then the facet F_2 has no non-integer ray intersections in its relative interior. Therefore, any possible choice of this facet such that no ray points to $\text{rel int}(F_2) \setminus \mathbb{Z}^2$ will give a triangle that yields the same vector $\gamma(B)$.

Hence, there are only polynomially many possibilities for Case b.

Case c. We now count the vectors $\gamma(B)$ such that $M(B)$ satisfies the conditions of Case c with respect to our set of rays $\{r^1, \dots, r^k\}$. Consider any such $M(B)$. Then there exist r^{j_1}, r^{j_2} such that $p^{j_1}, p^{j_2} \in F_1$ and $\#([p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2) \geq 2$, where p^{j_1}, p^{j_2} are the ray intersections for r^{j_1}, r^{j_2} , respectively. Moreover, p^{j_1} is an integer point on the facet F_3 . There are $2 \times \binom{k}{2}$ ways to choose r^{j_1}, r^{j_2} from the set $\{r^1, \dots, r^k\}$ with r^{j_1} pointing from f to $F_1 \cap F_3$ (see Fig. 16).

We next choose $\text{aff}(F_1)$ as the affine hull of a facet of $(C(r^{j_1}, r^{j_2}))_I$, which can be done because of Lemma 3.11. Since p^{j_1} is the integer point that r^{j_1} points to, one of the facets of $(C(r^{j_1}, r^{j_2}))_I$ that is incident to the vertex p^{j_1} is unbounded and lies on the infinite ray $f + \mathbb{R}_+ r^{j_1}$, while the other facet is bounded. Since F_1 must be bounded, there is a unique choice of $\text{aff}(F_1)$ as the affine hull of the bounded facet of $(C(r^{j_1}, r^{j_2}))_I$ that is incident to the vertex p^{j_1} .

Now we pick the integer points y^2, y^4 where y^2 is the integer point on the facet F_2 of $M(B)$ and y^4 is the integer point in the relative interior of F_3 that is closest to p^{j_1} . This analysis is the same as with Cases b_1 and b_2 . In Case c_1 , these points are uniquely determined by f . In Case c_2 , these are uniquely determined by one of the rays pointing between them. There are $O(k)$ ways of choosing this ray.

The statement of Case c implies that either there is also a corner ray pointing to $F_1 \cap F_2$, or $P \cap \text{rel int}(F_2) \setminus \mathbb{Z}^2 = \emptyset$.

If there is a corner ray pointing to $F_1 \cap F_2$, then the triangle is uniquely determined by the two corner rays, $\text{aff}(F_1)$, y^2 , and y^4 .

On the other hand, if $P \cap \text{rel int}(F_2) \setminus \mathbb{Z}^2 = \emptyset$, then F_2 can be chosen in any possible way such that no ray points to $\text{rel int}(F_2) \setminus \mathbb{Z}^2$. Then the triangle is uniquely determined by r^1 , $\text{aff}(F_1)$, $\text{aff}(F_2)$, y^2 , and y^4 .

Therefore, there are only polynomially many possibilities for Case c.

Case d. We consider Type 2 triangles with a corner ray r^{j_1} pointing from f to $F_1 \cap F_3$. We label the closest integer point in $\text{rel int}(F_3)$ to $F_1 \cap F_3$ as y^3 , and the next closest integer point in $\text{rel int}(F_3)$ as y^4 . Also, since $P \cap (y^3, y^4) \neq \emptyset$, there exists a ray r^{j_3} that points from f through (y^3, y^4) (we use the notation r^{j_3} to remind ourselves that it points to F_3). Moreover, the condition that no two ray intersections on F_3 can contain two (or more) integer points between them implies that the ray intersections on F_3 are contained in the segment $[F_1 \cap F_3, y^4]$. As before, y^1 and y^2 will denote the integer points on the facets F_1 and F_2 .

Case d₁ and Case d₂. For these two cases, there exists a ray r^{j_2} that points from f through (y^1, y^2) to F_1 (for example, in Case d_1 this will be the corner ray pointing from f to $F_1 \cap F_2$). Observe that $[p^{j_1}, p^{j_2}] \cap \mathbb{Z}^2 = \{y^1\}$ and $[p^{j_1}, p^{j_3}] \cap \mathbb{Z}^2 = \{y^3\}$, where y^1 and y^3 lie in (p^{j_1}, p^{j_2}) and (p^{j_1}, p^{j_3}) , respectively. We now count the choices of these triangles.

First pick rays $r^{j_1}, r^{j_2}, r^{j_3}$, for which there are $\binom{k}{3}$ ways to do this. Pick y^1 as a vertex of $(C(r^{j_1}, r^{j_2}))_I$ and pick y^3 as a vertex of $(C(r^{j_1}, r^{j_3}))_I$, utilizing Lemma 3.11 (i). By Lemma 3.10, there are only polynomially many ways to do this.

