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SECOND-ORDER PARAMETER-FREE DUALITY MODELS IN SEMI-INFINITE MINMAX FRACTIONAL PROGRAMMING

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□ *In this article, we construct and discuss several second-order parameter-free duality models for a semi-infinite minmax fractional programming problem and establish appropriate duality results under various generalized second-order $(\mathscr{F}, \beta, \phi, \rho, \theta)$-univexity assumptions.*

Keywords Discrete minmax fractional programming; Duality theorems; Generalized $(\mathscr{F}, \beta, \phi, \rho, \theta)$-univex functions; Infinitely many equality and inequality constraints; Parameter-free duality models; Semi-infinite programming.

Mathematics Subject Classification 49N15; 90C26; 90C30; 90C32; 90C34; 90C47.

1. INTRODUCTION

In this article, we state and prove numerous second-order nonparametric duality results under various generalized $(\mathscr{F}, \beta, \phi, \rho, \theta)$-sounivexity assumptions for the following semi-infinite discrete minmax fractional programming problem:

$$(P) \quad \text{Minimize } \max_{1 \le i \le p} \frac{f_i(x)}{g_i(x)}$$

subject to

$$G_j(x, t) \le 0 \quad \text{for all } t \in T_j, \;\; j \in \underline{q},$$
$$H_k(x, s) = 0 \quad \text{for all } s \in S_k, \;\; k \in \underline{r},$$
$$x \in X,$$

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where p, q, and r are positive integers, X is a nonempty open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), for each $j \in \underline{q} \equiv \{1, 2, \ldots, q\}$ and $k \in \underline{r}$, T_j, and S_k are compact subsets of complete metric spaces, for each $i \in \underline{p}$, f_i and g_i are real-valued functions defined on X, for each $j \in \underline{q}$, $z \to G_j(z, t)$ is a real-valued function defined on X for all $t \in T_j$, for each $k \in \underline{r}$, $z \to H_k(z, s)$ is a real-valued function defined on X for all $s \in S_k$, for each $j \in \underline{q}$ and $k \in \underline{r}$, $t \to G_j(x, t)$ and $s \to H_k(x, s)$ are continuous real-valued functions defined, respectively, on T_j and S_k for all $x \in X$, and for each $i \in \underline{p}$, $g_i(x) > 0$ for all x satisfying the constraints of (P).

The present study is essentially a continuation of the investigation initiated in the companion articles [73, 74]. In [73], some information about discrete minmax fractional programming is presented, the current status of semi-infinite programming is briefly discussed and numerous key references are cited, an overview of the concept of invexity and some of its extensions is given, and a fairly large number of sets of global nonparametric sufficient optimality results under various generalized (η, ρ)-invexity assumptions are established. In [74], a number of first-order nonparametric duality models are formulated and various weak, strong, and strict converse duality theorems are proved under appropriate generalized (η, ρ)-invexity conditions. For the necessary background material and preliminaries, the reader is referred to [73, 74]. Here we shall make use of some much broader classes of generalized convex functions to construct several second-order nonparametric duality models for (P) and prove appropriate duality theorems.

Second-order duality for a conventional nonlinear programming problem of the form

$$(P_0) \quad \text{Minimize } f(x) \text{ subject to } g_i(x) \leq 0, \quad i \in \underline{m}, \quad x \in \mathbb{R}^n,$$

where f and g_i, $i \in \underline{m}$, are real-valued functions defined on $\mathbb{R}^n$, was initially considered by Mangasarian [44]. The idea underlying his approach to constructing a second-order dual problem was based on taking linear and quadratic approximations of the objective and constraint functions about an arbitrary but fixed point, forming the Wolfe dual of the approximated problem, and then letting the fixed point to vary. More specifically, he formulated the following second-order dual problem for (P_0):

$$(D_0) \quad \text{Maximize } f(y) + \sum_{i=1}^{m} u_i g_i(y) - \frac{1}{2}\left\langle z, \left[\nabla^2 f(y) + \sum_{i=1}^{m} u_i \nabla^2 g_i(y)\right] z\right\rangle$$

subject to

$$\nabla f(y) + \sum_{i=1}^{m} u_i \nabla g_i(y) + \left[\nabla^2 f(y) + \sum_{i=1}^{m} u_i \nabla^2 g_i(y)\right] z = 0,$$
$$y, z \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad u \geq 0,$$

where $\nabla F(y)$ and $\nabla^2 F(y)$ are, respectively, the gradient and Hessian of the function $F : \mathbb{R}^n \to \mathbb{R}$ evaluated at y, and $\langle a, b \rangle$ denotes the inner product of the vectors a and b. Imposing somewhat complicated conditions on f, g_i, $i \in \underline{m}$, and z, he proved weak, strong, and converse duality theorems for (P_0) and (D_0).

Reconsidering Mangasarian's second-order dual problem, Mond [49] established some duality results under relatively simpler conditions involving a certain second-order generalization of the concept of convexity, pointed out some possible computational advantages of second-order duality results, and also studied a pair of second-order symmetric dual problems. Subsequently, Mond's original notion of second-order convexity was generalized by other authors in different ways and utilized for establishing various second-order duality results for several classes of nonlinear programming problems. For brief accounts of the evolution of these generalized second-order convexity concepts, the reader is referred to [3, 14, 33, 52, 53], and for more information about second- and higher-order duality results, the reader may consult [1–11, 13, 14, 16–31, 33–40, 43–50, 52–54, 56–71, 75, 76].

Although presently various second-order duality results exist in the related literature for several classes of mathematical programming problems with a finite number of constraints, so far no such results are available for any kind of **semi-infinite** minmax fractional programming problems. To the best of our knowledge, all the second-order nonparametric duality results established in this article are new in the area of **semi-infinite programming**.

The rest of this article is organized as follows. In Section 2, we recall a few definitions and auxiliary results that will be needed in the sequel. In Section 3, we consider two second-order nonparametric duality models with simple constraint structures and establish duality under appropriate $(\mathcal{F}, \beta, \phi, \rho, \theta)$-sounivexity hypotheses. In Section 4, we present two parameter-free duality models with relatively more flexible constraint structures for which duality can be proved under a greater variety of generalized $(\mathcal{F}, \beta, \phi, \rho, \theta)$-sounivexity conditions. We continue our discussion of second-order duality in Section 5 where we use a certain partitioning scheme and construct two generalized second-order nonparametric duality models and obtain several duality results under various generalized $(\mathcal{F}, \beta, \phi, \rho, \theta)$-sounivexity assumptions. Finally, in Section 6 we summarize our main results and also point out some further research opportunities arising from certain modifications of the principal problem model considered in this article.

Evidently, all the duality results established in this article can easily be modified and restated for each one of the following seven classes of nonlinear programming problems, which are special cases of (P):

$$(P1) \quad \underset{x \in \mathbb{F}}{\text{Minimize}} \; \frac{f_1(x)}{g_1(x)};$$

$(P2)$ $\underset{x\in\mathbb{F}}{\text{Minimize}}\ \max_{1\leq i\leq p} f_i(x);$

$(P3)$ $\underset{x\in\mathbb{F}}{\text{Minimize}}\ f_1(x),$

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (P), that is,

$$\mathbb{F} = \{x \in X : G_j(x, t) \leq 0 \text{ for all } t \in T_j, j \in \underline{q},$$
$$H_k(x, s) = 0 \text{ for all } s \in S_k, k \in \underline{r}\};$$

$(P4)$ Minimize $\max_{1\leq i\leq p} \dfrac{f_i(x)}{g_i(x)}$

subject to

$$\widetilde{G}_j(x) \leq 0, \quad j \in \underline{q}, \quad \widetilde{H}_k(x) = 0, \quad k \in \underline{r}, \quad x \in X,$$

where f_i and g_i, $i \in \underline{p}$, are as defined in the description of (P), and $\tilde{G}_j$, $j \in \underline{q}$, and $\widetilde{H}_k$, $k \in \underline{r}$, are real-valued functions defined on X;

$(P5)$ $\underset{x\in\mathbb{G}}{\text{Minimize}}\ \dfrac{f_1(x)}{g_1(x)};$

$(P6)$ $\underset{x\in\mathbb{G}}{\text{Minimize}}\ \max_{1\leq i\leq p} f_i(x);$

$(P7)$ $\underset{x\in\mathbb{G}}{\text{Minimize}}\ f_1(x),$

where $\mathbb{G}$ is the feasible set of $(P4)$, that is,

$$\mathbb{G} = \{x \in X : \widetilde{G}_j(x) \leq 0, j \in \underline{q}, \widetilde{H}_k(x) = 0, k \in \underline{r}\}.$$

Since in most cases these results can easily be altered and rephrased for each one of the above seven problems, we shall not state them explicitly.

2. PRELIMINARIES

In this section, we recall, for convenience of reference, the definitions of certain classes of generalized convex functions which will be needed in the sequel. For a brief discussion of the origins and predecessors of these functions as well as numerous relevant references, the reader is referred to [73].

Definition 2.1. Let f be a differentiable real-valued function defined on $\mathbb{R}^n$. Then f is said to be η-invex (invex with respect to η) at y if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$,

$$f(x) - f(y) \geq \langle \nabla f(y), \eta(x, y)\rangle;$$

f is said to be η-invex on $\mathbb{R}^n$ if the above inequality holds for all $x, y \in \mathbb{R}^n$.

From this definition it is clear that every differentiable real-valued convex function is invex with $\eta(x,y) = x - y$. This generalization of the concept of convexity was originally proposed by Hanson [32], who showed that for a nonlinear programming problem of the form

$$\text{Minimize } f(x) \text{ subject to } g_i(x) \leq 0, \quad i \in \underline{m}, \quad x \in \mathbb{R}^n,$$

where the differentiable functions $f, g_i : \mathbb{R}^n \to \mathbb{R}, i \in \underline{m}$, are invex with respect to the same function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the Karush-Kuhn-Tucker necessary optimality conditions are also sufficient. The term *invex* (for *in*variant con*vex*) was coined by Craven [15] to signify the fact that the invexity property, unlike convexity, remains invariant under bijective coordinate transformations.

In a similar manner, one can readily define η-pseudoinvex and η-quasiinvex functions as generalizations of differentiable pseudoconvex and quasiconvex functions.

The notion of invexity has been generalized in several directions. For our present purposes, we shall need two simple extensions of invexity, namely, ρ-invexity and $\mathscr{F}$-convexity which were originally defined in [34, 41], respectively.

Let η be a function from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$, and let h be a differentiable real-valued function defined on $\mathbb{R}^n$.

Definition 2.2. The function h is said to be (η, ρ)-invex at x^* if there exists $\rho \in \mathbb{R}$ such that for each $x \in \mathbb{R}^n$,

$$h(x) - h(x^*) \geq \langle \nabla h(x^*), \eta(x, x^*) \rangle + \rho \|x - x^*\|^2.$$

An $\mathscr{F}$-convex function is defined in terms of a sublinear function, that is, a function that is subadditive and positively homogeneous.

Definition 2.3. A function $\mathscr{F} : \mathbb{R}^n \to \mathbb{R}$ is said to be *sublinear*(*superlinear*) if $\mathscr{F}(x + y) \leq (\geq) \mathscr{F}(x) + \mathscr{F}(y)$ for all $x, y \in \mathbb{R}^n$, and $\mathscr{F}(ax) = a\mathscr{F}(x)$ for all $x \in \mathbb{R}^n$ and $a \in \mathbb{R}_+ \equiv [0, \infty)$.

Now combining the definitions of $\mathscr{F}$-convex and (η, ρ)-invex functions, we can define $(\mathscr{F}, \rho)$-convex, $(\mathscr{F}, \rho)$-pseudoconvex, and $(\mathscr{F}, \rho)$-quasiconvex functions.

Let g be a differentiable real-valued function defined on $\mathbb{R}^n$, and assume that for each $x, y \in \mathbb{R}^n$, the function $\mathscr{F}(x, y; \cdot) : \mathbb{R}^n \to \mathbb{R}$ is sublinear.

Definition 2.4. The function g is said to be $(\mathscr{F}, \rho)$-*convex* at y if there exists a real number ρ such that for each $x \in \mathbb{R}^n$,

$$g(x) - g(y) \geq \mathscr{F}(x, y; \nabla g(y)) + \rho \|x - y\|^2.$$

Definition 2.5. The function g is said to be $(\mathcal{F}, \rho)$-*pseudoconvex* at y if there exists a real number ρ such that for each $x \in \mathbb{R}^n$,

$$\mathcal{F}(x, y; \nabla g(y)) \geq -\rho \|x - y\|^2 \Rightarrow g(x) \geq g(y).$$

Definition 2.6. The function g is said to be $(\mathcal{F}, \rho)$-*quasiconvex* at y if there exists a real number ρ such that for each $x \in \mathbb{R}^n$,

$$g(x) \leq g(y) \Rightarrow \mathcal{F}(x, y; \nabla g(y)) \leq -\rho \|x - y\|^2.$$

The foregoing classes of generalized convex functions have been utilized for establishing numerous sets of sufficient optimality conditions and a great variety of duality results for several categories of static and dynamic optimization problems. For recent surveys and syntheses of these results, the reader is referred to [42, 55].

Another significant generalization of the notion of invexity, called *univexity*, which subsumes a number of previously proposed types of generalized convex functions, was proposed in [12]. We recall the definitions of univex, pseudounivex, and quasiunivex functions.

Let h be a differentiable real-valued function defined on $\mathbb{R}^n$, let η be a function from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$, let Φ be a real-valued function defined on $\mathbb{R}$, and let b be a function from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}_+ \backslash \{0\} \equiv (0, \infty)$.

Definition 2.7. The function h is said to be *univex* at y with respect to η, Φ, and b if for each $x \in \mathbb{R}^n$,

$$b(x, y)\Phi(h(x) - h(y)) \geq \langle \nabla h(y), \eta(x, y) \rangle.$$

Definition 2.8. The function h is said to be *pseudounivex* at y with respect to η, Φ, and b if for each $x \in \mathbb{R}^n$,

$$\langle \nabla h(y), \eta(x, y) \rangle \geq 0 \Rightarrow b(x, y)\Phi(h(x) - h(y)) \geq 0.$$

Definition 2.9. The function h is said to be *quasiunivex* at y with respect to η, Φ, and b if for each $x \in \mathbb{R}^n$,

$$b(x, y)\Phi(h(x) - h(y)) \leq 0 \Rightarrow \langle \nabla h(y), \eta(x, y) \rangle \leq 0.$$

The concept of second-order convexity generalizing that of convexity, as mentioned earlier, was originally proposed by Mond [49]. Subsequently, this concept was extended to the class of invex functions by Hanson [33] who demonstrated its use in establishing some duality relations in nonlinear programming. Following are slightly modified versions of the classes of second-order invex functions introduced in [33].

