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FULL LENGTH PAPER

On semidefinite programming relaxations of maximum k-section

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Abstract We derive a new semidefinite programming bound for the maximum k-section problem. For $k = 2$ (i.e. for maximum bisection), the new bound is at least as strong as a well-known bound by Poljak and Rendl (SIAM J Optim 5(3):467–487, 1995). For $k \geq 3$ the new bound dominates a bound of Karisch and Rendl (Topics in semidefinite and interior-point methods, 1998). The new bound is derived from a recent semidefinite programming bound by De Klerk and Sotirov for the more general quadratic assignment problem, but only requires the solution of a much smaller semidefinite program.

Keywords Maximum bisection · Maximum section · Semidefinite programming · Coherent configurations · Strongly regular graph

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1 Introduction

A k-section (or k-equipartition) of a (weighted) graph is a partition of the vertex set of the graph into k sets of equal cardinality. The weight (or cost) of a k-section is the sum of the weights of all edges that connect vertices in different sets of the partition. Thus the maximum (resp. minimum) k-section problem is to find a k-section of maximum (resp. minimum) weight in a given weighted graph.

An equivalent formulation that will be useful to us is as follows. Let

$$K_{\underbrace{m,\ldots,m}_{k_{\text{times}}}}$$

denote a complete multipartite graph with k color classes all of size m. The maximum (resp. minimum) k-section problem is to find a $K_{m,\ldots,m}$ subgraph of maximum (resp. minimum) weight in a given weighted, complete graph on $|V| = mk$ vertices.

The maximum k-section problem is NP-hard for $k \geq 2$ [8]. For *maximum bisection* ($k = 2$), a polynomial-time approximation ratio of 0.7016 is known [14] (see also [7] and [26]). In other words, the randomized algorithm in [14] generates a bisection of the graph of expected weight at least 0.7016 times that of a maximum bisection.

Andersson [3] proposed an $1 - 1/k + ck^3$-approximation algorithm for maximum k-section (see also [18]), where c is some (unknown) absolute constant.

We also mention that the maximum and minimum k-section problems are different in terms of approximability (although both are NP-hard).

All the above mentioned approximation results involve semidefinite programming (SDP) relaxations. In this paper we therefore revisit SDP relaxations for max k-section, and establish relationships between several SDP bounds from the literature.

In particular we propose a new SDP bound for the maximum (or minimum) k-section problem, obtained from an SDP bound of the more general quadratic assignment problem as proposed by De Klerk and Sotirov [4]. We show that the new relaxation is at least as good as the relaxation due to Poljak and Rendl [20] for $k = 2$ (maximum bisection). For $k \geq 3$, we prove it is at least as good as a bound introduced by Karisch and Rendl [18]. Moreover, the computation of the new SDP bound may be done much more efficiently than that of the general bound of De Klerk and Sotirov in [4], since it only requires the solution of a much smaller semidefinite program.

1.1 Outline

This paper is structured as follows. After a summary of notation we review some known SDP relaxations of max k-section (Sect. 2). Subsequently, we review how max k-section may be reformulated as a quadratic assignment problem (QAP), and review some SDP relaxations of QAP problems in Sect. 3. The SDP relaxations of QAP lead to large relaxations of max k-section, and to reduce the size of these SDP problems one has to exploit algebraic symmetry. The necessary algebraic background for this is given in Sect. 4. In Sect. 5 we derive the new SDP bound for max k-section from the QAP relaxation, by performing symmetry reduction. Theoretical comparisons with existing bounds are carried out in Sect. 6, and in Sect. 7 we show how certain SDP bounds simplify for strongly regular graphs. Finally, numerical examples are presented in Sect. 8.

1.2 Notation

The space of $p \times q$ matrices is denoted by $\mathbb{R}^{p\times q}$, the space of $k \times k$ symmetric matrices is denoted by $\mathbb{S}^{k\times k}$. For index sets $\alpha, \beta \subset \{1, \ldots, n\}$, we denote the submatrix that contains the rows of A indexed by α and the columns indexed by β as $A(\alpha, \beta)$. If $\alpha = \beta$, the principal submatrix $A(\alpha, \alpha)$ of A is abbreviated as $A(\alpha)$. The ith row of a matrix C we denote by $C(i, :)$.

We use I_n to denote the identity matrix of order n, and J_n the $n \times n$ all-ones matrix. We omit the subscript if the order is clear from the context. Also, $E_{ij} = e_i e_j^T$ where e_i is the i-th standard basis vector. The all ones vector will be denoted by e.

The vec operator stacks the columns of a matrix, while the Diag operator maps an n-vector to an $n \times n$ diagonal matrix in the obvious way. Similarly, $\text{diag}(A)$ denotes the vector obtained by extracting the diagonal of a square matrix A. The set of $n \times n$ permutations matrices is denoted by Π_n.