Claim ε The vector $\gamma(B)$ is uniquely determined by the choices of $r^{j_1}, r^{j_2}, r^{j_3}, y^1$, and y^3 in Case d_1 and Case d_2 .

Proof First note that $[y^2, y^4]$ is necessarily parallel to $[y^1, y^3]$. Therefore, regardless of the choice of y^2, y^4 , the half-space $H_{2,4}$ is already determined by $[y^1, y^3]$. Recall that, by assumption, $f \notin H_{2,4}$. Define the family of splits

$$\mathcal{S} = \left\{ S \in \mathbb{R}^{3 \times 2} \mid \begin{array}{l} y^1, y^3 \in M(S), \text{ and } M(S) \cap \text{int}(H_{2,4}) \neq \emptyset, \\ M(S) \text{ is a maximal lattice-free split} \end{array} \right\}.$$

For any distinct $M(S_1), M(S_2)$ with $S_1, S_2 \in \mathcal{S}$, since they both contain $[y^1, y^3]$, we find that $M(S_1) \cap M(S_2) \setminus H_{2,4} = \emptyset$. Since $f \notin H_{2,4}$, there exists a unique $M(S)$ with $S \in \mathcal{S}$ such that $f \in M(S)$. Therefore, the unique choices for y^2, y^4 are the two points given by $\partial M(S) \cap \partial H_{2,4}$.

Now we show how to choose $M(B)$. The affine hull of facet F_3 is determined by the segment $[y^3, y^4] \subset F_3$, and $\text{aff}(F_1)$ is determined by $[p^{j_1}, y^1] \subset F_1$, where p^{j_1} is the corner ray intersection of r^{j_1} on F_3 . Lastly, $\text{aff}(F_2)$ must be chosen. For Case d_1 , r^{j_2} is chosen as a corner ray pointing from f to $F_1 \cap F_2$, and therefore, $\text{aff}(F_2)$ is determined by the ray intersection p^{j_2} of r^{j_2} on F_1 , and by y^2 , i.e., by the segment $[p^{j_2}, y^2]$. For Case d_2 , any choice of $\text{aff}(F_2)$ such that there are no rays pointing from f to $\text{rel int}(F_2) \setminus \mathbb{Z}^2$ and such that $f \in M(B)$ will yield the same vector $\gamma(B)$, thus, we only need to consider one such triangle. \square

Since the vector $\gamma(B)$ is uniquely determined by these choices, there are only polynomially many possibilities for this case.

Case d_3 . For this case, there exists a corner ray r^{j_2} that points from f to $F_1 \cap F_2$. Since $P \not\subset H_{2,4}$, there also exists a ray r^{j_4} such that it points from f through (y^2, y^4) . Since r^{j_1} is a corner ray pointing from f to $F_1 \cap F_3$ and the ray intersections are contained in $[F_1 \cap F_3, y^4]$, r^{j_4} must be chosen to point from f to F_2 .

We now count triangles of this description. First pick rays $r^{j_1}, r^{j_2}, r^{j_3}, r^{j_4}$ from the set $\{r^1, \dots, r^k\}$. There are at most $\binom{k}{4}$ ways to do this. According to Lemma 3.11 (i), pick y^1 as a vertex of $(C(r^{j_1}, r^{j_2}))_I$, y^2 as a vertex of $(C(r^{j_2}, r^{j_4}))_I$, and y^3 as a vertex of $(C(r^{j_1}, r^{j_3}))_I$. By Lemma 3.10, there are only polynomially many ways to do this. Then y^4 is uniquely determined since y^1, y^2, y^3, y^4 form an area 1 parallelogram. The affine hull of F_3 is uniquely determined, since it runs along $[y^3, y^4]$. Since r^{j_1} is a corner ray pointing to $F_1 \cap F_3$, the choice of y^1 fixes $\text{aff}(F_1)$. Finally, since r^{j_2} is a corner ray pointing to $F_1 \cap F_2$, the choice of y^2 fixes $\text{aff}(F_2)$. Therefore, there are only polynomially many Type 2 triangles satisfying the conditions of this case.