Let f be a twice differentiable real-valued function defined on $\mathbb{R}^n$.

Definition 2.10. The function f is said to be second-order η-invex (invex with respect to η) at x^* if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$,

$$f(x) - f(x^*) \geq \langle \nabla f(x^*) + \nabla^2 f(x^*)z, \eta(x, x^*)\rangle - \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle.$$

Definition 2.11. The function f is said to be second-order η-pseudoinvex at x^* if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n (x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \eta(x, x^*)\rangle \geq 0 \Rightarrow f(x) \geq f(x^*) - \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle.$$

Definition 2.12. The function f is said to be second-order η-quasiinvex at x^* if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$,

$$f(x) \leq f(x^*) - \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle \Rightarrow \langle \nabla f(x^*) + \nabla^2 f(x^*)z, \eta(x, x^*)\rangle \leq 0.$$

Finally, we are in a position to give our definitions of generalized second-order $(\mathcal{F}, \beta, \phi, \rho, \theta)$-univex functions. They are formulated by combining Definitions 2.1–2.12. We shall use the word **sounivex**, for second-order **univex**.

Let $x^* \in X$ and assume that the function $f : X \to \mathbb{R}$ is twice differentiable at x^*.

Definition 2.13. The function f is said to be *(strictly)* $(\mathcal{F}, \beta, \phi, \rho, \theta)$-*sounivex* at x^* if there exist functions $\beta : X \times X \to \mathbb{R}_+ \backslash \{0\}, \phi : \mathbb{R} \to \mathbb{R}, \rho : X \times X \to \mathbb{R}, \theta : X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{F}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X (x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\begin{aligned}\phi\left(f(x) - f(x^*) + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle\right) &(>) \\ \geq \mathcal{F}(x, x^*; \beta(x, x^*)[\nabla f(x^*) + \nabla^2 f(x^*)z]) &+ \rho(x, x^*)\|\theta(x, x^*)\|^2.\end{aligned}$$

Definition 2.14. The function f is said to be *(strictly)* $(\mathcal{F}, \beta, \phi, \rho, \theta)$-*pseudosounivex* at x^* if there exist functions β: $X \times X \to \mathbb{R}_+ \backslash \{0\}, \phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta : X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{F}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X (x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\begin{aligned}&\mathcal{F}(x, x^*; \beta(x, x^*)[\nabla f(x^*) + \nabla^2 f(x^*)z]) \geq -\rho(x, x^*)\|\theta(x, x^*)\|^2 \\ &\Rightarrow \phi\left(f(x) - f(x^*) + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle\right) (>) \geq 0.\end{aligned}$$

Definition 2.15. The function f is said to be *(prestrictly)* $(\mathcal{F}, \beta, \phi, \rho, \theta)$-*quasisounivex* at x^* if there exist functions $\beta : X \times X \to \mathbb{R}_+ \backslash \{0\}$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta : X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{F}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X$ and $z \in \mathbb{R}^n$,

$$\phi\left(f(x) - f(x^*) + \frac{1}{2}\langle z, \nabla^2 f(x^*) z\rangle\right) (<) \leq 0$$
$$\Rightarrow \mathcal{F}\big(x, x^*; \beta(x, x^*)[\nabla f(x^*) + \nabla^2 f(x^*) z]\big) \leq -\rho(x, x^*)\|\theta(x, x^*)\|^2.$$

From the above definitions it is clear that if f is $(\mathcal{F}, \beta, \phi, \rho, \theta)$-sounivex at x^*, then it is both $(\mathcal{F}, \beta, \phi, \rho, \theta)$-pseudosounivex and $(\mathcal{F}, \beta, \phi, \rho, \theta)$-quasisounivex at x^*, if f is $(\mathcal{F}, \beta, \phi, \rho, \theta)$-quasisounivex at x^*, then it is prestrictly $(\mathcal{F}, \beta, \phi, \rho, \theta)$-quasisounivex at x^*, and if f is strictly $(\mathcal{F}, \beta, \phi, \rho, \theta)$-pseudosounivex at x^*, then it is $(\mathcal{F}, \beta, \phi, \rho, \theta)$-quasisounivex at x^*.

In the proofs of the duality theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. For example, $(\mathcal{F}, \beta, \phi, \rho, \theta)$-quasisounivexity can be defined in the following equivalent way:

f is said to be $(\mathcal{F}, \beta, \phi, \rho, \theta)$-quasisounivex at x^* if there exist functions $\beta : X \times X \to \mathbb{R}_+ \backslash \{0\}$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta : X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{F}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X$ and $z \in \mathbb{R}^n$,

$$\mathcal{F}(x, x^*; \beta(x, x^*)[\nabla f(x^*) + \nabla^2 f(x^*) z]) > -\rho(x, x^*)\|\theta(x, x^*)\|^2$$
$$\Rightarrow \phi(f(x) - f(x^*) + \frac{1}{2}\langle z, \nabla^2 f(x^*) z\rangle) > 0.$$

Needless to say that the new classes of generalized convex functions characterized in Definitions 2.13–2.15 contain a variety of special cases that can easily be identified by appropriate choices of the functions $\mathcal{F}$, β, ϕ, ρ, and θ.

We conclude this section by recalling a set of nonparametric necessary optimality conditions for (P).

Theorem 2.1 ([73]). *Let $x^* \in \mathbb{F}$, let the functions f_i and g_i, $i \in \underline{p}$, be continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $\xi \to G_j(\xi, t)$ be continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $\xi \to H_k(\xi, s)$ be continuously differentiable at x^* for all $s \in S_k$. If x^* is an optimal solution of (P), if the generalized Abadie constraint qualification holds at x^*, and if the set cone$\{\nabla G_j(x^*, t) : t \in \widehat{T}_j(x^*), j \in \underline{q}\} +$ span$\{\nabla H_k(x^*, s) : s \in S_k, k \in \underline{r}\}$ is closed, then there exist $u^* \in U$ and integers ν_0 and ν, with $0 \leq \nu_0 \leq \nu \leq n + 1$, such that there exist ν_0 indices j_m, with $1 \leq j_m \leq q$, together with ν_0 points $t^m \in$*

$\widehat{T}_{j_m}(x^*)$, $m \in \underline{v_0}$, $v - v_0$ *indices* k_m, *with* $1 \leq k_m \leq r$, *together with* $v - v_0$ *points* $s^m \in S_{k_m}$ *for* $m \in \underline{v}\backslash\underline{v_0}$, *and* v *real numbers* v_m^* *with* $v_m^* > 0$ *for* $m \in \underline{v_0}$, *with the property that*

$$\sum_{i=1}^{p} u_i^*[\Gamma(x^*, u^*)\nabla f_i(x^*) - \Phi(x^*, u^*)\nabla g_i(x^*)]$$

$$+ \sum_{m=1}^{v_0} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m=v_0+1}^{v} v_m^* \nabla H_{k_m}(x^*, s^m) = 0,$$

$$u_i^*[\Gamma(x^*, u^*)f_i(x^*) - \Phi(x^*, u^*)g_i(x^*)] = 0, \quad i \in \underline{p},$$

$$\max_{1\leq i\leq p} \frac{f_i(x^*)}{g_i(x^*)} = \frac{\Phi(x^*, u^*)}{\Gamma(x^*, u^*)},$$

where $U = \{u \in \mathbb{R}^p : u \geq 0, \sum_{i=1}^{p} u_i = 1\}$, $\Phi(x^*, u^*) = \sum_{i=1}^{p} u_i^* f_i(x^*)$, $\Gamma(x^*, u^*) = \sum_{i=1}^{p} u_i^* g_i(x^*)$, *and* $\underline{v}\backslash\underline{v_0}$ *is the complement of the set* $\underline{v_0}$ *relative to the set* $\underline{v}$.

We shall call x a *normal* feasible solution of (P) if x satisfies all the constraints of (P), if the generalized Abadie constraint qualification holds at x, and if the set $cone\{\nabla G_j(x, t) : t \in \widehat{T}_j(x), j \in \underline{q}\} + span\{\nabla H_k(x, s) : s \in S_k, k \in \underline{r}\}$ is closed. The form and contents of the necessary optimality conditions given in the above theorem provide clear guidelines for formulating numerous Wolfe- and Mond-Weir-type second-order duality models for (P). The rest of this article is devoted to investigating various nonparametric duality results for (P). These duality results are based on Theorem 2.1 and the nonparametric sufficiency and duality results discussed in [73, 74].

In the remainder of this article, we shall assume that the functions f_i, g_i, $i \in \underline{p}$, $\xi \to G_j(\xi, t)$, and $\xi \to H_k(\xi, s)$ are twice continuously differentiable on $\overline{X}$ for all $t \in T_j$, $j \in \underline{q}$, and all $s \in S_k$, $k \in \underline{r}$.

3. DUALITY MODEL I

In this section, we consider two parameter-free dual problems for (P) with relatively simple constraint structures and prove weak, strong, and strict converse duality theorems under appropriate $(\mathcal{F}, \beta, \phi, \rho, \theta)$-sounivexity conditions. More general duality models and results for (P) will be discussed in the subsequent sections.

Let

$$\mathbb{H} = \{(y, z, u, v, \nu, \nu_0, J_{\nu_0}, K_{\nu\backslash\nu_0}, \bar{t}, \bar{s}) : y \in X; z \in \mathbb{R}^n; u \in U;$$

$$0 \leq \nu_0 \leq \nu \leq n+1; v \in \mathbb{R}^{\nu}, v_i > 0, 1 \leq i \leq \nu_0;$$

$$J_{\nu_0} = (j_1, j_2, \ldots, j_{\nu_0}), 1 \leq j_i \leq q; K_{\nu\backslash\nu_0} = (k_{\nu_0+1}, \ldots, k_\nu), 1 \leq k_i \leq r;$$
$$\bar{t} = (t^1, t^2, \ldots, t^{\nu_0}), t^i \in T_{j_i}; \bar{s} = (s^{\nu_0+1}, \ldots, s^\nu), s^i \in S_{k_i}\}.$$

Consider the following two problems:

$$(DI) \qquad \sup_{(y,z,u,v,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in \mathbb{H}} \frac{\sum_{i=1}^{p} u_i f_i(y) + \sum_{m=1}^{\nu_0} v_m G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(y, s^m)}{\sum_{i=1}^{p} u_i g_i(y)}$$

subject to

$$\Gamma(y, u)\Big[\sum_{i=1}^{p} u_i \nabla f_i(y) + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y, s^m)\Big]$$
$$- [\Phi(y, u) + \Lambda(y, v, \bar{t}, \bar{s})] \sum_{i=1}^{p} u_i \nabla g_i(y)$$
$$+ \Big\{\Gamma(y, u)\Big[\sum_{i=1}^{p} u_i \nabla^2 f_i(y) + \sum_{m=1}^{\nu_0} v_m \nabla^2 G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla^2 H_{k_m}(y, s^m)\Big]$$
$$- [\Phi(y, u) + \Lambda(y, v, \bar{t}, \bar{s})] \sum_{i=1}^{p} u_i \nabla^2 g_i(y)\Big\} z = 0, \tag{3.1}$$

$$-\frac{1}{2}\Big\langle z, \Big\{\Gamma(y, u)\Big[\sum_{i=1}^{p} u_i \nabla^2 f_i(y) + \sum_{m=1}^{\nu_0} v_m \nabla^2 G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla^2 H_{k_m}(y, s^m)\Big]$$
$$- [\Phi(y, u) + \Lambda(y, v, \bar{t}, \bar{s})] \sum_{i=1}^{p} u_i \nabla^2 g_i(y)\Big\} z\Big\rangle \geq 0, \tag{3.2}$$

where $\Lambda(y, v, \bar{t}, \bar{s}) = \sum_{m=1}^{\nu_0} v_m G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(y, s^m)$;

$$(\widetilde{D}I) \qquad \sup_{(y,z,u,v,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in \mathbb{H}} \frac{\sum_{i=1}^{p} u_i f_i(y) + \sum_{m=1}^{\nu_0} v_m G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(y, s^m)}{\sum_{i=1}^{p} u_i g_i(y)}$$

subject to (3.2) and

$$\mathscr{F}\Big(x, y; \Gamma(y, u)\Big[\sum_{i=1}^{p} u_i \nabla f_i(y) + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y, s^m)\Big]$$
$$- [\Phi(y, u) + \Lambda(y, v, \bar{t}, \bar{s})] \sum_{i=1}^{p} u_i \nabla g_i(y)$$

$$+\left\{\Gamma(y,u)\left[\sum_{i=1}^{p}u_i\nabla^2 f_i(y)+\sum_{m=1}^{v_0}v_m\nabla^2 G_{j_m}(y,t^m)+\sum_{m=v_0+1}^{v}v_m\nabla^2 H_{k_m}(y,s^m)\right]\right.$$

$$\left.\left.-\,[\Phi(y,u)+\Lambda(y,v,\bar{t},\bar{s})]\sum_{i=1}^{p}u_i\nabla^2 g_i(y)\right\}z\right)\geq 0 \quad \text{for all } x\in\mathbb{F}, \quad (3.3)$$

where $\mathcal{F}(x,y;\cdot)$ is a sublinear function from $\mathbb{R}^n$ to $\mathbb{R}$.

Comparing (DI) and $(\widetilde{DI})$, we see that $(\widetilde{DI})$ is relatively more general than (DI) in the sense that any feasible solution of (DI) is also feasible for $(\widetilde{DI})$, but the converse is not necessarily true. Furthermore, we observe that (3.1) is a system of n equations, whereas (3.3) is a single inequality. Clearly, from a computational point of view, (DI) is preferable to $(\widetilde{DI})$ because of the dependence of (3.3) on the feasible set of (P).

Despite these apparent differences, it turns out that the statements and proofs of all the duality theorems for (P)–(DI) and (P)–$(\widetilde{DI})$ are almost identical and, therefore, we shall consider only the pair (P)–(DI).

In the sequel, we shall make frequent use of the following auxiliary result, which provides an alternative expression for the objective function of (P).

Lemma 3.1 ([72]). *For each $x\in X$,*

$$\varphi(x)\equiv\max_{1\leq i\leq p}\frac{f_i(x)}{g_i(x)}=\max_{u\in U}\frac{\sum_{i=1}^{p}u_i f_i(x)}{\sum_{i=1}^{p}u_i g_i(x)}.$$

The next two theorems show that (DI) is a dual problem for (P).