The *Kronecker product* $A \otimes B$ of matrices $A \in \mathbb{R}^{p\times q}$ and $B \in \mathbb{R}^{r\times s}$ is defined as the $pr \times qs$ matrix composed of pq blocks of size $r \times s$, with block ij given by $a_{ij}B$ where $i = 1, \ldots, p$ and $j = 1, \ldots, q$.

The following properties of the Kronecker product will be used in the paper, see e.g. [12] (we assume that the dimensions of the matrices appearing in these identities are such that all expressions are well defined):

$$(A \otimes B)(C \otimes D) = AC \otimes BD \tag{1}$$

$$\text{trace}(A \otimes B) = \text{trace}(A)\text{trace}(B). \tag{2}$$

Moreover, following the notation we just introduced, it can be easily verified that for any column vectors $v, w \in \mathbb{R}^n$:

$$\text{Diag}(\text{vec}(vw^T)) = \text{Diag}(w) \otimes \text{Diag}(v). \tag{3}$$

2 Some known SDP relaxations of maximum *k*-section

The following SDP relaxation of the maximum bisection problem on a graph on $|V| = 2m$ vertices and with edge weight matrix W is due to Poljak and Rendl [20] (see also Frieze and Jerrum [7] and Ye [26]):

$$\max \left\{ \frac{1}{4}\text{trace}(W(J_{2m} - X)) \mid \text{diag}(X) = e,\; Xe = 0,\; X \succeq 0 \right\}. \tag{4}$$

To see that this is a relaxation of the maximum bisection problem, set $X = vv^T$, where $v \in \{-1, 1\}^{2m}$ gives the optimal bisection of the vertex set.

An SDP relaxation of the max k-section problem (here $|V| = km$) due to Karisch and Rendl [18] (see also Wolkowicz and Zhao [27]) is the following:

$$\max \left\{ \frac{1}{2}\text{trace}(W(J_{km} - X)) \mid \text{diag}(X) = e,\ Xe = me,\ X \succeq 0,\ X \geq 0 \right\}. \quad (5)$$

Throughout the paper we will refer to the SDP relaxation in (5) as $k - GP_{R2}$, the name given in [18].

The two relaxations (4) and (5) give the same bound if $k = 2$ (for maximum bisection). We include a proof for completeness.

Theorem 1 *If $k = 2$, the relaxation $2 - GP_{R2}$ is equivalent to the Frieze-Jerrum relaxation from* (4).

Proof Given an optimal solution X of (4), set $\overline{X} := \frac{1}{2}(J_{2m} + X)$. Note that $\overline{X} \geq 0$, since $X \geq -J_{2m}$ as implied by the constraints in (4). Also, $\text{diag}(\overline{X}) = e$ and $\overline{X} \succeq 0$. Moreover, since $Xe = 0$ we have $\overline{X}e = \frac{1}{2}J_{2m}e + Xe = me$. It is straightforward (by construction) to see that the two objective values are equal.

Conversely, assume that X is feasible for (5) and set $\overline{X} := 2X - J_{2m}$. We have $\text{diag}(\overline{X}) = e$ and $\overline{X}e = 2Xe - J_{2m}e = 2me - 2me = 0$.

Since $X \succeq 0$ we have $\lambda_{\min}(X) \geq 0$. Moreover e is an eigenvector of X with corresponding eigenvalue m. From the eigenvalue decomposition of X one has:

$$X = \frac{1}{2}J_{2m} + \sum_{i=2}^{n} \lambda_i q_i q_i^T,$$

where $\lambda_i \geq 0$ and q_i are the eigenvalues and eigenvectors of X respectively. It follows that $\overline{X} \equiv 2X - J_{2m} \succeq 0$. It is also easy to see that the two objectives coincide, by construction. □

2.1 Additional inequalities and stronger relaxations

The so-called MET inequalities model the fact that, if vertices i and j belong to the same cluster in an equipartition, and a vertex k belongs to the same cluster as i, then j and k should also belong to the same cluster. It is useful to note that the entry X_{ij} in the matrix variable in (5) is the relaxation of a binary variable that is one if vertices i and j belong to the same partition of the k-section, and zero otherwise.

Defining the MET polyhedron as

$$\text{MET} = \left\{ X \in \mathbb{S}^{n \times n} \ : \ X_{ij} + X_{ij} \leq 1 + X_{jk} \quad \forall i, j, k \in \{1, \ldots, n\} \right\},$$

one may therefore add the constraint $X \in \text{MET}$ to problem $k - GP_{R2}$. We will refer to the resulting SDP relaxation as SDP-MET, the name used by Lisser and Rendl [19].

The authors of [19] also studied the weaker linear programming relaxation, where the constraint $X \succeq 0$ is dropped from SDP-MET, and the constraint $X \leq J$ is added. (The latter constraint is redundant in SDP-MET.)