This concludes the proof of the fact that there are only a polynomial (in the binary encoding sizes of f, r^1, \dots, r^k) number of vectors $\gamma(B)$ such that $M(B)$ is a Type 2 triangle satisfying Cases a, b, c and d. \square

3.3 Proof of Proposition 3.1

Proof of Proposition 3.1 Let $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$ using the sets \mathcal{E}_i from Propositions 3.13, 3.14, 3.15 and 3.17. We show that for any $\gamma \in \Delta \setminus \mathcal{E}$, γ is dominated by some $\gamma' \in \Delta$, or γ is a strict convex combination of some $\gamma_1, \gamma_2 \in \Delta$. If $\gamma \notin \Pi \cup \Delta_1 \cup \Delta_2 \cup \Delta_3$, then γ cannot be realized by a maximal lattice-free split or triangle and so $\gamma = \gamma(B)$ for some $B \in \mathbb{R}^{3 \times 2}$ such that $M(B)$ is not a maximal lattice-free convex set. This implies that there exists $B' \in \mathbb{R}^{3 \times 2}$ such that $M(B')$ is a maximal lattice-free convex set containing $M(B)$ and γ is dominated by $\gamma(B')$.

So we consider $\gamma \in \Pi \cup \Delta_1 \cup \Delta_2 \cup \Delta_3$. Observe that $\Pi \cup \Delta_1 \cup \Delta_2 \cup \Delta_3 = \Pi \cup (\Delta_1 \setminus (\Pi \cup \Delta_2)) \cup \Delta_3 \cup (\Delta_2 \setminus (\Delta_3 \cup \Pi))$ and so γ is in one of the sets $\Pi, \Delta_1 \setminus (\Pi \cup \Delta_2), \Delta_3$ or $\Delta_2 \setminus (\Delta_3 \cup \Pi)$. Since $\gamma \notin \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$, we have

that γ is in one of the four sets $(\Pi \setminus \mathcal{E}_0)$, $(\Delta_1 \setminus (\mathcal{E}_1 \cup \Pi \cup \Delta_2))$, $(\Delta_3 \setminus \mathcal{E}_3)$ or $(\Delta_2 \setminus (\mathcal{E}_2 \cup \Delta_3 \cup \Pi))$. Now it follows from Propositions 3.13, 3.14, 3.15 and 3.17 that γ is dominated by some $\gamma' \in \Delta$, or γ is a strict convex combination of some $\gamma^1, \gamma^2 \in \Delta$. Furthermore, the cardinality $\#\mathcal{E}$, being bounded above by the sum of the cardinalities of \mathcal{E}_i , $i = 0, \dots, 3$, is polynomial in the binary encoding sizes of f, r^1, \dots, r^k .

4 Proof of Theorem 1.3

In this section, we will complete the proof of Theorem 1.3. As stated in the introduction, the only result we will need from Sect. 3 is Theorem 3.2. Apart from this, we will utilize the results proved in Sect. 2. We first state the following proposition.

Proposition 4.1 *Let $\|\cdot\|$ be a norm on the space of matrices $\mathbb{R}^{3 \times 2}$. Let \mathcal{B} be a family of matrices in $\mathbb{R}^{3 \times 2}$. If there exists $\varepsilon > 0$ such that $\mathfrak{B}(f, \varepsilon) \subseteq M(B)$ for all $B \in \mathcal{B}$, then there exists a real number M depending only on ε such that $\|B\| < M$ for all $B \in \mathcal{B}$.*

Proof Since $\mathfrak{B}(f, \varepsilon) \subseteq M(B)$, the point $f + \varepsilon b^i \in M(B)$, where b^i is the i -th row of B . Therefore, $b^i \cdot (f + \varepsilon b^i - f) \leq 1$. Therefore, $\|b^i\|_2 \leq \frac{1}{\sqrt{\varepsilon}}$. Since this holds for every row b^i , there exists M depending only on ε such that $\|B\| < M$. \square

We will use the following set to derive a bound on a sequence of matrices to show there exists a convergent subsequence. For any vector $\gamma \in \mathbb{R}_+^k$, define

$$M_\gamma = \text{conv}(\{f\} \cup \{f + \frac{1}{\gamma_j} r^j \mid \gamma_j \neq 0\}) + \text{cone}(\{r^j \mid \gamma_j = 0\}).$$

Observation 4.2 *For all $B \in \mathbb{R}^{3 \times 2}$ we have the inclusion $M_{\psi(B)} \subseteq M(B)$.*

Proof Clearly $f \in M(B)$. Next observe that $f + \frac{1}{\psi_B(r^j)} r^j \in M(B)$ if $\psi_B(r^j) > 0$. Finally, $\psi_B(r^j) = 0$ implies that r^j is in the recession cone of $M(B)$. The claim follows. \square

Theorem 4.3 *Assume that $f \in \mathbb{Q}^2$ and $r^j \in \mathbb{Q}^2$ for all indices $j \in \{1, \dots, k\}$. If $\text{cone}(\{r^1, \dots, r^k\}) = \mathbb{R}^2$, then Δ' has a polynomial (in the binary encoding sizes of f, r^1, \dots, r^k) number of extreme points.*

Proof Consider any extreme point x of Δ' . By Observation 2.5, $x \in \text{cl}(\text{conv}(\Delta))$. By Lemma 2.6, there exists a sequence (a^n) of points from Δ such that (a^n) converges to x . \square

Claim ζ There exists a bounded sequence of matrices $(B_n) \in \mathbb{R}^{3 \times 2}$ such that $\gamma(B_n) = a^n$ and $M(B_n)$ is lattice-free.