Theorem 3.1 (Weak Duality). *Let x and $w\equiv(y,z,u,v,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})$ be arbitrary feasible solutions of (P) and (DI), respectively, and assume that $\Phi(y,u)+\Lambda(y,v,\bar{t},\bar{s})\geq 0, \Gamma(y,u)>0$, and either one of the following two sets of hypotheses is satisfied:*

(a) (i) *for each $i\in\underline{p}, f_i$ is $(\mathcal{F},\beta,\phi,\bar{\rho}_i,\theta)$-sounivex and $-g_i$ is $(\mathcal{F},\beta,\phi,\tilde{\rho}_i,\theta)$-sounivex at y;*

(ii) *for each $m\in\underline{\nu_0}, \xi\to G_{j_m}(\xi,t^m)$ is $(\mathcal{F},\beta,\phi,\hat{\rho}_m,\theta)$-sounivex at y;*

(iii) *for each $m\in\underline{\nu}\backslash\underline{\nu_0}, \xi\to v_m H_{k_m}(\xi,s^m)$ is $(\mathcal{F},\beta,\phi,\breve{\rho}_m,\theta)$-sounivex at y;*

(iv) *ϕ is superlinear and $\phi(a)\geq 0\Rightarrow a\geq 0$;*

(v) $\sum_{i=1}^{p}u_i\{\Gamma(y,u)\bar{\rho}_i(x,y)+[\Phi(y,u)+\Lambda(y,v,\bar{t},\bar{s})]\tilde{\rho}_i(x,y)\}+\sum_{m=1}^{\nu_0}v_m\hat{\rho}_m(x,y)\Gamma(y,u)+\sum_{m=\nu_0+1}^{\nu}\breve{\rho}_m(x,y)\Gamma(y,u)\geq 0;$

(b) *the Lagrangian-type function*

$$\xi\to L(\xi,y,u,v,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})$$

$$= \Gamma(y,u)\Big[\sum_{i=1}^{p} u_i f_i(\xi) + \sum_{m=1}^{v_0} v_m G_{j_m}(\xi, t^m) + \sum_{m=v_0+1}^{v} v_m H_{k_m}(\xi, s^m)\Big]$$

$$- [\Phi(y,u) + \Lambda(y,v,\bar{t},\bar{s})] \sum_{i=1}^{p} u_i g_i(\xi)$$

is $(\mathscr{F}, \beta, \phi, 0, \theta)$-pseudosounivex at y and $\phi(a) \geq 0 \Rightarrow a \geq 0$.

Then $\varphi(x) \geq \psi_1(w)$, where ψ_1 is the objective function of (DI).

Proof. (a) Using the hypotheses specified in (i)–(iii), we have

$$\phi\left(f_i(x) - f_i(y) + \frac{1}{2}\langle z, \nabla^2 f_i(y) z\rangle\right)$$
$$\geq \mathscr{F}(x,y;\beta(x,y)[\nabla f_i(y) + \nabla^2 f_i(y)z]) + \bar{\rho}_i(x,y)\|\theta(x,y)\|^2, \quad i \in \underline{p}, \tag{3.4}$$

$$\phi\left(-g_i(x) + g_i(y) - \frac{1}{2}\langle z, \nabla^2 g_i(y) z\rangle\right)$$
$$\geq \mathscr{F}(x,y;\beta(x,y)[-\nabla g_i(y) - \nabla^2 g_i(y)z]) + \tilde{\rho}_i(x,y)\|\theta(x,y)\|^2, \quad i \in \underline{p}, \tag{3.5}$$

$$\phi\left(G_{j_m}(x,t^m) - G_{j_m}(y,t^m) + \frac{1}{2}\langle z, \nabla^2 G_{j_m}(y,t^m) z\rangle\right)$$
$$\geq \mathscr{F}(x,y;\beta(x,y)[\nabla G_{j_m}(y,t^m) + \nabla^2 G_{j_m}(y,t^m)z]$$
$$+ \hat{\rho}_m(x,y)\|\theta(x,y)\|^2, \quad m \in \underline{v_0}, \tag{3.6}$$

$$\phi\left(v_m H_{k_m}(x,s^m) - v_m H_{k_m}(y,s^m) + \frac{1}{2}\langle z, v_m \nabla^2 H_{k_m}(y,s^m) z\rangle\right)$$
$$\geq \mathscr{F}(x,y;\beta(x,y)[v_m \nabla H_{k_m}(y,s^m) + v_m \nabla^2 H_{k_m}(y,s^m)z])$$
$$+ \breve{\rho}_m(x,y)\|\theta(x,y)\|^2, \quad m \in \underline{v}\backslash\underline{v_0}. \tag{3.7}$$

Now, multiplying (3.4) by $u_i\Gamma(y,u)$ and then summing over $i \in \underline{p}$, (3.5) by $u_i[\Phi(y,u) + \Lambda(y,u,v,\bar{t},\bar{s})]$ and then summing over $i \in \underline{p}$, (3.6) by $v_m\Gamma(y,u)$ and then summing over $m \in \underline{v_0}$, (3.7) by $\Gamma(y,u)$ and then summing over $m \in \underline{v}\backslash\underline{v_0}$, adding the resulting inequalities, and using the superlinearity of ϕ and sublinearity of $\mathscr{F}(x,y;\cdot)$, we obtain

$$\phi\Big(\sum_{i=1}^{p} u_i\{\Gamma(y,u) f_i(x) - [\Phi(y,u) + \Lambda(y,u,v,\bar{t},\bar{s})] g_i(x)\}$$

$$+ \Gamma(y,u)\Big[\sum_{m=1}^{v_0} v_m G_{j_m}(x,t^m) + \sum_{k=1}^{r} v_m H_{k_m}(x,s^m)\Big]$$

$$
\begin{aligned}
&- \Gamma(y, u)\Bigg[\sum_{i=1}^{p} u_i f_i(y) + \sum_{m=1}^{v_0} v_m G_{j_m}(y, t^m) + \sum_{m=v_0+1}^{v} v_m H_{k_m}(y, s^m) \Bigg] \\
&+ [\Phi(y, u) + \Lambda(y, v, \bar{t}, \bar{s})] \sum_{i=1}^{p} u_i g_i(y) \\
&+ \frac{1}{2}\Bigg\langle z, \Bigg\{ \Gamma(y, u)\Bigg[\sum_{i=1}^{p} u_i \nabla^2 f_i(y) + \sum_{m=1}^{v_0} v_m \nabla^2 G_{j_m}(y, t^m) + \sum_{m=v_0+1}^{v} v_m \nabla^2 H_{k_m}(y, s^m) \Bigg] \\
&\qquad - [\Phi(y, u) + \Lambda(y, v, \bar{t}, \bar{s})] \sum_{i=1}^{p} u_i \nabla^2 g_i(y) \Bigg\} z \Bigg\rangle\Bigg) \\
&\geq \mathscr{F}\Bigg(x, y; \Gamma(y, u)\Bigg[\sum_{i=1}^{p} u_i \nabla f_i(y) + \sum_{m=1}^{v_0} v_m \nabla G_{j_m}(y, t^m) + \sum_{m=v_0+1}^{v} v_m \nabla H_{k_m}(y, s^m) \Bigg] \\
&\quad - [\Phi(y, u) + \Lambda(y, v, \bar{t}, \bar{s})] \sum_{i=1}^{p} u_i \nabla g_i(y) \\
&\quad + \Bigg\{ \Gamma(y, u)\Bigg[\sum_{i=1}^{p} u_i \nabla^2 f_i(y) + \sum_{m=1}^{v_0} v_m \nabla^2 G_{j_m}(y, t^m) + \sum_{m=v_0+1}^{v} v_m \nabla^2 H_{k_m}(y, s^m) \Bigg] \\
&\qquad - [\Phi(y, u) + \Lambda(y, v, \bar{t}, \bar{s})] \sum_{i=1}^{p} u_i \nabla^2 g_i(y) \Bigg\} z \Bigg) \\
&\quad + \Bigg\{ \sum_{i=1}^{p} u_i \{ \Gamma(y, u) \bar{\rho}_i(x, y) + [\Phi(y, u) + \Lambda(y, v, \bar{t}, \bar{s})] \tilde{\rho}_i(x, y) \} \\
&\qquad + \sum_{m=1}^{v_0} v_m \hat{\rho}_m(x, y) \Gamma(y, u) + \sum_{m=v_0+1}^{v} \breve{\rho}_m(x, y) \Gamma(y, u) \Bigg\} \|\theta(x, y)\|^2.
\end{aligned}
$$

Because of the sublinearity of $\mathscr{F}(x, y; \cdot)$, (3.1), and (v), the above inequality reduces to

$$
\begin{aligned}
&\phi\Bigg(\sum_{i=1}^{p} u_i \{ \Gamma(y, u) f_i(x) - [\Phi(y, u) + \Lambda(y, u, v, \bar{t}, \bar{s})] g_i(x)] \} \\
&+ \Gamma(y, u)\Bigg[\sum_{m=1}^{v_0} v_m G_{j_m}(x, t^m) + \sum_{k=1}^{r} w_m H_{k_m}(x, s^m) \Bigg] \\
&- \Gamma(y, u)\Bigg[\sum_{i=1}^{p} u_i f_i(y) + \sum_{m=1}^{v_0} v_m G_{j_m}(y, t^m) + \sum_{m=v_0+1}^{v} v_m H_{k_m}(y, s^m) \Bigg]
\end{aligned}
$$

$$+ [\Phi(y,u)+\Lambda(y,v,\bar{t},\bar{s})]\sum_{i=1}^{p}u_i g_i(y)$$
$$+\frac{1}{2}\Big\langle z,\Big\{\Gamma(y,u)\Big[\sum_{i=1}^{p}u_i\nabla^2 f_i(y)+\sum_{m=1}^{\nu_0}v_m\nabla^2 G_{j_m}(y,t^m)+\sum_{m=\nu_0+1}^{\nu}v_m\nabla^2 H_{k_m}(y,s^m)\Big]$$
$$-[\Phi(y,u)+\Lambda(y,v,\bar{t},\bar{s})]\sum_{i=1}^{p}u_i\nabla^2 g_i(y)\Big\}z\Big\rangle\Big)\geq 0.$$

In view of the definitions of Φ,Γ, and Λ, and the fact that $\phi(a)\Rightarrow a\geq 0$, $x\in\mathbb{F}$, and (3.2) holds, we deduce from the above inequality that

$$\sum_{i=1}^{p}u_i\{\Gamma(y,u)f_i(x)-[\Phi(y,u)+\Lambda(y,v,\bar{t},\bar{s})]g_i(x)\}\geq 0. \tag{3.8}$$

Now, using this inequality and Lemma 3.1, we see that

$$\varphi(x)=\max_{d\in U}\frac{\sum_{i=1}^{p}d_i f_i(x)}{\sum_{i=1}^{p}d_i g_i(x)}\geq\frac{\sum_{i=1}^{p}u_i f_i(x)}{\sum_{i=1}^{p}u_i g_i(x)}\geq\frac{\Phi(y,u)+\Lambda(y,v,\bar{t},\bar{s})}{\Gamma(y,u)}=\psi_1(w).$$

(b) In view of the sublinearity of $\mathscr{F}(x,y;\cdot)$, positivity of $\beta(x,y)$, $(\mathscr{F},\beta,\phi,0,\theta)$-pseudosounivexity of $L(\cdot,y,u,v,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})$ at y, and (3.1) we conclude that

$$\phi(L(x,y,u,v,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})-L(y,y,u,v,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s}))$$
$$+\frac{1}{2}\langle z,\nabla^2 L(y,y,u,v,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})z\rangle)\geq 0.$$

But $\phi(a)\geq 0\Rightarrow a\geq 0$, and, hence, the above inequality yields

$$\begin{aligned}L(x,y,u,v,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})&\geq L(y,y,u,v,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\\&\quad-\frac{1}{2}\langle z,\nabla^2 L(y,y,u,v,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})z\rangle)\\&\geq 0,\end{aligned}$$

where the second inequality follows from the definitions of Φ,Γ, and Λ, and (3.2). Now proceeding as in the proof of part (a), we obtain (3.8) which leads, as seen above, to the desired conclusion that $\varphi(x)\geq\psi_1(w)$. □

Theorem 3.2 (Strong Duality). *Let x^* be a normal optimal solution of (P) and assume that either one of the two sets of conditions set forth in Theorem 3.1 is satisfied for all feasible solutions of (DI). Then there exist $u^*, v^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^*\backslash\nu_0^*}, \bar{t}^*$,* *and* $\bar{s}^*$ *such that* $w^* \equiv (x^*, z^* = 0, u^*, v^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^* \backslash \nu_0^*}, \bar{t}^*, \bar{s}^*)$ *is an optimal solution of (DI) and* $\varphi(x^*) = \psi_1(w^*)$.

Proof. Since x^* is a normal optimal solution of (P), by Theorem 2.1, there exist $u^* \in U$ and integers ν_0 and ν, with $0 \leq \nu_0 \leq \nu \leq n+1$, such that there exist ν_0 indices j_m, with $1 \leq j_m \leq q$, together with ν_0 points $t^m \in \widehat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \leq k_m \leq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$ for $m \in \underline{\nu} \backslash \underline{\nu_0}$, and ν real numbers $\bar{v}_m$ with $\bar{v}_m > 0$ for $m \in \underline{\nu_0}$, with the property that

$$\sum_{i=1}^{p} u_i^* [\Gamma(x^*, u^*) \nabla f_i(x^*) - \Phi(x^*, u^*) \nabla g_i(x^*)] + \sum_{m=1}^{\nu_0} \bar{v}_m \nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0+1}^{\nu} \bar{v}_m \nabla H_{k_m}(x^*, s^m) = 0, \tag{3.9}$$

$$\max_{1 \leq i \leq p} \frac{f_i(x^*)}{g_i(x^*)} = \frac{\Phi(x^*, u^*)}{\Gamma(x^*, u^*)}. \tag{3.10}$$

Since $\Lambda(x^*, \bar{v}, \bar{t}, \bar{s}) = 0$, (3.9) can be rewritten as follows:

$$\Gamma(x^*, u^*) \left[\sum_{i=1}^{p} u_i^* \nabla f_i(x^*) + \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla H_{k_m}(x^*, s^m) \right] - [\Phi(x^*, u^*) + \Lambda(x^*, v^*, \bar{t}, \bar{s})] \sum_{i=1}^{p} u_i^* \nabla g_i(x^*) = 0, \tag{3.11}$$

where $v_m^* = \bar{v}_m / \Gamma(x^*, u^*)$ for $m \in \underline{\nu}$. In as much as $x^* \in \mathbb{F}$ and hence

$$\sum_{m=1}^{\nu_0} v_m^* G_{j_m}(x^*, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* H_{k_m}(x^*, s^m) = 0,$$

from (3.10) and (3.11) it is clear that w^* is a feasible solution of (DI) and $\varphi(x^*) = \psi_1(w^*)$. If w^* were not optimal, then there would exist a feasible solution $\tilde{w} \equiv (\tilde{x}, \tilde{z}, \tilde{u}, \tilde{v}, \tilde{\lambda}, \tilde{\nu}, \tilde{\nu}_0, J_{\tilde{\nu}_0}, K_{\tilde{\nu} \backslash \tilde{\nu}_0}, \bar{\tilde{t}}, \bar{\tilde{s}})$ of (DI) such that $\psi_1(\tilde{w}) > \psi_1(x^*) = \varphi(x^*)$, contradicting Theorem 3.1. Therefore, w^* is an optimal solution of (DI). □

We also have the following converse duality result for (P) and (DI).