Another set of valid inequalities are the so-called independent set inequalities, that model the fact that, in every vertex set of size $k+1$, at least two of the vertices belong to the same partition in a k-section. The corresponding polyhedron is:

$$\text{INDEP} = \left\{ X \in \mathbb{S}^{n\times n} \ : \sum_{i<j,\ i,j\in\mathcal{I}} X_{ij} \geq 1 \ \ \forall \mathcal{I} \subset \{1,\ldots,n\},\ |\mathcal{I}| = k+1 \right\}.$$

The SDP relaxation where the constraint $X \in$ INDEP is added to SDP-MET is called $k - GP_{R3}$ in [18], and we will use the same name.

These (and other) valid inequalities have been used in branch-and-cut schemes for various graph partitioning problems; see Ambruster et al. [2], Ghaddar et al. [10] and Lisser and Rendl [19].

3 Max-*k* section as a quadratic assignment problem

The maximum k-section problem is a special case of the more general quadratic assignment problem (QAP):

$$\min_{X\in\Pi_n} \text{trace}(AX^TBX), \tag{6}$$

where A and B are given symmetric $n \times n$ matrices, and Π_n is the set of $n \times n$ permutation matrices.

To see this, consider the adjacency matrix of $K_{m,\ldots,m}$ (with any fixed labeling of the vertices), e.g.

$$A := (J_k - I_k) \otimes J_m \equiv \begin{pmatrix} 0_m & J_m & \cdots & J_m \\ J_m & 0_m & \ddots & \vdots \\ \vdots & \ddots & \ddots & J_m \\ J_m & \cdots & J_m & 0_m \end{pmatrix} \in \mathbb{S}^{km\times km}. \tag{7}$$

If X is a permutation matrix that defines a re-labeling of the vertices, then the adjacency matrix after re-labeling is X^TAX.

The QAP reformulation of max k-section on a complete graph with vertex set V ($|V| = km$) and matrix of edge weights W is therefore given by:

$$\frac{1}{2}\max_{X\in\Pi_{|V|}} \text{trace}(WX^TAX). \tag{8}$$

3.1 SDP relaxations of QAP

We distinguish between two SDP relaxations of QAP. One of them was studied by Povh and Rendl [21], and the second one by De Klerk and Sotirov [4].

The Povh-Rendl relaxation is as follows:

$$\left.\begin{array}{ll}\min & \operatorname{trace}(B \otimes A)Y \\ \text{s.t.} & \operatorname{trace}(I \otimes E_{jj})Y = 1, \operatorname{trace}(E_{jj} \otimes I)Y = 1 \quad j = 1, \ldots, n \\ & \operatorname{trace}(I \otimes (J - I) + (J - I) \otimes I)Y = 0 \\ & \operatorname{trace}(J_{n^2} Y) = n^2 \\ & Y \geq 0, \quad Y \in \mathbb{S}_+^{n^2 \times n^2}, \end{array}\right\} \tag{9}$$

where $I, J, E_{jj} \in \mathbb{R}^{n \times n}$. One may easily verify that (9) is indeed a relaxation of the QAP (6) by noting that a feasible point of (9) is given by

$$\widetilde{Y} := \operatorname{vec}(X)\operatorname{vec}(X)^T \quad \text{if } X \in \Pi_n,$$

and that the objective value of (9) at this point $\widetilde{Y}$ is precisely $\operatorname{trace}(AXBX^T)$.

The next relaxation for QAP that we discuss is due to De Klerk and Sotirov [4]. For the max k-section QAP (8), the underlying idea of this relaxation is that any given vertex may be assigned to any given partition, without loss of generality.

Formally, let $X \in \Pi_n$, and $r, s \in \{1, \ldots, n\}$ such that $X_{r,s} = 1$. Then let us denote $\alpha = \{1, \ldots, n\} \setminus r$ and $\beta = \{1, \ldots, n\} \setminus s$.

It was proven in [4] that the following SDP problem provides a lower bound for the QAP whenever the automorphism group of one of the data matrices (A or B) is transitive:

$$\left.\begin{array}{ll}\min & \operatorname{trace}(B(\beta) \otimes A(\alpha) + \operatorname{Diag}(\bar{c}))Y \\ \text{s.t.} & \operatorname{trace}(I \otimes E_{jj})Y = 1, \quad \operatorname{trace}(E_{jj} \otimes I)Y = 1 \quad j = 1, \ldots, n-1 \\ & \operatorname{trace}(I \otimes (J - I) + (J - I) \otimes I)Y = 0 \\ & \operatorname{trace}(J_{(n-1)^2} Y) = (n-1)^2 \\ & Y \geq 0, \quad Y \in \mathbb{S}_+^{(n-1)^2 \times (n-1)^2}, \end{array}\right\} \tag{10}$$

where $I, J, E_{jj} \in \mathbb{R}^{(n-1) \times (n-1)}$, and

$$\bar{c} := 2\operatorname{vec}(A(\alpha, \{r\})B(\{s\}, \beta)). \tag{11}$$

Note that this relaxation depends on the choice of r and s. If the automorphism group of A is transitive, then every choice of r yields the same bound. Moreover, the number of possible choices of s depends on the automorphism group of B: there are as many different bounds, as there are orbits of the automorphism group of B; see [4].