Proof Since (a^n) converges to x , there exists $N \in \mathbb{N}$ such that $a_i^n \leq x_i + 1$ for every $n \geq N$ and $i \in \{1, \dots, k\}$. Since (a^n) is a sequence in Δ , there exists a sequence of matrices (B_n) such that $(a^n) = (\gamma(B_n))$ and $M(B_n)$ is lattice-free for all $n \in \mathbb{N}$. Consider the sequence of polyhedra $M_{\gamma(B_n)}$. Let $\varepsilon = \frac{1}{1 + \max_i x_i}$.

By the definition of N , for every $n \geq N$, we have that $\frac{1}{a^n} \geq \varepsilon$. Since the conical hull of the rays r^1, \dots, r^k is \mathbb{R}^2 , this implies that there exists $\bar{\varepsilon}$ such that $\mathfrak{B}(f, \bar{\varepsilon}) \subseteq M_{\gamma(B_n)}$ for all $n \geq N$. By Observation 4.2, $M_{\gamma(B_n)} \subseteq M(B_n)$. Therefore, for every $n \geq N$, $\mathfrak{B}(f, \bar{\varepsilon}) \subseteq M(B_n)$. Proposition 4.1 implies that there exists a real number M depending only on $\bar{\varepsilon}$ such that $\|B_n\| \leq M$ for all $n \geq N$. This implies that (B_n) is a bounded sequence. \square

By the Bolzano–Weierstrass theorem, we can extract a convergent subsequence (\bar{B}_n) converging to a point \bar{B} . The map $B \mapsto \gamma(B)$ is continuous because $\psi_B(r)$ is continuous in B for every fixed r . Therefore, the sequence $(\gamma(\bar{B}_n))$ converges to $\gamma(\bar{B})$. By assumption $(a^n) = (\gamma(B_n))$ converges to x and therefore $\gamma(\bar{B}) = x$. Moreover, since $M(\bar{B}_n)$ is lattice-free for all $n \in \mathbb{N}$, $M(\bar{B})$ is also lattice-free and hence it is a lattice-free triangle or a lattice-free split. Thus, $x = \gamma(\bar{B}) \in \Delta$ since $M(\bar{B})$ is a lattice-free triangle or lattice-free split. Therefore, we have shown that for every extreme point x of Δ' , we have that $x \in \Delta$.

Let \mathcal{E} be the set from Theorem 3.2. Theorem 3.2 implies that the extreme point $x \notin \Delta \setminus \mathcal{E}$. Since we show that $x \in \Delta$, this implies $x \in \mathcal{E}$. Since $\#\mathcal{E}$ is polynomial in the encoding sizes of f, r^1, \dots, r^k , we have shown this property for the number of extreme points Δ' as well. \square

Corollary 4.4 *Assume that $f \in \mathbb{Q}^2$ and $r^j \in \mathbb{Q}^2$ for all indices $j \in \{1, \dots, k\}$. If $\text{cone}(\{r^1, \dots, r^k\}) = \mathbb{R}^2$, then the triangle closure T is a polyhedron with a polynomial (in the binary encoding sizes of f, r^1, \dots, r^k) number of facets.*

Proof Lemma 2.4 and Theorem 4.3 together imply the corollary. \square

We can now finally prove Theorem 1.3.

Proof of Theorem 1.3 If $\text{cone}(\{r^1, \dots, r^k\}) = \mathbb{R}^2$, then Corollary 4.4 gives the result. Otherwise we add rays $r^{k+1}, \dots, r^{k'}$ such that $\text{cone}(\{r^1, \dots, r^k, r^{k+1}, \dots, r^{k'}\}) = \mathbb{R}^2$. Now consider the system (1) with the rays $r^1, \dots, r^{k'}$. We can similarly define the triangle closure T' for this extended system. Given a matrix $B \in \mathbb{R}^{3 \times 2}$, let $\alpha(B) = (\psi_B(r^i))_{i=1}^{k'} \in \mathbb{R}^{k'}$ (we will continue to use $\gamma(B) = (\psi_B(r^i))_{i=1}^k$). So T' is defined as

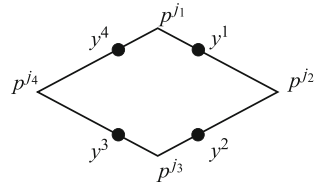
$$T' = \{s \in \mathbb{R}_+^{k'} \mid \alpha(B) \cdot s \geq 1 \text{ for all } B \text{ such that } M(B) \text{ is a lattice-free triangle}\}. \quad \square$$

Claim η We have $T' \cap \{s_{k+1} = 0, \dots, s_{k'} = 0\} = T \times \{0^{k'-k}\}$.