Theorem 3.3 (Strict Converse Duality). *Let* x^* *be a normal optimal solution of (P), let* $\tilde{w} \equiv (\tilde{x}, \tilde{z}, \tilde{u}, \tilde{v}, \tilde{\nu}, \tilde{\nu}_0, J_{\tilde{\nu}_0}, K_{\tilde{\nu} \backslash \tilde{\nu}_0}, \bar{\tilde{t}}, \bar{\tilde{s}})$ *be an optimal solution of (DI), and assume that either one of the following two sets of conditions is satisfied:*

(a) *The assumptions specified in part (a) of Theorem* 3.1 *are satisfied for the feasible solution* $\tilde{w}$ *of (DI),* $\phi(a) > 0 \Rightarrow a > 0$, f_i *is strictly* $(\mathscr{F}, \beta, \phi, \bar{\rho}_i, \theta)$*-sounivex at* $\tilde{x}$ *for at least one* $i \in \underline{p}$ *with the corresponding component* $\tilde{u}_i$ *of* $\tilde{u}$ *positive, or* $-g_i$ *is strictly* $(\mathscr{F}, \beta, \bar{\phi}, \tilde{\rho}_i, \theta)$*-sounivex at* $\tilde{x}$ *for at least one* $i \in \underline{p}$ *with the corresponding component* $\tilde{u}_i$ *of* $\tilde{u}$ *positive, or* $\xi \to G_{j_m}(\xi, \tilde{t}^m)$ *is strictly* $(\mathscr{F}, \beta, \phi, \hat{\rho}_i, \theta)$*-sounivex at* $\tilde{x}$ *for at least one* $m \in \underline{\tilde{\nu}_0}$, *or* $\xi \to \tilde{v}_m H_{k_m}(\xi, \tilde{s}^m)$ *is strictly* $(\mathscr{F}, \beta, \phi, \breve{\rho}_i, \theta)$*-sounivex at* $\tilde{x}$ *for at least one* $m \in \underline{\tilde{\nu}} \backslash \underline{\tilde{\nu}_0}$, *or*

$$\sum_{i=1}^{p} \tilde{u}_i\{\Gamma(\tilde{x}, \tilde{u})\bar{\rho}_i(x^*, \tilde{x}) + [\Phi(\tilde{x}, \tilde{u}) + \Lambda(\tilde{x}, \tilde{v}, \bar{\tilde{t}}, \bar{\tilde{s}})]\tilde{\rho}_i(x^*, \tilde{x})\}$$
$$+ \sum_{m=1}^{\tilde{\nu}_0} \tilde{v}_m \hat{\rho}_m(x^*, \tilde{x})\Gamma(\tilde{x}, \tilde{u}) + \sum_{m=\tilde{\nu}_0+1}^{\tilde{\nu}} \breve{\rho}_m(x^*, \tilde{x})\Gamma(\tilde{x}, \tilde{u}) > 0.$$

(b) *The assumptions specified in part (b) of Theorem* 3.1 *are satisfied for the feasible solution* $\tilde{w}$ *of (DI) and the function* $\xi \to L(\xi, \tilde{x}, \tilde{u}, \tilde{v}, \tilde{\nu}, \tilde{\nu}_0, J_{\tilde{\nu}_0}, K_{\tilde{\nu}\backslash\tilde{\nu}_0}, \bar{\tilde{t}}, \bar{\tilde{s}})$ *is strictly* $(\mathscr{F}, \beta, \phi, 0, \theta)$*-pseudosounivex at* $\tilde{x}$, *and* $\phi(a) > 0 \Rightarrow a > 0$.

Then $\tilde{x} = x^*$ *and* $\varphi(x^*) = \psi_1(\tilde{w})$.

Proof. (a): Suppose to the contrary that $\tilde{x} \neq x^*$. Since x^* is a normal optimal solution of (P), by Theorem 2.1, there exist $u^*, v^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^*\backslash\nu_0^*}, \bar{t}^*$, and $\bar{s}^*$ such that $w^* \equiv (x^*, z^* = 0, u^*, v^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^*\backslash\nu_0^*}, \bar{t}^*, \bar{s}^*)$ is an optimal solution of (DI) and $\varphi(x^*) = \psi_1(w^*)$. Now proceeding as in the proof of Theorem 3.1 (with x replaced by x^* and w by $\tilde{w}$) and using any one of the conditions set forth above, we arrive at the strict inequality

$$\sum_{i=1}^{p} \tilde{u}_i\{\Gamma(\tilde{x}, \tilde{u})f_i(x^*) - [\Phi(\tilde{x}, \tilde{u}) + \Lambda(\tilde{x}, \tilde{v}, \bar{\tilde{t}}, \bar{\tilde{s}})]g_i(x^*)\} > 0.$$

Using Lemma 3.1 and this inequality, as in the proof of Theorem 3.1, we obtain $\varphi(x^*) > \psi_1(\tilde{w})$, which contradicts the fact that $\varphi(x^*) = \psi_1(w^*) \leq \psi_1(\tilde{w})$. Therefore, we conclude that $\tilde{x} = x^*$ and $\varphi(x^*) = \psi_1(\tilde{w})$.

(b): The proof is similar to that of part (a). □

4. DUALITY MODEL II

In this section, we consider two duality models with special constraint structures that allow for a greater variety of generalized $(\mathscr{F}, \beta, \phi, \rho, \theta)$-sounivexity conditions under which duality can be established.

Consider the following two problems: (DII)

$$\sup_{(y,z,u,v,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in \mathbb{H}} \frac{\sum_{i=1}^{p} u_i f_i(y) + \sum_{m=1}^{\nu_0} v_m G_{j_m}(y,t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(y,s^m)}{\sum_{i=1}^{p} u_i g_i(y)}$$

subject to

$$\Gamma(y,u)\Bigg[\sum_{i=1}^{p} u_i \nabla f_i(y) + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y,t^m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y,s^m)\Bigg]$$

$$- [\Phi(y,u) + \Lambda(y,v,\bar{t},\bar{s})] \sum_{i=1}^{p} u_i \nabla g_i(y)$$

$$+ \Bigg\{\Gamma(y,u)\Bigg[\sum_{i=1}^{p} u_i \nabla^2 f_i(y) + \sum_{m=1}^{\nu_0} v_m \nabla^2 G_{j_m}(y,t^m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla^2 H_{k_m}(y,s^m)\Bigg]$$

$$- [\Phi(y,u) + \Lambda(y,v,\bar{t},\bar{s})] \sum_{i=1}^{p} u_i \nabla^2 g_i(y)\Bigg\} z = 0, \tag{4.1}$$

$$\Gamma(y,u) f_i(y) - [\Phi(y,u) + \Lambda(y,v,\bar{t},\bar{s})] g_i(y) - \frac{1}{2}\langle z, \{\Gamma(y,u)\nabla^2 f_i(y)$$

$$- [\Phi(y,u) + \Lambda(y,v,\bar{t},\bar{s})]\nabla^2 g_i(y)\} z\rangle \geq 0, \quad i \in \underline{p}, \tag{4.2}$$

$$G_{j_m}(y,t^m) - \frac{1}{2}\langle z, \nabla^2 G_{j_m}(y,t^m) z\rangle \geq 0, \quad m \in \underline{\nu_0}, \tag{4.3}$$

$$v_m H_{k_m}(y,s^m) - \tfrac{1}{2}\langle z, v_m \nabla^2 H_{k_m}(y,s^m) z\rangle \geq 0, \quad m \in \underline{\nu}\backslash\underline{\nu_0}; \tag{4.4}$$

$(\widetilde{D}II)$

$$\sup_{(y,z,u,v,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in \mathbb{H}} \frac{\sum_{i=1}^{p} u_i f_i(y) + \sum_{m=1}^{\nu_0} v_m G_{j_m}(y,t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(y,s^m)}{\sum_{i=1}^{p} u_i g_i(y)}$$

subject to (3.3) and (4.2)–(4.4).

The remarks made about the relationships between (DI) and $(\widetilde{D}I)$ are, of course, also applicable to (DII) and $(\widetilde{D}II)$.

The next two theorems show that (DII) is a dual problem for (P).

Theorem 4.1 (Weak Duality). *Let x and $w \equiv (y, z, u, v, \nu, \nu_0, J_{\nu_0}, K_{\nu\backslash\nu_0}, \bar{t}, \bar{s})$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that $\Phi(y,u) + \Lambda(y,v,\bar{t},\bar{s}) \geq 0, \Gamma(y,u) > 0$, and any one of the following five sets of hypotheses is satisfied:*

(a) (i) *for each $i \in \underline{p}, f_i$ is $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$-sounivex and $-g_i$ is $(\mathcal{F}, \beta, \bar{\phi}, \tilde{\rho}_i, \theta)$-sounivex at $y, \bar{\phi}$ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

(ii) *the function* $\xi \to G_{j_m}(\xi, t^m)$ *is* $(\mathscr{F}, \beta, \hat{\phi}_m, \hat{\rho}_m, \theta)$*-quasisounivex at* $y, \hat{\phi}_m$ *is increasing, and* $\hat{\phi}_m(0) = 0$ *for each* $m \in \underline{v_0}$;

(iii) *the function* $\xi \to v_m H_{k_m}(\xi, s^m)$ *is* $(\mathscr{F}, \beta, \breve{\phi}_m, \breve{\rho}_m, \theta)$*-quasisounivex at* $y, \breve{\phi}_m$ *is increasing, and* $\breve{\phi}_m(0) = 0$ *for each* $m \in \underline{v}\backslash\underline{v_0}$;

(iv) $\rho^*(x, y) + \sum_{m=1}^{v_0} v_m \hat{\rho}_m(x, y) + \sum_{m=v_0+1}^{v} \breve{\rho}_m(x, y) \geq 0$, *where* $\rho^*(x, y) = \sum_{i=1}^{p} u_i\{\Gamma(y, u)\bar{\rho}_i(x, y) + [\Phi(y, u) + \Lambda(y, v, \bar{t}, \bar{s})]\tilde{\rho}_i(x, y)\}$;

(b) (i) *for each* $i \in \underline{p}, f_i$ *is* $(\mathscr{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$*-sounivex and* $-g_i$ *is* $(\mathscr{F}, \beta, \bar{\phi}, \tilde{\rho}_i, \theta)$*-sounivex at* $y, \bar{\phi}$ *is superlinear, and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) *the function* $\xi \to \sum_{m=1}^{v_0} v_m G_{j_m}(\xi, t^m)$ *is* $(\mathscr{F}, \beta, \hat{\phi}, \hat{\rho}, \theta)$*-quasisounivex at* $y, \hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$;

(iii) *the function* $\xi \to v_m H_{k_m}(\xi, s^m)$ *is* $(\mathscr{F}, \beta, \breve{\phi}_m, \breve{\rho}_m, \theta)$*-quasisounivex at* $y, \breve{\phi}_m$ *is increasing, and* $\breve{\phi}_m(0) = 0$ *for each* $m \in \underline{v}\backslash\underline{v_0}$;

(iv) $\rho^*(x, y) + \hat{\rho}(x, y) + \sum_{m=v_0+1}^{v} \breve{\rho}_m(x, y) \geq 0$;

(c) (i) *for each* $i \in \underline{p}, f_i$ *is* $(\mathscr{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$*-sounivex and* $-g_i$ *is* $(\mathscr{F}, \beta, \bar{\phi}, \tilde{\rho}_i, \theta)$*-sounivex at* $y, \bar{\phi}$ *is superlinear, and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) *the function* $\xi \to G_{j_m}(\xi, t^m)$ *is* $(\mathscr{F}, \beta, \hat{\phi}_m, \hat{\rho}_m, \theta)$*-quasisounivex at* $y, \hat{\phi}_m$ *is increasing, and* $\hat{\phi}_m(0) = 0$ *for each* $m \in \underline{v_0}$;

(iii) *the function* $\xi \to \sum_{m=v_0+1}^{v} v_m H_{k_m}(\xi, s^m)$ *is* $(\mathscr{F}, \beta, \breve{\phi}, \breve{\rho}, \theta)$*-quasisounivex at* $y, \breve{\phi}$ *is increasing, and* $\breve{\phi}(0) = 0$;

(iv) $\rho^*(x, y) + \sum_{m=1}^{v_0} v_m \hat{\rho}_m(x, y) + \breve{\rho}(x, y) \geq 0$;

(d) (i) *for each* $i \in \underline{p}, f_i$ *is* $(\mathscr{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$*-sounivex and* $-g_i$ *is* $(\mathscr{F}, \beta, \bar{\phi}, \tilde{\rho}_i, \theta)$*-sounivex at* $y, \bar{\phi}$ *is superlinear, and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) *the function* $\xi \to \sum_{m=1}^{v_0} v_m G_{j_m}(\xi, t^m)$ *is* $(\mathscr{F}, \beta, \hat{\phi}, \hat{\rho}, \theta)$*-quasisounivex at* $y, \hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$;

(iii) *the function* $\xi \to \sum_{m=v_0+1}^{v} v_m H_{k_m}(\xi, s^m)$ *is* $(\mathscr{F}, \beta, \breve{\phi}, \breve{\rho}, \theta)$*-quasisounivex at* $y, \breve{\phi}$ *is increasing, and* $\breve{\phi}(0) = 0$;

(iv) $\rho^*(x, y) + \hat{\rho}(x, y) + \breve{\rho}(x, y) \geq 0$;

(e) (i) *for each* $i \in \underline{p}, f_i$ *is* $(\mathscr{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$*-sounivex and* $-g_i$ *is* $(\mathscr{F}, \beta, \bar{\phi}, \tilde{\rho}_i, \theta)$*-sounivex at* $y, \bar{\phi}$ *is superlinear, and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) *the function* $\xi \to \sum_{m=1}^{v_0} v_m G_{j_m}(\xi, t^m) + \sum_{m=v_0+1}^{v} v_m H_{k_m}(\xi, s^m)$ *is* $(\mathscr{F}, \beta, \hat{\phi}, \hat{\rho}, \theta)$*-quasisounivex at* $y, \hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$;

(iii) $\rho^*(x, y) + \hat{\rho}(x, y) \geq 0$.