Proof Consider any point $s \in T' \cap \{s_{k+1} = 0, \dots, s_{k'} = 0\}$ and let $s^k = (s_1, \dots, s_k)$ be the truncation of s to the first k coordinates. Consider any $a \in \mathbb{R}^k$ such that $a = \gamma(B)$ for some matrix B where $M(B)$ is a lattice-free triangle. Consider $a' = \alpha(B)$. Clearly, $a'_i = a_i$ for $i \in \{1, \dots, k\}$. Since $a' \cdot s \geq 1$ and $a' \cdot s = a \cdot s^k$, we have that $a \cdot s^k \geq 1$. So, $s \in T \times \{0^{k'-k}\}$.

For the reverse inclusion, consider a point $s \in T \times \{0^{k'-k}\}$ and let $s^k = (s_1, \dots, s_k)$ be the truncation of s to the first k coordinates. Consider any $a' \in \mathbb{R}^{k'}$ such that $a' = \alpha(B)$ for some matrix B where $M(B)$ is a lattice-free triangle. Let $a = \gamma(B)$. As before, $a'_i = a_i$ for $i \in \{1, \dots, k\}$. Since $a \cdot s^k \geq 1$ and $a' \cdot s = a \cdot s^k$, we have that $a' \cdot s \geq 1$. So, $s \in T' \cap \{s_{k+1} = 0, \dots, s_{k'} = 0\}$.

Fig. 17 Example of a quadrilateral for which the ratio condition does *not* hold, i.e., there exists $t > 0$ satisfying (9). Here $\dim \mathcal{N}(\mathcal{B}) \neq 0$



Since $\text{cone}(\{r^1, \dots, r^k, r^{k+1}, \dots, r^{k'}\}) = \mathbb{R}^2$, Corollary 4.4 says that T' is a polyhedron with a polynomial (in the binary encoding sizes of f, r^1, \dots, r^k) number of facets. Since $T' \cap \{s_{k+1} = 0, \dots, s_{k'} = 0\} = T \times \{0^{k'-k}\}$, this shows that T is a polyhedron with a polynomial number of facets. \square

This concludes the part of the paper which deals with the result that the triangle closure is a polyhedron.

5 Proof of Theorems 1.4 and 1.5

We now complete our second result showing that the mixed integer hull $\text{conv}(R_f)$ has only polynomially many facets. We first make a counting argument for quadrilaterals that is similar to the counting arguments in Sect. 3.2. For quadrilaterals, Cornuéjols and Margot [12] defined the *ratio condition* as a necessary and sufficient condition to yield an extreme inequality when all corner rays are present. Suppose $p^{j_1}, p^{j_2}, p^{j_3}, p^{j_4}$ are the corner ray intersections assigned in a counter-clockwise orientation, and y^i is the integer point contained in $[p^{j_i}, p^{j_{i+1}}]$, where we set $j_5 = j_1$. The ratio condition holds if there does not exist a scalar $t > 0$ such that

$$\frac{\|y^i - p^{j_i}\|_2}{\|y^i - p^{j_{i+1}}\|_2} = \begin{cases} t & \text{for } i = 1, 3 \\ \frac{1}{t} & \text{for } i = 2, 4. \end{cases} \tag{9}$$

This is illustrated in Fig. 17.

Proposition 5.1 (Counting quadrilaterals) *There exists a finite set $\mathcal{E}_4 \subseteq \mathbb{R}_+^k$ such that if $\gamma(B) \notin \mathcal{E}_4$ for some $B \in \mathbb{R}^{4 \times 2}$ where $M(B)$ is a maximal lattice-free quadrilateral, then $\gamma(B)$ is not extreme. Moreover, the cardinality of \mathcal{E}_4 is bounded by a polynomial in the binary encoding size of f, r^1, \dots, r^k . Specifically, \mathcal{E}_4 can be chosen as the set of all $\gamma(B)$ such that $M(B)$ is a maximal lattice-free quadrilateral satisfying one of the following:*

Case a. $P \subset \mathbb{Z}^2$.

Case b. $M(B)$ has four distinct corner rays and the ratio condition holds.

Proof Step 1. Suppose that $\gamma(B)$ is extreme and that we are not in Case a. Lemma 3.9 shows that all four corner rays must exist. Suppose, for the sake of contradiction, that the ratio condition does not hold. Lemma 8.1 in Appendix 8 then shows that $\dim \mathcal{N}(\mathcal{B}) \geq 1$. Let $\bar{A} = (\bar{a}^1; \bar{a}^2; \bar{a}^3; \bar{a}^4) \in \mathcal{N}(\mathcal{B}) \setminus \{0\}$. Since $\bar{A} \neq 0$, we have $\bar{a}^i \neq 0$ for some $i = 1, \dots, 4$. Since there are 4 corner rays, $p^j \in F_i$ for some

$j = 1, \dots, k$. Lastly, since $\{y^i\} = Y_i$, Lemma 3.7 shows that for some $\varepsilon > 0$, $\gamma(B)$ is a strict convex combination of $\gamma(B + \varepsilon\bar{A})$ and $\gamma(B - \varepsilon\bar{A})$ and $M(B + \varepsilon\bar{A})$, $M(B - \varepsilon\bar{A})$ are lattice-free quadrilaterals. Therefore, $\gamma(B)$ is not extreme, which is a contradiction. Hence, the ratio condition must hold.

Step 2. We now bound the number of possible vectors in \mathcal{E}_4 .

Case a. Observation 3.12 shows that there is a unique $\gamma \in \mathcal{E}_3$ corresponding to this case.

Case b. $M(B)$ has all four corner rays and the ratio condition holds. By Lemma 8.1, the quadrilateral we construct must be uniquely determined by the choice of four corner rays and choice of four integer points that lie on the facets of the quadrilateral. There are $O(k^4)$ ways to pick four rays $r^{j_1}, r^{j_2}, r^{j_3}, r^{j_4}$ to be corner rays. By Lemma 3.11 (i), the four integer points y^1, y^2, y^3, y^4 are such that y^i is a vertex of $(C(r^{j_i}, r^{j_{i+1}}))_I$, with $i = 1, 2, 3$ and y^4 a vertex of $(C(r^{j_4}, r^{j_1}))_I$. Lemma 3.10 shows that there are polynomially many possibilities for y^1, \dots, y^4 . \square

Proof of Theorem 1.4 We now introduce the set Γ of all vectors $\gamma(B)$ that come from arbitrary (not necessarily maximal) lattice-free polyhedra in \mathbb{R}^2 ,

$$\Gamma = \bigcup_{n \in \mathbb{N}} \{ \gamma(B) \mid B \in \mathbb{R}^{n \times 2} \text{ such that } M(B) \text{ is a lattice-free convex set} \}.$$

Since we consider $B \in \mathbb{R}^{n \times 2}$ for all $n \in \mathbb{N}$, this includes all $\gamma(B)$ such that $M(B)$ is a lattice-free split, triangle, or quadrilateral and all other polyhedra that are lattice-free in \mathbb{R}^2 . It is easy to verify (see Lemma 1.6 and its proof in [12]) that $\text{conv}(R_f)$ is a polyhedron of *blocking type* (see Section 9.2 in [19] for a discussion of blocking polyhedra). In fact, because of the correspondence (4) between valid inequalities for the mixed integer hull and lattice-free sets, one can show that Γ is actually the *blocking polyhedron* of $\text{conv}(R_f)$, i.e., $\text{conv}(R_f) = \{s \in \mathbb{R}_+^k \mid \gamma \cdot s \geq 1 \text{ for all } \gamma \in \Gamma\}$ and $\Gamma = \{\gamma \in \mathbb{R}_+^k \mid \gamma \cdot s \geq 1 \text{ for all } s \in \text{conv}(R_f)\}$. Hence, by Theorem 9.2(iv) in [19], the facets of $\text{conv}(R_f)$ are given by $\gamma \cdot s \geq 1$ where γ is an extreme point of Γ . So we need to enumerate the extreme points of Γ . Moreover, if γ is an extreme point of Γ , then there does not exist $\gamma' \in \Gamma$ such that γ is dominated by γ' . By Propositions 3.13, 3.14, 3.15, 3.17 and 5.1, the extreme points of Γ can only be in $\mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4$, whose cardinalities are bounded above by a polynomial in the binary encoding sizes of f, r^1, \dots, r^k . \square

Proof of Theorem 1.5 As established in the proof of Theorem 1.4, we only need to search in the set $\mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4$ to find the facet-defining inequalities. The conditions defining these five sets and the counting arguments presented in Sect. 3 and Proposition 5.1 can be converted into an algorithm for enumerating all the points in $\mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4$. This relies on the algorithm by Hartmann [15] for computing the facets and vertices of integer hulls, which runs in polynomial time in fixed dimension (cf. Lemma 3.10). Thus we generate a set of valid inequalities (of polynomial cardinality) that is a superset of all the facets. We can then use standard LP techniques to select the facet-defining ones from these. \square

6 Conclusion

We conclude this paper with a discussion of some interesting open problems.