Then $\varphi(x) \geq \psi_2(w)$, *where* ψ_2 *is the objective function of (DII).*

Proof. (a): Proceeding as in the proof of part (a) of Theorem 3.1, we see that our assumptions in (i) lead to the following inequality:

$$\phi\bigg(\sum_{i=1}^{p} u_i\{\Gamma(y, u)f_i(x) - [\Phi(y, u) + \Lambda(y, v, \bar{t}, \bar{s})]g_i(x)$$
$$- \{\Gamma(y, u)f_i(y) - [\Phi(y, u) + \Lambda(y, v, \bar{t}, \bar{s})]g_i(y)]\}\}$$

$$+\frac{1}{2}\left\langle z, \sum_{i=1}^{p} u_i\{\Gamma(y,u)\nabla^2 f_i(y) - [\Phi(y,u) + \Lambda(y,v,\bar{t},\bar{s})]\nabla^2 g_i(y)\}z\right\rangle\Bigg)$$

$$\geq \mathscr{F}\Bigg(x, y; \beta(x,y)\sum_{i=1}^{p} u_i\{\Gamma(y,u)\nabla\}f_i(y)\} - [\Phi(y,u) + \Lambda(y,v,\bar{t},\bar{s})]\nabla g_i(y)$$

$$+ \{\Gamma(y,u)\nabla^2 f_i(y) - [\Phi(y,u) + \Lambda(y,v,\bar{t},\bar{s})]\nabla^2 g_i(y)\}z\Bigg) + \sum_{i=1}^{p} u_i$$

$$\times \{\Gamma(y,u)\bar{\rho}_i(x,y) + [\Phi(y,u) + \Lambda(y,v,\bar{t},\bar{s})]\tilde{\rho}_i(x,y)\}\|\theta(x,y)\|^2. \quad (4.5)$$

From the primal feasibility of x, dual feasibility of w, and (4.3) we see that

$$G_{j_m}(x,t^m) \leq 0 \leq G_{j_m}(y,t^m) - \frac{1}{2}\langle z, \nabla^2 G_{j_m}(y,t^m)z\rangle$$

for each $m \in \underline{v_0}$, and, hence, in view of the properties of the functions ϕ_m, we get

$$\phi_j(G_{j_m}(x,t^m) - [G_{j_m}(y,t^m) - \frac{1}{2}\langle z, \nabla^2 G_{j_m}(y,t^m)z\rangle]) \leq 0,$$

which, in view of (ii), implies that

$$\mathscr{F}(x,y;\beta(x,y)[\nabla G_{j_m}(y,t^m) + \nabla^2 G_{j_m}(y,t^m)z]) \leq -\hat{\rho}_m(x,y)\|\theta(x,y)\|^2.$$

As $v_m > 0$ for each $m \in \underline{v_0}$ and $\mathscr{F}(x,y;\cdot)$ is sublinear, the above inequalities yield

$$\mathscr{F}\Bigg(x, y; \beta(x,y)\sum_{m=1}^{v_0} v_m[\nabla G_{j_m}(y,t^m) + \nabla^2 G_{j_m}(y,t^m)z]\Bigg)$$

$$\leq -\sum_{m=1}^{v_0} v_m\hat{\rho}_m(x,y)\|\theta(x,y)\|^2. \quad (4.6)$$

Similarly, from the primal feasibility of x, dual feasibility of w, (4.4), and (iii) we deduce that

$$\mathscr{F}\Bigg(x, y; \beta(x,y)\sum_{m=v_0+1}^{v} v_m[\nabla H_{k_m}(y,s^m) + \nabla^2 H_{k_m}(y,s^m)z]\Bigg)$$

$$\leq -\sum_{m=v_0+1}^{v} \breve{\rho}_m(x,y)\|\theta(x,y)\|^2. \quad (4.7)$$

Combining (4.1) and (4.5)–(4.7), and using (iv) and the sublinearity of $\mathscr{F}(x, y; \cdot)$, we obtain

$$\phi\Bigg(\sum_{i=1}^{p} u_i\Bigg\{\Gamma(y,u)f_i(x) - [\Phi(y,u) + \Lambda(y,v,\bar{t},\bar{s})]g_i(x)$$
$$- \{\Gamma(y,u)f_i(y) - [\Phi(y,u) + \Lambda(y,v,\bar{t},\bar{s})]g_i(y)]\}\Bigg\}$$
$$+ \frac{1}{2}\Bigg\langle z, \Bigg\{\sum_{i=1}^{p} u_i\{\Gamma(y,u)\nabla^2 f_i(y) - [\Phi(y,u) + \Lambda(y,v,\bar{t},\bar{s})]\nabla^2 g_i(y,s^m)\Bigg\}z\Bigg\rangle\Bigg) \geq 0.$$

Since $\phi(a) \geq 0 \Rightarrow a \geq 0$ and (4.2) holds, the above inequality reduces to

$$\sum_{i=1}^{p} u_i\{\Gamma(y,u)f_i(x) - [\Phi(y,u) + \Lambda(y,v,\bar{t},\bar{s})]g_i(x)\} \geq 0. \tag{4.8}$$

Now using (4.8) and Lemma 3.1, we obtain the weak duality inequality as follows:

$$\varphi(x) = \max_{d\in U} \frac{\sum_{i=1}^{p} d_i f_i(x)}{\sum_{i=1}^{p} d_i g_i(x)} \geq \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)} \geq \frac{\Phi(y,u) + \Lambda(y,v,\bar{t},\bar{s})}{\Gamma(y,u)} = \psi_2(w).$$

(b)–(e): The proofs are similar to that of part (a). □

Theorem 4.2 (Strong Duality). *Let x^* be a normal optimal solution of (P) and assume that any one of the five sets of conditions set forth in Theorem 4.1 is satisfied for all feasible solutions of (DII). Then there exist $u^*, v^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^*\backslash\nu_0^*}, \bar{t}^*$, and $\bar{s}^*$ such that $(x^*, z^* = 0, u^*, v^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^*\backslash\nu_0^*}, \bar{t}^*, \bar{s}^*)$ is an optimal solution of (DII) and $\varphi(x^*) = \psi_2(x^*)$.*

Proof. The proof is similar to that of Theorem 3.2. □

Theorem 4.3 (Strict Converse Duality). *Let x^* be a normal optimal solution of (P), let $\tilde{w} \equiv (\tilde{x}, \tilde{z}, \tilde{u}, \tilde{v}, \tilde{\nu}, \tilde{\nu}_0, J_{\tilde{\nu}_0}, K_{\tilde{\nu}\backslash\tilde{\nu}_0}, \bar{\tilde{t}}, \bar{\tilde{s}})$ be an optimal solution of (DII), and assume that any one of the following five sets of conditions is satisfied:*

(a) *The assumptions specified in part (a) of Theorem 4.1 are satisfied for the feasible solution $\tilde{w}$ of (DII). Moreover, $\bar{\phi}(a) > 0 \Rightarrow a > 0, f_i$ is strictly $(\mathscr{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$-sounivex at $\tilde{x}$ for at least one $i \in \underline{p}$ with the corresponding component $\tilde{u}_i$ of $\tilde{u}$ positive, or $-g_i$ is strictly $(\mathscr{F}, \bar{\beta}, \bar{\phi}, \tilde{\rho}_i, \theta)$-sounivex at $\tilde{x}$ for at least one $i \in \underline{p}$ with the corresponding component $\tilde{u}_i$ of $\tilde{u}$ positive, or $\xi \rightarrow G_{j_m}(\xi, \tilde{t}^m)$ is strictly $(\mathscr{F}, \beta, \hat{\phi}_m, \hat{\rho}_m, \theta)$-pseudosounivex at $\tilde{x}$ for at least* *one* $m \in \underline{\tilde{v}_0}$, *or* $\xi \to \tilde{v}_m H_{k_m}(\xi, \tilde{s}^m)$ *is strictly* $(\mathcal{F}, \beta, \breve{\phi}_m, \breve{\rho}_m, \theta)$*-pseudosounivex at* $\tilde{x}$ *for at least one* $m \in \underline{\tilde{v}} \backslash \underline{\tilde{v}_0}$, *or* $\rho^*(x^*, \tilde{x}) + \sum_{m=1}^{\tilde{v}_0} \tilde{v}_m \hat{\rho}_m(x^*, \tilde{x}) + \sum_{m=\tilde{v}_0+1}^{\tilde{v}} \breve{\rho}_m(x^*, \tilde{x}) > 0$, *where* $\rho^*(x^*, \tilde{x}) = \sum_{i=1}^{p} \tilde{u}_i \{\Gamma(\tilde{x}, \tilde{u}) \bar{\rho}_i(x^*, \tilde{x}) + [\Phi(\tilde{x}, \tilde{u}) + \Lambda(\tilde{x}, \tilde{v}, \bar{\bar{t}}, \bar{\bar{s}})] \tilde{\rho}_i(x^*, \tilde{x})$.

(b) *The assumptions specified in part (b) of Theorem 4.1 are satisfied for the feasible solution* $\tilde{w}$ *of (DII). Moreover,* $\bar{\phi}(a) > 0 \Rightarrow a > 0$, f_i *is strictly* $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$*-sounivex at* $\tilde{x}$ *for at least one* $i \in \underline{p}$ *with the corresponding component* $\tilde{u}_i$ *of* $\tilde{u}$ *positive, or* $-g_i$ *is strictly* $(\mathcal{F}, \bar{\beta}, \bar{\phi}, \tilde{\rho}_i, \theta)$*-sounivex at* $\tilde{x}$ *for at least one* $i \in \underline{p}$ *with the corresponding component* $\tilde{u}_i$ *of* $\tilde{u}$ *positive, or* $\xi \to \sum_{m=1}^{\tilde{v}_0} \tilde{v}_m G_{j_m}(\xi, \tilde{t}^m)$ *is strictly* $(\mathcal{F}, \beta, \hat{\phi}, \hat{\rho}, \theta)$*-pseudosounivex at* $\tilde{x}$, *or* $\xi \to \tilde{v}_m H_{k_m}(\xi, \tilde{s}^m)$ *is strictly* $(\mathcal{F}, \beta, \breve{\phi}_m, \breve{\rho}_m, \theta)$*-pseudosounivex at* $\tilde{x}$ *for at least one* $m \in \underline{\tilde{v}} \backslash \underline{\tilde{v}_0}$, *or* $\rho^*(x^*, \tilde{x}) + \hat{\rho}(x^*, \tilde{x}) + \sum_{m=\tilde{v}_0+1}^{\tilde{v}} \breve{\rho}_m(x^*, \tilde{x}) > 0$.

(c) *The assumptions specified in part (c) of Theorem 4.1 are satisfied for the feasible solution* $\tilde{w}$ *of (DII). Moreover,* $\bar{\phi}(a) > 0 \Rightarrow a > 0$, f_i *is strictly* $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$*-sounivex at* $\tilde{x}$ *for at least one* $i \in \underline{p}$ *with the corresponding component* $\tilde{u}_i$ *of* $\tilde{u}$ *positive, or* $-g_i$ *is strictly* $(\mathcal{F}, \bar{\beta}, \bar{\phi}, \tilde{\rho}_i, \theta)$*-sounivex at* $\tilde{x}$ *for at least one* $i \in \underline{p}$ *with the corresponding component* $\tilde{u}_i$ *of* $\tilde{u}$ *positive, or* $\xi \to G_{j_m}(\xi, \tilde{t}^m)$ *is strictly* $(\mathcal{F}, \beta, \hat{\phi}_m, \hat{\rho}_m, \theta)$*-pseudosounivex at* $\tilde{x}$ *for at least one* $m \in \underline{\tilde{v}_0}$, *or* $\xi \to \sum_{m=\tilde{v}_0+1}^{\tilde{v}} \tilde{v}_m H_{k_m}(\xi, \tilde{s}^m)$ *is strictly* $(\mathcal{F}, \beta, \breve{\phi}, \breve{\rho}, \theta)$*-pseudosounivex at* $\tilde{x}$, *or* $\rho^*(x^*, \tilde{x}) + \sum_{m=1}^{\tilde{v}_0} \tilde{v}_m \hat{\rho}_m(x^*, \tilde{x}) + \breve{\rho}(x^*, \tilde{x}) > 0$.

(d) *The assumptions specified in part (d) of Theorem 4.1 are satisfied for the feasible solution* $\tilde{w}$ *of (DII). Moreover,* $\phi(\bar{a}) > 0 \Rightarrow a > 0$, f_i *is strictly* $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$*-sounivex at* $\tilde{x}$ *for at least one* $i \in \underline{p}$ *with the corresponding component* $\tilde{u}_i$ *of* $\tilde{u}$ *positive, or* $-g_i$ *is strictly* $(\mathcal{F}, \bar{\beta}, \bar{\phi}, \tilde{\rho}_i, \theta)$*-sounivex at* $\tilde{x}$ *for at least one* $i \in \underline{p}$ *with the corresponding component* $\tilde{u}_i$ *of* $\tilde{u}$ *positive, or* $\xi \to \sum_{m=1}^{\tilde{v}_0} \tilde{v}_m G_{j_m}(\xi, \tilde{t}^m)$ *is strictly* $(\mathcal{F}, \beta, \hat{\phi}, \hat{\rho}, \theta)$*-pseudosounivex at* $\tilde{x}$, *or* $\xi \to \sum_{m=\tilde{v}_0+1}^{\tilde{v}} \tilde{v}_m H_{k_m}(\xi, \tilde{s}^m)$ *is strictly* $(\mathcal{F}, \beta, \breve{\phi}, \breve{\rho}_m, \theta)$*-pseudosounivex at* $\tilde{x}$, *or* $\rho^*(x^*, \tilde{x}) + \hat{\rho}(x^*, \tilde{x}) + \breve{\rho}(x^*, \tilde{x}) > 0$.