Generalized triangle closures A drawback of the triangle closure result presented in this paper is that it only applies to a system with two integer variables. Here is one way to generalize to general mixed integer linear programs. Consider the polyhedron $C = \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : Px + Qy \leq d\}$ for some matrices $P \in \mathbb{R}^{m \times p}$, $Q \in \mathbb{R}^{m \times q}$. We are interested in the mixed integer hull $C_I = \text{conv}(C \cap (\mathbb{Z}^p \times \mathbb{R}^q))$. We define the generalized triangle closure in the following way. Consider any two-dimensional lattice subspace of \mathbb{Z}^p and consider a lattice-free triangle in this subspace. Let \mathcal{T} be the family of all such triangles from all possible two-dimensional lattice subspaces. For any $T \in \mathcal{T}$, let b^1, \dots, b^p be a lattice basis for \mathbb{Z}^p such that b^1 and b^2 are a basis for the lattice subspace containing T . Let $L(T)$ be the linear subspace spanned by b^3, \dots, b^p . Then $R(T) = (T \oplus L(T)) \times \mathbb{R}^q$ is a polyhedron which contains no point from $\mathbb{Z}^p \times \mathbb{R}^q$ in its interior. We define the *generalized triangle closure* as

$$\bigcap_{T \in \mathcal{T}} \text{conv}(C \setminus \text{int}(R(T))).$$

We would like to show that the generalized triangle closure is also a polyhedron.

Quadrilateral closures Even for the case $m = 2$, one can ask about the quadrilateral closure. If one considers this to be the intersection of inequalities derived from all possible lattice-free quadrilaterals (and not just the maximal ones), then this convex set can be seen to be the same as the mixed integer hull $\text{conv}(R_f)$, since every maximal lattice-free triangle and split can be arbitrarily well approximated by a quadrilateral (but not necessarily a maximal one). Of course, $\text{conv}(R_f)$ is known to be a polyhedron, and so the question for the quadrilateral closure becomes interesting only if we restrict ourselves to the maximal quadrilaterals. We conjecture that this is also a polyhedron, but it does not seem to be an immediate corollary of the results of this paper.

On a related note, we mention that in this paper, we included non-maximal triangles to define the triangle closure. However, one can see that if we restrict ourselves to only maximal triangles, we would obtain the same convex set. This is because an inequality derived from a non-maximal triangle is equal to or dominated by one derived from a maximal triangle or a split, and a split can be obtained as the limit of maximal triangles. In this respect, the quadrilateral closure differs from the triangle closure: an inequality derived from a non-maximal quadrilateral may not be equal to or dominated by an inequality from a maximal quadrilateral or a limit of such inequalities.

Higher dimensions Many of the tools described in Sect. 3.1 readily extend to $m \geq 3$. This can be used to study the extremality of inequalities arising from maximal lattice-free convex sets in higher dimension. Unfortunately, it is unclear what results can be obtained in higher dimensions due to the difficult task of first classifying all maximal lattice-free convex sets in higher dimensions. Such a classification for $m = 3$ is only partially known [4] while even less is known for $m > 3$. That said, the tools given

in this paper may be found useful for studying specific classes of lattice-free convex sets in higher dimensions such as simplices or cross-polytopes. Studying such classes of inequalities may produce stronger valid inequalities simply because more rows of the simplex tableau are utilized.

7 Appendix: Uniqueness of a triangle defined by 3 corner rays and a point on the relative interior of each facet

Proposition 7.1 *Any triangle defined by 3 corner rays and 3 points (one on the relative interior of each facet) is uniquely defined.*

Proof The space of triangles with these three corner rays and 3 points is exactly the tilting space of any such triangle satisfying this. For convenience we define $\bar{y}^i := y^i - f$ and $\bar{p}^i := p^i - f$, where p^i are the ray intersections. Then $\bar{p}^i = \frac{1}{\psi_B(r^i)}r^i$.

We want to show that the solution to the following systems of equations is unique.

$$\begin{matrix} a^1 \cdot \bar{y}^1 = 1 \\ a^1 \cdot \bar{p}^2 = a^2 \cdot \bar{p}^2 \\ a^2 \cdot \bar{y}^2 = 1 \\ a^2 \cdot \bar{p}^3 = a^3 \cdot \bar{p}^3 \\ a^3 \cdot \bar{y}^3 = 1 \\ a^3 \cdot \bar{p}^1 = a^1 \cdot \bar{p}^1 \end{matrix} \Rightarrow \begin{bmatrix} \bar{y}^1 & & & & & \\ \bar{p}^2 & -\bar{p}^2 & & & & \\ & \bar{y}^2 & & & & \\ & \bar{p}^3 & -\bar{p}^3 & & & \\ & & \bar{y}^3 & & & \\ -\bar{p}^1 & & & \bar{p}^1 & & \end{bmatrix} \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

We then write this down as a matrix equation where every vector in the matrix is a row vector of size 2, therefore we have a 6×6 matrix. We will analyze the determinant of the matrix.

Since the points $\bar{y}^1, \bar{y}^2, \bar{y}^3$ are on the interior of each facet, they can be written as convex combinations of $\bar{p}^1, \bar{p}^2, \bar{p}^3$.

$$\begin{aligned} \bar{y}^1 &= \frac{1}{\alpha'}\bar{p}^1 + \frac{\alpha}{\alpha'}\bar{p}^2 & \bar{p}^1 &= \alpha'\bar{y}^1 - \alpha\bar{p}^2 \\ \bar{y}^2 &= \frac{1}{\beta'}\bar{p}^2 + \frac{\beta}{\beta'}\bar{p}^3 & \bar{p}^2 &= \beta'\bar{y}^2 - \beta\bar{p}^3 \\ \bar{y}^3 &= \frac{1}{\gamma'}\bar{p}^3 + \frac{\gamma}{\gamma'}\bar{p}^1 & \bar{p}^3 &= \gamma'\bar{y}^3 - \gamma\bar{p}^1 \end{aligned}$$

Therefore, we can perform row reduction on the last row. Just tracking the last row, we have

$$[-\bar{p}^1 \ 0 \ \bar{p}^1] \rightarrow [0 \ \alpha\bar{p}^2 \ \bar{p}^1] \rightarrow [0 \ 0 \ \bar{p}^1 - \alpha\beta\bar{p}^3].$$

This matrix now has an upper block triangular form, and the determinant is easily computed as

$$\det(\bar{y}^1; \bar{p}^2) \det(\bar{y}^2; \bar{p}^3) \det(\bar{y}^3; \bar{p}^1 - \alpha\beta\bar{p}^3).$$

and note that $\bar{Y} = \bar{X}S$. Since y^1, \dots, y^4 form a parallelogram by Theorem 1.2, $\bar{y}^1 + \bar{y}^3 = \bar{y}^2 + \bar{y}^4$, or equivalently, $\bar{Y}u = 0$ where $u = (1; -1; 1; -1)$. This implies that $\bar{Y}u = \bar{X}Su = 0$. Since we assume that it is not true that $\alpha = \frac{1}{\beta} = \gamma = \frac{1}{\delta}$, it follows that $Su \neq 0$. Therefore, $Su \in \ker(\bar{X}) \setminus \{0\}$, where \ker denotes the kernel. Some simple linear algebra, made more explicit in [12, Lemma 3.5, proof of Theorem 3.10], shows that $\ker(\bar{X}) \subseteq \text{im}(S)$, where $\text{im}(S)$ denotes the column space, and that

$$\text{rank}(\bar{X}S) = \text{rank}(S) - \dim(\ker(\bar{X}) \cap \text{im}(S))$$

from which it follows that $\text{rank}(S) = 4$. Therefore, $\det(S) = d$ for some $d \neq 0$, where $\det(S) = \frac{1 - \alpha\beta\gamma\delta}{\alpha'\beta'\gamma'\delta'}$. Therefore

$$\alpha\beta\gamma\delta = 1 - d\alpha'\beta'\gamma'\delta'$$

and since $\alpha', \beta', \gamma', \delta' \neq 0$, we conclude that $\alpha\beta\gamma\delta \neq 1$.

We have shown that the ratio condition holds if and only if $\dim \mathcal{N}(\mathcal{P}) = 0$. Lastly, note that the set $\{M(B + \bar{A}) \mid \bar{A} \in \mathcal{N}(\mathcal{P})\}$ is the set of all quadrilaterals defined by the four corner rays $r^{j_1}, r^{j_2}, r^{j_3}, r^{j_4}$ and the four integer points y^1, y^2, y^3, y^4 . Hence, $\dim \mathcal{N}(\mathcal{P}) = 0$ if and only if $M(B)$ is the unique quadrilateral defined by these corner rays and integer points.

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