(e) *The assumptions specified in part (e) of Theorem 4.1 are satisfied for the feasible solution* $\tilde{w}$ *of (DII). Moreover,* $\bar{\phi}(a) > 0 \Rightarrow a > 0$, f_i *is strictly* $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$*-sounivex at* $\tilde{x}$ *for at least one* $i \in \underline{p}$ *with the corresponding component* $\tilde{u}_i$ *of* $\tilde{u}$ *positive, or* $-g_i$ *is strictly* $(\mathcal{F}, \bar{\beta}, \bar{\phi}, \tilde{\rho}_i, \theta)$*-sounivex at* $\tilde{x}$ *for at least one* $i \in \underline{p}$ *with the corresponding component* $\tilde{u}_i$ *of* $\tilde{u}$ *positive, or* $\xi \to \sum_{m=1}^{\tilde{v}_0} \tilde{v}_m G_{j_m}(\xi, \tilde{t}^m) + \sum_{m=\tilde{v}_0+1}^{\tilde{v}} \tilde{v}_m H_{k_m}(\xi, \tilde{s}^m)$ *is strictly* $(\mathcal{F}, \beta, \hat{\phi}, \hat{\rho}, \theta)$*-pseudosounivex at* $\tilde{x}$, *or* $\rho^*(x^*, \tilde{x}) + \hat{\rho}(x^*, \tilde{x}) > 0$.

Then $\tilde{x} = x^*$ *and* $\varphi(x^*) = \psi_2(\tilde{w})$.

Proof. The proof is similar to that of Theorem 3.3. □

In Theorem 4.1, separate $(\mathcal{F}, \beta, \phi, \rho, \theta)$-sounivexity assumptions were imposed on the functions f_i and $-g_i$, $i \in \underline{p}$. It is possible to formulate and prove a multitude of additional sets of second-order nonparametric duality results in which various generalized $(\mathcal{F}, \beta, \phi, \rho, \theta)$-sounivexity requirements are placed on various combinations of the functions $\xi \to \Gamma(y, u)f_i(\xi) - [\Phi(y, u) + \Lambda(y, v, \bar{t}, \bar{s})]g_i(\xi)$, $i \in \underline{p}$, $\xi \to \sum_{i=1}^{p} u_i\{\Gamma(y, u)f_i(\xi) - [\Phi(y, u) + \Lambda(y, v, \bar{t}, \bar{s})]g_i(\xi)\}$, $\xi \to G_{j_m}(\xi, t^m)$, $\xi \to v_m H_{k_m}(\xi, s^m)$, $\xi \to \sum_{m=1}^{v_0} v_m G_{j_m}(\xi, t^m)$, $\xi \to \sum_{m=v_0+1}^{v} v_m H_{k_m}(\xi, s^m)$, $\xi \to \sum_{m=1}^{v_0} v_m G_{j_m}(\xi, t^m) + \sum_{m=v_0+1}^{v} v_m H_{k_m}(\xi, s^m)$, $\xi \to \Gamma(y, u)f_i(\xi) - [\Phi(y, u) + \Lambda(y, v, \bar{t}, \bar{s})]g_i(\xi) + \sum_{m=1}^{v_0} v_m G_{j_m}(\xi, t^m) + \sum_{m=v_0+1}^{v} v_m H_{k_m}(\xi, s^m)$, $i \in \underline{p}$, and $\xi \to \sum_{i=1}^{p} u_i\{\Gamma(y, u)f_i(\xi) - [\Phi(y, u) + \Lambda(y, v, \bar{t}, \bar{s})] g_i(\xi)\} + \sum_{m=1}^{v_0} v_m G_{j_m}(\xi, t^m) + \sum_{m=v_0+1}^{v} v_m H_{k_m}(\xi, s^m)$. However, for the sake of avoiding excessive repetition, these results will not be treated separately. Furthermore, it turns out that a great majority of these duality results can be obtained as special cases of some generalized duality theorems which will be discussed in the next section. It is important to point out that Theorems 4.1–4.3 are not subsumed by these more general duality results.

5. DUALITY MODEL III

In this section, we discuss several families of second-order nonparametric duality results under various generalized $(\mathcal{F}, \beta, \phi, \rho, \theta)$-sounivexity hypotheses imposed on certain combinations of the problem functions. This is accomplished by employing a certain partitioning scheme which was originally proposed in [51] for the purpose of constructing generalized dual problems for nonlinear programming problems. For this we need some additional notation.

Let v_0 and v be integers, with $1 \leq v_0 \leq v \leq n+1$, and let $\{J_0, J_1, \ldots, J_M\}$ and $\{K_0, K_1, \ldots, K_M\}$ be partitions of the sets $\underline{v_0}$ and $\underline{v} \backslash \underline{v_0}$, respectively; thus, $J_i \subseteq \underline{v_0}$ for each $i \in \underline{M} \cup \{0\}$, $J_i \cap J_j = \emptyset$ for each $i, j \in \underline{M} \cup \{0\}$ with $i \neq j$, and $\cup_{i=0}^{M} J_i = \underline{v_0}$. Obviously, similar properties hold for $\{K_0, K_1, \ldots, K_M\}$. Moreover, if m_1 and m_2 are the numbers of the partitioning sets of $\underline{v_0}$ and $\underline{v} \backslash \underline{v_0}$, respectively, then $M = \max\{m_1, m_2\}$ and $J_i = \emptyset$ or $K_i = \emptyset$ for $i > \min\{m_1, m_2\}$.

In addition, we use the real-valued functions $\xi \to \Psi(\xi, y, u, v, \bar{t}, \bar{s})$ and $\xi \to \Lambda_\tau(\xi, v, \bar{t}, \bar{s})$ defined, for fixed $y, u, v, v, v_0, J_{v_0}, K_{v \backslash v_0}, \bar{t}$, and $\bar{s}$, on X as follows:

$$\Psi(\xi, y, u, v, \bar{t}, \bar{s}) = \Gamma(y, u)\left[\sum_{i=1}^{p} u_i f_i(\xi) + \sum_{m \in J_0} v_m G_{j_m}(\xi, t^m) + \sum_{m \in K_0} v_m H_{k_m}(\xi, s^m)\right]$$

$$- [\Psi(y, u) + \Lambda_0(y, v, \bar{t}, \bar{s})] \sum_{i=1}^{p} u_i g_i(\xi),$$

$$\Lambda_\tau(\xi, v, \bar{t}, \bar{s}) = \sum_{m \in J_\tau} v_m G_{j_m}(\xi, t^m) + \sum_{m \in K_\tau} v_m H_{k_m}(\xi, s^m), \quad \tau \in \underline{M} \cup \{0\}.$$

Making use of the sets and functions defined above, we can state our general parameter-free duality models as follows: $(DIII)$

$$\sup_{(y,z,u,v,\nu,\nu_0,J_{\nu_0},K_{\nu\setminus\nu_0},\bar{t},\bar{s})\in \mathbb{H}} \frac{\sum_{i=1}^p u_i f_i(y) + \sum_{m\in J_0} v_m G_{j_m}(y, t^m) + \sum_{m \in K_0} v_m H_{k_m}(y, s^m)}{\sum_{i=1}^p u_i g_i(y)}$$

subject to

$$\begin{aligned}
&\sum_{i=1}^p u_i \Bigg\{ \Gamma(y,u) \Bigg[\nabla f_i(y) + \sum_{m\in J_0} v_m \nabla G_{j_m}(y, t^m) + \sum_{m\in K_0} w_m \nabla H_{k_m}(y, s^m) \Bigg] \\
&\qquad - [\Phi(y,u) + \Lambda_0(y, v, \bar{t}, \bar{s})] \nabla g_i(y) \Bigg\} \\
&\quad + \sum_{m\in \underline{\nu_0}\setminus J_0} v_m \nabla G_{j_m}(y, t^m) + \sum_{m\in(\underline{\nu}\setminus\underline{\nu_0})\setminus K_0} v_m \nabla H_{k_m}(y, s^m) \\
&\quad + \Bigg\{ \sum_{i=1}^p u_i \Bigg\{ \Gamma(y,u) \Bigg[\nabla^2 f_i(y) + \sum_{m\in J_0} v_m \nabla^2 G_{j_m}(y, t^m) + \sum_{m\in K_0} v_m \nabla^2 H_{k_m}(y, s^m) \Bigg] \\
&\qquad - [\Phi(y,u) + \Lambda_0(y, v, \bar{t}, \bar{s})] \nabla^2 g_i(y) \Bigg\} \\
&\qquad + \sum_{m\in \underline{\nu_0}\setminus J_0} v_m \nabla^2 G_{j_m}(y, t^m) + \sum_{m\in(\underline{\nu}\setminus\underline{\nu_0})\setminus K_0} v_m \nabla^2 H_{k_m}(y, s^m) \Bigg\} z = 0, \qquad (5.1)
\end{aligned}$$

$$\begin{aligned}
&\Gamma(y,u) \Bigg[\sum_{i=1}^p u_i f_i(y) + \sum_{m\in J_0} v_m G_{j_m}(y, t^m) + \sum_{m\in K_0} w_m H_{k_m}(y, s^m) \Bigg] \\
&\quad - [\Phi(y,u) + \Lambda_0(y, v, \bar{t}, \bar{s})] \sum_{i=1}^p u_i g_i(y) \\
&\quad - \frac{1}{2} \Bigg\langle z, \Bigg\{ \Gamma(y,u) \Bigg[\sum_{i=1}^p u_i \nabla^2 f_i(y) + \sum_{m\in J_0} v_m \nabla^2 G_{j_m}(y, t^m) + \sum_{m\in K_0} v_m \nabla^2 H_{k_m}(y, s^m) \Bigg] \\
&\qquad - [\Phi(y,u) + \Lambda_0(y, v, \bar{t}, \bar{s})] \sum_{i=1}^p u_i \nabla^2 g_i(y) \Bigg\} z \Bigg\rangle \geq 0, \quad i \in \underline{p}, \qquad (5.2)
\end{aligned}$$

$$\sum_{m\in J_\tau} v_m G_{j_m}(y,t^m) + \sum_{m\in K_\tau} v_m H_{k_m}(y,s^m)$$
$$- \frac{1}{2}\left\langle z, \left[\sum_{m\in J_\tau} v_m \nabla^2 G_{j_m}(y,t^m) + \sum_{m\in K_\tau} v_m \nabla^2 H_{k_m}(y,s^m)\right] z\right\rangle \geq 0, \quad \tau \in \underline{M}; \quad (5.3)$$

$(\widetilde{D}III)$

$$\sup_{(y,z,u,v,\nu,\nu_0,J_{\nu_0},K_{\nu\setminus\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \frac{\sum_{i=1}^{p} u_i f_i(y) + \sum_{m\in J_0} v_m G_{j_m}(y,t^m) + \sum_{m\in K_0} v_m H_{k_m}(y,s^m)}{\sum_{i=1}^{p} u_i g_i(y)}$$

subject to (5.2), (5.3), and

$$\mathcal{F}\Bigg(x,y;\sum_{i=1}^{p} u_i\Bigg\{\Gamma(y,u)\Bigg[\nabla f_i(y) + \sum_{m\in J_0} v_m \nabla G_{j_m}(y,t^m) + \sum_{m\in K_0} v_m \nabla H_{k_m}(y,s^m)\Bigg]$$
$$- [\Phi(y,u) + \Lambda_0(y,v,\bar{t},\bar{s})]\nabla g_i(y)\Bigg\}$$
$$+ \sum_{m\in \underline{\nu_0}\setminus J_0} v_m \nabla G_{j_m}(y,t^m) + \sum_{m\in(\underline{\nu}\setminus\underline{\nu_0})\setminus K_0} v_m \nabla H_{k_m}(y,s^m)$$
$$+ \Bigg\{\sum_{i=1}^{p} u_i\Bigg\{\Gamma(y,u)\Bigg[\nabla^2 f_i(y) + \sum_{m\in J_0} v_m \nabla^2 G_{j_m}(y,t^m) + \sum_{m\in K_0} v_m \nabla^2 H_{k_m}(y,s^m)\Bigg]$$
$$- [\Phi(y,u) + \Lambda_0(y,v,\bar{t},\bar{s})]\nabla^2 g_i(y)\Bigg\}$$
$$+ \sum_{m\in \underline{\nu_0}\setminus J_0} v_m \nabla^2 G_{j_m}(y,t^m) + \sum_{m\in(\underline{\nu}\setminus\underline{\nu_0})\setminus K_0} v_m \nabla^2 H_{k_m}(y,s^m)\Bigg\} z\Bigg) \geq 0 \quad \text{for all } x \in \mathbb{F},$$

where $\mathcal{F}(x,y;\cdot)$ is a sublinear function from $\mathbb{R}^n$ to $\mathbb{R}$.

The remarks and observations made earlier about the relationships between (DI) and $(\widetilde{D}I)$ are, of course, also valid for $(DIII)$ and $(\widetilde{D}III)$.

The next two theorems show that $(DIII)$ is a dual problem for (P).

Theorem 5.1 (Weak Duality). *Let x and $w \equiv (y,z,u,v,\nu,\nu_0,J_{\nu_0},K_{\nu\setminus\nu_0},\bar{t},\bar{s})$ be arbitrary feasible solutions of (P) and (DIII), respectively, and assume that any one of the following four sets of hypotheses is satisfied:*

(a) (i) $\xi \to \Psi(\xi,y,u,v,\bar{t},\bar{s})$ *is* $(\mathcal{F},\beta,\bar{\phi},\bar{\rho},\theta)$*-pseudosounivex at* y, *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) *for each* $\tau \in \underline{M}$, $\xi \to \Lambda_\tau(\xi,v,\bar{t},\bar{s})$ *is* $(\mathcal{F},\beta,\tilde{\phi}_\tau,\tilde{\rho}_\tau,\theta)$*-quasisounivex at* y, $\tilde{\phi}_\tau$ *is increasing, and* $\tilde{\phi}_\tau(0) = 0$*;*

(iii) $\bar{\rho}(x,y) + \sum_{\tau=1}^{M} \tilde{\rho}_\tau(x,y) \geq 0$*;*

(b) (i) $\xi \to \Psi(\xi, y, u, v, \bar{t}, \bar{s})$ *is prestrictly* $(\mathscr{F}, \beta, \bar{\phi}, \bar{\rho}, \theta)$*-quasisounivex at* y, $\bar{\phi}(a)$ *is strictly increasing, and* $\bar{\phi}(0) = 0$;

(ii) *for each* $\tau \in \underline{M}, \xi \to \Lambda_\tau(\xi, v, \bar{t}, \bar{s})$ *is* $(\mathscr{F}, \beta, \tilde{\phi}_\tau, \tilde{\rho}_\tau, \theta)$*-quasisounivex at* $y, \tilde{\phi}_\tau$ *is increasing, and* $\tilde{\phi}_\tau(0) = 0$;

(iii) $\bar{\rho}(x, y) + \sum_{\tau=1}^{M} \tilde{\rho}_\tau(x, y) > 0$;

(c) (i) $\xi \to \Psi(\xi, y, u, v, \bar{t}, \bar{s})$ *is prestrictly* $(\mathscr{F}, \beta, \bar{\phi}, \bar{\rho}, \theta)$*-quasisounivex at* y, $\bar{\phi}(a)$ *is strictly increasing, and* $\bar{\phi}(0) = 0$;

(ii) *for each* $\tau \in \underline{M}, \xi \to \Lambda_\tau(\xi, v, \bar{t}, \bar{s})$ *is strictly* $(\mathscr{F}, \beta, \tilde{\phi}_\tau, \tilde{\rho}_\tau, \theta)$*-pseudosounivex at* $y, \tilde{\phi}_\tau$ *is increasing, and* $\tilde{\phi}_\tau(0) = 0$;

(iii) $\bar{\rho}(x, y) + \sum_{\tau=1}^{M} \tilde{\rho}_\tau(x, y) \geq 0$;

(d) (i) $\xi \to \Psi(\xi, y, u, v, \bar{t}, \bar{s})$ *is prestrictly* $(\mathscr{F}, \beta, \bar{\phi}, \bar{\rho}, \theta)$*-quasisounivex at* y, $\bar{\phi}(a)$ *is strictly increasing, and* $\bar{\phi}(0) = 0$;

(ii) *for each* $\tau \in \underline{M_1}, \xi \to \Lambda_\tau(\xi, v, \bar{t}, \bar{s})$ *is* $(\mathscr{F}, \beta, \tilde{\phi}_\tau, \tilde{\rho}_\tau, \theta)$*-quasisounivex at* y, *for each* $\tau \in \underline{M_2} \neq \emptyset, \xi \to \Lambda_\tau(\xi, v, \bar{t}, \bar{s})$ *is strictly* $(\mathscr{F}, \beta, \tilde{\phi}_\tau, \tilde{\rho}_\tau, \theta)$*-pseudosounivex at* y, *and for each* $\tau \in \underline{M}, \tilde{\phi}_\tau$ *is increasing and* $\tilde{\phi}_\tau(0) = 0$, *where* $\{\underline{M_1}, \underline{M_2}\}$ *is a partition of* $\underline{M}$;

(iii) $\bar{\rho}(x, y) + \sum_{\tau=1}^{M} \tilde{\rho}_\tau(x, y) \geq 0$.

Then $\varphi(x) \geq \psi_3(w)$, *where* ψ_3 *is the objective function of* (*DIII*).

Proof. (a): It is clear that (5.1) can be expressed as follows:

$$\mathscr{F}\Bigg(x, y; \beta(x, y)\Bigg\{\sum_{i=1}^{p} u_i\Bigg\{\Gamma(y, u)\Bigg[\nabla f_i(y) + \sum_{m \in J_0} v_m \nabla G_{j_m}(y, t^m) + \sum_{m \in K_0} v_m \nabla H_{k_m}(y, s^m)\Bigg]$$
$$- [\Phi(y, u) + \Lambda_0(y, v, \bar{t}, \bar{s})]\nabla g_i(y)\Bigg\}$$
$$+ \Bigg\{\sum_{i=1}^{p} u_i\Bigg\{\Gamma(y, u)\Bigg[\nabla^2 f_i(y) + \sum_{m \in J_0} v_m \nabla^2 G_{j_m}(y, t^m) + \sum_{m \in K_0} v_m \nabla^2 H_{k_m}(y, s^m)\Bigg]$$
$$- [\Phi(y, u) + \Lambda_0(y, v, \bar{t}, \bar{s})]\nabla^2 g_i(y)\Bigg\}z\Bigg\}\Bigg)$$
$$+ \mathscr{F}\Bigg(x, y; \beta(x, y)\sum_{\tau=1}^{M}\Bigg\{\sum_{m \in J_\tau} v_m \nabla G_{j_m}(y, t^m) + \sum_{m \in K_\tau} v_m \nabla H_{k_m}(y, s^m)$$
$$+ \Bigg[\sum_{m \in J_\tau} v_m \nabla^2 G_{j_m}(y, t^m) + \sum_{m \in K_\tau} v_m \nabla^2 H_{k_m}(y, s^m)\Bigg]z\Bigg\}\Bigg) \geq 0. \tag{5.4}$$

Since for each $\tau \in \underline{M}$,

$$\begin{aligned}\Lambda_\tau(x, v, \bar{t}, \bar{s}) &= \sum_{m\in J_\tau} v_m G_{j_m}(x, t^m) + \sum_{m\in K_\tau} v_m H_{k_m}(x, s^m)\\ &\leq 0 \text{ (by the primal feasibility of x and positivity of } v_m, m \in \underline{v_0})\\ &\leq \Lambda_\tau(y, v, \bar{t}, \bar{s}) - \frac{1}{2}\langle z, \nabla^2\Lambda_\tau(y, v, \bar{t}, \bar{s})z\rangle \quad \text{(by (5.3))},\end{aligned}$$

and, hence,

$$\tilde{\phi}_\tau(\Lambda_\tau(x, v, \bar{t}, \bar{s}) - \Lambda_\tau(y, v, \bar{t}, \bar{s}) + \frac{1}{2}\langle z, \nabla^2\Lambda_\tau(y, v, \bar{t}, \bar{s})z\rangle) \leq 0,$$

it follows from (ii) that

$$\begin{aligned}&\mathscr{F}\Bigg(x, y; \beta(x, y)\Bigg\{\sum_{m\in J_\tau} v_m \nabla G_{j_m}(y, t^m) + \sum_{m\in K_\tau} v_m \nabla H_{k_m}(y, s^m)\\ &\quad + \Bigg[\sum_{m\in J_\tau} v_m \nabla^2 G_{j_m}(y, t^m) + \sum_{m\in K_\tau} v_m \nabla^2 H_{k_m}(y, s^m)\Bigg]z\Bigg\}\Bigg) \leq -\tilde{\rho}_\tau(x, y)\|\theta(x, y)\|^2.\end{aligned}$$

Summing over $\tau \in \underline{M}$ and using the sublinearity of $\mathscr{F}(x, y; \cdot)$, we obtain

$$\begin{aligned}&\mathscr{F}\Bigg(x, y; \beta(x, y)\sum_{\tau=1}^{M}\Bigg\{\sum_{m\in J_\tau} v_m \nabla G_{j_m}(y, t^m) + \sum_{m\in K_\tau} v_m \nabla H_{k_m}(y, s^m)\\ &\quad + \Bigg[\sum_{m\in J_\tau} v_m \nabla^2 G_{j_m}(y, t^m) + \sum_{m\in K_\tau} v_m \nabla^2 H_{k_m}(y, s^m)\Bigg]z\Bigg\}\Bigg) \leq -\sum_{\tau=1}^{M}\tilde{\rho}_\tau(x, y)\|\theta(x, y)\|^2.\end{aligned}$$

Combining this inequality with (5.4), and using (iii), we get

$$\begin{aligned}&\mathscr{F}\Bigg(x, y; \beta(x, y)\Bigg\{\sum_{i=1}^{p} u_i\Bigg\{\Gamma(y, u)\Bigg[\nabla f_i(y) + \sum_{m\in J_0} v_m \nabla G_{j_m}(y, t^m) + \sum_{m\in K_0} v_m \nabla H_{k_m}(y, s^m)\Bigg]\\ &\qquad - [\Phi(y, u) + \Lambda_0(y, v, \bar{t}, \bar{s})]\nabla g_i(y)\Bigg\}\\ &\quad + \sum_{i=1}^{p} u_i\Bigg\{\Gamma(y, u)\Bigg[\nabla^2 f_i(y) + \sum_{m\in J_0} v_m \nabla^2 G_{j_m}(y, t^m) + \sum_{m\in K_0} v_m \nabla^2 H_{k_m}(y, s^m)\Bigg]\\ &\qquad - [\Phi(y, u) + \Lambda_0(y, v, \bar{t}, \bar{s})]\nabla^2 g_i(y)\Bigg\}z\Bigg\}\Bigg)\\ &\quad \geq \sum_{\tau=1}^{M}\tilde{\rho}_\tau(x, y)\|\theta(x, y)\|^2 \geq -\bar{\rho}(x, y)\|\theta(x, y)\|^2,\end{aligned}$$

which by virtue of (i) implies that

$$\bar{\phi}(\Psi(x, y, u, v, \bar{t}, \bar{s}) - \Psi(y, y, u, v, \lambda, \bar{t}, \bar{s}) + \frac{1}{2}\langle z, \nabla^2\Psi(y, y, u, v, \bar{t}, \bar{s})z\rangle) \geq 0.$$

But $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$, and hence this inequality becomes

$$\Psi(x, y, u, v, \bar{t}, \bar{s}) \geq \Psi(y, y, u, v, \bar{t}, \bar{s}) - \frac{1}{2}\langle z, \nabla^2\Psi(y, y, u, v, \bar{t}, \bar{s})z\rangle \geq 0,$$

where the second inequality follows from the dual feasibility of w and (5.2). In view of the primal feasibility of x, the above inequality reduces to (4.8), which in turn leads to the desired conclusion that $\varphi(x) \geq \psi_3(w)$.

(b): The proof is similar to that of part (a).

(c): Suppose to the contrary that $\varphi(x) < \psi_3(w)$. This implies that for each $i \in \underline{p}$,

$$\Gamma(x, y)f_i(x) - [\Phi(x, y) + \Lambda_0(y, v, \bar{t}, \bar{s})]g_i(x) < 0. \tag{5.5}$$

Keeping in mind that $v_m > 0$ for each $m \in \underline{v_0}$, we have

$$\begin{aligned}
&\Psi(x, y, u, v, \bar{t}, \bar{s}) \\
&\quad = \sum_{i=1}^{p} u_i \Bigg\{\Gamma(y, u)\Bigg[f_i(x) + \sum_{m \in J_0} v_m G_{j_m}(x, t^m) + \sum_{m \in K_0} v_m H_{k_m}(x, s^m)\Bigg] \\
&\qquad\qquad - [\Phi(y, u) + \Lambda_0(y, v, \bar{t}, \bar{s})]g_i(x)\Bigg\}, \\
&\quad \leq \sum_{i=1}^{p} u_i\{\Gamma(y, u)f_i(x) - [\Phi(y, u) + \Lambda_0(y, v, \bar{t}, \bar{s})]g_i(x)\} \\
&\qquad \text{(by the primal feasibility of } x) < 0 \text{ (by (5.5))} \\
&\quad \leq \Psi(y, y, u, v, \bar{t}, \bar{s}) - \frac{1}{2}\langle z, \nabla^2\Psi(y, y, u, v, \bar{t}, \bar{s})z\rangle \text{ (by (5.2))}
\end{aligned}$$

and, hence,

$$\bar{\phi}(\Psi(x, y, u, v, \bar{t}, \bar{s}) - \Psi(y, y, u, v, \bar{t}, \bar{s}) + \frac{1}{2}\langle z, \nabla^2\Psi(y, y, u, v, \bar{t}, \bar{s})z\rangle) < 0,$$

which, in view of (i), implies that

$$\begin{aligned}
\mathscr{F}\Bigg(x, y; \beta(x, y)\Bigg\{\sum_{i=1}^{p} u_i\Bigg\{\Gamma(y, u)\Bigg[\nabla f_i(y) + \sum_{m \in J_0} v_m \nabla G_{j_m}(y, t^m) + \sum_{m \in K_0} v_m \nabla H_{k_m}(y, s^m)\Bigg] \\
- [\Phi(y, u) + \Lambda_0(y, v, \bar{t}, \bar{s})]\nabla g_i(y)\Bigg\}
\end{aligned}$$

$$+\sum_{i=1}^{p} u_i\left\{\Gamma(y,u)\left[\nabla^2 f_i(y)+\sum_{m\in J_0} v_m\nabla^2 G_{j_m}(y,t^m)+\sum_{m\in K_0} v_m\nabla^2 H_{k_m}(y,s^m)\right]\right.$$
$$\left.\left.\left.-[\Phi(y,u)+\Lambda_0(y,v,\bar{t},\bar{s})]\nabla^2 g_i(y)\right\}z\right\}\right)\leq -\bar{\rho}(x,y)\|\theta(x,y)\|^2. \qquad (5.6)$$

Proceeding as in the proof of part (a), our assumptions in (ii) lead to

$$\mathscr{F}\left(x,y;\beta(x,y)\left\{\sum_{\tau=1}^{M}\left\{\sum_{m\in J_\tau} v_m\nabla G_{j_m}(y,t^m)+\sum_{m\in K_\tau} v_m\nabla H_{k_m}(y,s^m)\right.\right.\right.$$
$$\left.\left.\left.+\left[\sum_{m\in J_\tau} v_m\nabla^2 G_{j_m}(y,t^m)+\sum_{m\in K_\tau} v_m\nabla^2 H_{k_m}(y,s^m)\right]z\right\}\right\}\right)<-\sum_{\tau=1}^{M}\tilde{\rho}_\tau(x,y)\|\theta(x,y)\|^2,$$

which, when combined with (5.4), gives the following strict inequality:

$$\mathscr{F}\left(x,y;\beta(x,y)\left\{\sum_{i=1}^{p} u_i\left\{\Gamma(y,u)\left[\nabla f_i(y)+\sum_{m\in J_0} v_m\nabla G_{j_m}(y,t^m)+\sum_{m\in K_0} v_m\nabla H_{k_m}(y,s^m)\right]\right.\right.\right.$$
$$\left.-[\Phi(y,u)+\Lambda_0(y,v,\bar{t},\bar{s})]\nabla g_i(y)\right\}$$
$$+\sum_{i=1}^{p} u_i\left\{\Gamma(y,u)\left[\nabla^2 f_i(y)+\sum_{m\in J_0} v_m\nabla^2 G_{j_m}(y,t^m)+\sum_{m\in K_0} v_m\nabla^2 H_{k_m}(y,s^m)\right]\right.$$
$$\left.\left.\left.-[\Phi(y,u)+\Lambda_0(y,v,\bar{t},\bar{s})]\nabla^2 g_i(y)\right\}z\right\}\right)>\sum_{\tau=1}^{M}\tilde{\rho}_\tau(x,y)\|\theta(x,y)\|^2.$$

In view of (iii), this inequality contradicts (5.6). Hence, we conclude that $\varphi(x)\geq\psi_3(w)$.

(d): The proof is similar to that of part (c). □

Theorem 5.2 (Strong Duality). *Let x^* be a normal optimal solution of (P) and assume that any one of the four sets of conditions set forth in Theorem 5.1 is satisfied for all feasible solutions of (DIII). Then there exist $u^*, v^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^*\backslash\nu_0^*}, \bar{t}^*$, and $\bar{s}^*$ such that $w^*\equiv(x^*, z^*=0, u^*, v^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^*\backslash\nu_0^*}, \bar{t}^*, \bar{s}^*)$ is an optimal solution of (DIII) and $\varphi(x^*)=\psi_3(x^*)$.*

Proof. Since x^* is a normal optimal solution of (P), by Theorem 2.1, there exist $u^*\in U$ and integers ν_0 and ν, with $0\leq\nu_0\leq\nu\leq n+1$, such that there exist ν_0 indices j_m, with $1\leq j_m\leq q$, together with ν_0 points $t^m\in\widehat{T}_{j_m}(x^*)$, $m\in\underline{\nu_0}$, $\nu-\nu_0$ indices k_m, with $1\leq k_m\leq r$, together with $\nu-\nu_0$

points $s^m \in S_{k_m}$ for $m \in \underline{v}\backslash\underline{v_0}$, and v real numbers $\bar{v}_m$ with $\bar{v}_m > 0$ for $m \in \underline{v_0}$, with the property that

$$\sum_{i=1}^{p} u_i^*[\Gamma(x^*, u^*)\nabla f_i(x^*) - \Phi(x^*, u^*)\nabla g_i(x^*)] + \sum_{m=1}^{v_0} \bar{v}_m \nabla G_{j_m}(x^*, t^m) + \sum_{m=v_0+1}^{v} \bar{v}_m \nabla H_{k_m}(x^*, s^m) = 0, \tag{5.7}$$

$$u_i^*[\Gamma(x^*, u^*)f_i(x^*) - \Phi(x^*, u^*)g_i(x^*)] = 0, \quad i \in \underline{p}, \tag{5.8}$$

$$\max_{1\leq i\leq p} \frac{f_i(x^*)}{g_i(x^*)} = \frac{\Phi(x^*, u^*)}{\Gamma(x^*, u^*)}. \tag{5.9}$$

Since $\Lambda(x^*, \bar{v}, \bar{t}, \bar{s}) = 0$, (5.7) and (5.8) can be rewritten as follows:

$$\sum_{i=1}^{p} u_i^* \Bigg\{ \Gamma(x^*, u^*) \Bigg[\nabla f_i(x^*) + \sum_{m\in J_0} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m\in K_0} v_m^* \nabla H_{k_m}(x^*, s^m) \Bigg] - [\Phi(x^*, u^*) + \Lambda_0(x^*, v^*, \bar{t}, \bar{s})]\nabla g_i(x^*) \Bigg\} + \sum_{m\in \underline{v_0^*}\backslash J_0} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m\in(\underline{v^*}\backslash\underline{v_0^*})\backslash K_0} v_m^* \nabla H_{k_m}(x^*, s^m) = 0, \tag{5.10}$$

$$u_i^*\{\Gamma(x^*, u^*)f_i(x^*) - [\Phi(x^*, u^*) + \Lambda_0(x^*, v^*, \bar{t}, \bar{s})]g_i(x^*)\} = 0, \quad i \in \underline{p}, \tag{5.11}$$

where $v_m^* = \bar{v}_m/\Gamma(x^*, u^*)$ for each $m \in J_0$, $v_m^* = \bar{v}_m$ for each $m \in \underline{v_0^*}\backslash J_0$, $v_m^* = \bar{v}_m/\Gamma(x^*, u^*)$ for each $m \in K_0$, and $v_m^* = \bar{v}_m$ for each $m \in (\underline{v^*}\backslash\underline{v_0^*})\backslash K_0$. Inasmuch as $x^* \in \mathbb{F}$ and, hence,

$$\sum_{m=1}^{v_0} v_m^* G_{j_m}(x^*, t^m) + \sum_{m=v_0+1}^{v} v_m^* H_{k_m}(x^*, s^m) = 0,$$

from (5.9)–(5.11) it is clear that w^* is a feasible solution of (*DIII*) and $\varphi(x^*) = \psi_3(w^*)$. If w^* were not optimal, then there would exist a feasible solution $\tilde{w} \equiv (\tilde{x}, \tilde{z}, \tilde{u}, \tilde{v}, \tilde{\lambda}, \tilde{v}, \tilde{v_0}, J_{\tilde{v}_0}, K_{\tilde{v}\backslash\tilde{v}_0}, \bar{\bar{t}}, \bar{\bar{s}})$ of (*DIII*) such that $\psi_3(\tilde{w}) > \psi_3(w^*) = \varphi(x^*)$, contradicting Theorem 5.1. Therefore, w^* is an optimal solution of (*DIII*). □

Theorem 5.3 (Strict Converse Duality). *Let x^* be a normal optimal solution of (P), let $\tilde{w} \equiv (\tilde{x}, \tilde{z}, \tilde{u}, \tilde{v}, \tilde{v}, \tilde{v_0}, J_{\tilde{v}_0}, K_{\tilde{v}\backslash\tilde{v}_0}, \bar{\bar{t}}, \bar{\bar{s}})$ be an optimal solution of (DIII), and assume that any one of the following four sets of conditions holds:*

(a) *The assumptions specified in part (a) of Theorem* 5.1 *are satisfied for the feasible solution $\tilde{w}$ of (DIII), $\bar{\phi}(a) > 0 \Rightarrow a > 0$, and the function $\xi \to \Psi(\xi, \tilde{x}, \tilde{u}, \tilde{v}, \bar{\bar{t}}, \bar{\bar{s}})$ is strictly $(\mathscr{F}, \beta, \bar{\phi}, \bar{\rho}, \theta)$-pseudosounivex at $\tilde{x}$.*

(b) *The assumptions specified in part (b) of Theorem* 5.1 *are satisfied for the feasible solution* $\tilde{w}$ *of (DIII),* $\bar{\phi}(a) > 0 \Rightarrow a > 0$, *and the function* $\xi \to \Psi(\xi, \tilde{x}, \tilde{u}, \tilde{v}, \bar{\tilde{t}}, \bar{\tilde{s}})$ *is* $(\mathscr{F}, \beta, \bar{\phi}, \bar{\rho}, \theta)$*-quasisounivex at* $\tilde{x}$.

(c) *The assumptions specified in part (c) of Theorem* 5.1 *are satisfied for the feasible solution* $\tilde{w}$ *of (DIII),* $\bar{\phi}(a) > 0 \Rightarrow a > 0$, *and the function* $\xi \to \Psi(\xi, \tilde{x}, \tilde{u}, \tilde{v}, \bar{\tilde{t}}, \bar{\tilde{s}})$ *is* $(\mathscr{F}, \beta, \bar{\phi}, \bar{\rho}, \theta)$*-quasisounivex at* $\tilde{x}$.

(d) *The assumptions specified in part (d) of Theorem* 5.1 *are satisfied for the feasible solution* $\tilde{w}$ *of (DIII),* $\bar{\phi}(a) > 0 \Rightarrow a > 0$, *and the function* $\xi \to \Psi(\xi, \tilde{x}, \tilde{u}, \tilde{v}, \bar{\tilde{t}}, \bar{\tilde{s}})$ *is* $(\mathscr{F}, \beta, \bar{\phi}, \bar{\rho}, \theta)$*-quasisounivex at* $\tilde{x}$.

Then $\tilde{x} = x^*$ *and* $\varphi(x^*) = \psi_3(\tilde{e})$.

Proof. Since x^* is a normal optimal solution of (P), by Theorem 2.1, there exist $u^*, v^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^* \setminus \nu_0^*}, \bar{t}^*$, and $\bar{s}^*$ such that $w^* \equiv (x^*, z^* = 0, u^*, v^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu^* \setminus \nu_0^*}, \bar{t}^*, \bar{s}^*)$ is a feasible solution of $(DIII)$ and $\varphi(x^*) = \psi_3(w^*)$.

(a): Suppose to the contrary that $\tilde{x} \neq x^*$. Now proceeding as in the proof of part (a) of Theorem 5.1 (with x replaced by x^* and w by $\tilde{w}$), we arrive at the inequality

$$\mathscr{F}\Bigg(x^*, \tilde{x}; \beta(x^*, \tilde{x})\Bigg\{\sum_{i=1}^{p}\tilde{u}_i\Bigg\{\Gamma(\tilde{x}, \tilde{u})\Bigg[\nabla f_i(\tilde{x}) + \sum_{m\in J_0}\tilde{v}_m \nabla G_{j_m}(\tilde{x}, \tilde{t}^m) + \sum_{m\in K_0}\tilde{v}_m \nabla H_{k_m}(\tilde{x}, \tilde{s}^m)\Bigg]$$
$$- [\Phi(\tilde{x}, \tilde{u}) + \Lambda_0(\tilde{x}, \tilde{v}, \bar{\tilde{t}}, \bar{\tilde{s}})]\nabla g_i(\tilde{x})\Bigg\}$$
$$+ \sum_{i=1}^{p} \tilde{u}_i\Bigg\{\Gamma(\tilde{x}, \tilde{u})\Bigg[\nabla^2 f_i(\tilde{x}) + \sum_{m\in J_0} \tilde{v}_m \nabla^2 G_{j_m}(\tilde{x}, \tilde{t}^m) + \sum_{m\in K_0} \tilde{v}_m \nabla^2 H_{k_m}(\tilde{x}, \tilde{s}^m)\Bigg]$$
$$- [\Phi(\tilde{x}, \tilde{u}) + \Lambda_0(\tilde{x}, \tilde{v}, \bar{\tilde{t}}, \bar{\tilde{s}})]\nabla^2 g_i(\tilde{x})\Bigg\}\tilde{z}\Bigg\}\Bigg) \geq -\bar{\rho}(x^*, \tilde{x})\|\theta(x^*, \tilde{x})\|^2,$$

which, in view of our strict $(\mathscr{F}, \beta, \bar{\phi}, \bar{\rho}, \theta)$-pseudosounivexity hypothesis, implies that

$$\bar{\phi}\left(\Psi(x^*, \tilde{x}, \tilde{u}, \tilde{v}, \bar{\tilde{t}}, \bar{\tilde{s}}) - \Psi(\tilde{x}, \tilde{x}, \tilde{u}, \tilde{v}, \bar{\tilde{t}}, \bar{\tilde{s}}) + \frac{1}{2}\langle \tilde{z}, \nabla^2 \Psi(\tilde{x}, \tilde{x}, \tilde{u}, \tilde{v}, \bar{\tilde{t}}, \bar{\tilde{s}})\tilde{z}\rangle\right) > 0.$$

But $\bar{\phi}(a) > 0 \Rightarrow a > 0$, and, hence, this inequality becomes

$$\Psi(x^*, \tilde{x}, \tilde{u}, \tilde{v}, \bar{\tilde{t}}, \bar{\tilde{s}}) > \Psi(\tilde{x}, \tilde{x}, \tilde{u}, \tilde{v}, \bar{\tilde{t}}, \bar{\tilde{s}}) - \frac{1}{2}\langle \tilde{z}, \nabla^2 \Psi(\tilde{x}, \tilde{x}, \tilde{u}, \tilde{v}, \bar{\tilde{t}}, \bar{\tilde{s}})\tilde{z}\rangle \geq 0,$$

where the second inequality follows from the dual feasibility of $\tilde{w}$ and (5.2). In view of the primal feasibility of x^*, the above inequality reduces to

$$\sum_{i=1}^{p} \tilde{u}_i\{\Gamma(\tilde{x}, \tilde{u})f_i(x^*) - [\Phi(\tilde{x}, \tilde{u}) + \Lambda_0(\tilde{x}, \tilde{v}, \bar{\tilde{t}}, \bar{\tilde{s}})]g_i(x^*)\} > 0.$$

Using this inequality along with Lemma 3.1, as in the proof of Theorem 4.1, we get $\varphi(x^*) > \psi_3(\tilde{w})$ which contradicts the fact that $\varphi(x^*) = \psi_3(w^*) \leq \psi_3(\tilde{w})$.

(b)–(d): The proofs are similar to that of part (a). □

As pointed out earlier, the duality models $(DIII)$ and $(\widetilde{D}III)$ can be viewed as two families of dual problems for (P) whose members can easily be identified by appropriate choices of the partitioning sets J_μ and K_μ, $\mu \in \underline{M} \cup \{0\}$. These two families contain a vast number of interesting and important dual problems for (P), which include various dual problems in conventional nonlinear programming, fractional programming, and minmax programming problems. It appears that all these dual problems and the corresponding duality theorems are new in the area of semi-infinite programming.

6. CONCLUDING REMARKS

Using a direct nonparametric approach, in this article we have formulated six second-order dual problems and proved appropriate duality theorems under a variety of generalized $(\mathscr{F}, \beta, \phi, \rho, \theta)$-sounivexity conditions for a discrete minmax fractional programming problem. Since each one of these six dual problems and the related duality results can be modified and restated for each one of the seven special cases of the prototype problem (P) designated as $(P1)$–$(P7)$ in Section 1, evidently they provide a fairly large number of second-order duality results for several classes of semi-infinite as well as conventional nonlinear programming problems. Moreover, the methods used in this article can be used as a guide to extend the results to other classes of mathematical programming problems. For example, employing similar techniques, one can investigate the second-order duality aspects of the following semi-infinite multiobjective programming problem:

$$\text{Minimize } \varphi(x) = (\varphi_1(x), \ldots, \varphi_p(x)) = \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\right)$$

subject to

$$
\begin{aligned}
G_j(x,t) \leq 0 \quad & \text{for all } t \in T_j, \ \ j \in \underline{q}, \\
H_k(x,s) = 0 \quad & \text{for all } s \in S_k, \ \ k \in \underline{r}, \\
& x \in X.
\end{aligned}
$$

We shall investigate this and some other related problems in the future.

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