Let us now consider a matrix Y with the type of block structure that appears in (9) and (10):

$$Y := \begin{pmatrix} Y^{11} & \ldots & Y^{1p} \\ \vdots & \ddots & \vdots \\ Y^{p1} & \ldots & Y^{pp} \end{pmatrix}, \tag{12}$$

where p is a given integer and $Y^{ij} \in \mathbb{R}^{p \times p}$ for $i, j = 1, \ldots, p$.

Lemma 1 ([21]) *A matrix Y of the form* (12) *that is feasible for* (9) *(resp.* (10)*) satisfies:*

$$trace(Y^{ii}) = 1 \quad i = 1, \dots, p, \tag{13}$$

$$\sum_{i=1}^{p} diag(Y^{ii}) = e, \tag{14}$$

$$e^T Y^{ij} = diag(Y^{jj})^T \; i, \quad j = 1, \dots, p, \tag{15}$$

$$\sum_{i=1}^{p} Y^{ij} = e diag(Y^{jj})^T \quad j = 1, \dots, p, \tag{16}$$

for $p = n$ (resp. $p = n - 1$). Moreover, Y^{jj} is a diagonal matrix for all $j = 1, \dots, p$, and Y^{ij} has a zero diagonal if for all $i, j \in \{1, \dots, p\}$ and $i \neq j$.

In what follows, we will reduce the size of the SDP relaxation (10) for the QAP formulation of max k-section. In doing so, we will exploit the algebraic symmetry of the data matrices, i.e. the symmetry of the graph $K_{m-1,m,\dots,m}$. To this end, we will require some results from algebraic combinatorics. The necessary background is given in the next section.

4 Coherent configurations

Coherent configurations were introduced in [15], and are defined as follows.

Definition 1 (*Coherent configuration*) Assume that a given set of zero-one $n \times n$ matrices $\{A_1, \dots, A_d\}$ has the following properties:

(1) $\sum_{i \in \mathcal{I}} A_i = I$ for some index set $\mathcal{I} \subset \{1, \dots, d\}$ and $\sum_{i=1}^{d} A_i = J$.
(2) $A_i^T \in \{A_1, \dots, A_d\}$ for each i;
(3) $A_i A_j \in \text{span}\{A_1, \dots, A_d\}$ for all i, j.

Then $\{A_1, \dots, A_d\}$ is called a *coherent configuration*.

Thus, a coherent configuration is a basis of zero-one matrices of a (possibly non-commutative) matrix $*$-algebra. Moreover, in general, any matrix $*$-algebra has a canonical block-diagonal structure. This is a consequence of the structural theorem for matrix $*$-algebras. In stating the theorem, we require the following notation for the direct sum of two matrix algebras $\mathcal{A}_1$ and $\mathcal{A}_2$:

$$\mathcal{A}_1 \oplus \mathcal{A}_2 := \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \; A_1 \in \mathcal{A}_1, \; A_2 \in \mathcal{A}_2 \right\}.$$

Theorem 2 (Wedderburn [25]; see also §2.2 in [11]) *If $\mathcal{A} \subset \mathbb{C}^{n \times n}$ is a matrix $*$-algebra, then there exist a unitary matrix U and positive integers p and n_i, t_i $(i = 1, \dots, p)$ such that*

$$U^* \mathcal{A} U = \oplus_{i=1}^{p} t_i \odot \mathbb{C}^{n_i \times n_i}$$

where

$$t_i \odot \mathbb{C}^{n_i \times n_i} := \left\{ I_{t_i} \otimes M \mid M \in \mathbb{C}^{n_i \times n_i} \right\} \quad (i = 1, \ldots, p).$$

Based on the theorem, we define the $*$-isomorphism:

$$\phi : \mathcal{A} \mapsto \oplus_{i=1}^{p} \mathbb{C}^{n_i \times n_i}$$

for later use. We now give two examples of coherent configurations that we will use later for the k-section problem.

Example 1 Consider the coherent configuration associated with the complete bipartite graph $K_{k,\ell}$, where k and ℓ are given integers.

The coherent configuration has dimension 6 and consists of the following matrices:

$$A_1 = \begin{pmatrix} I_k & 0_{k\times\ell} \\ 0_{\ell\times k} & 0_{\ell\times\ell} \end{pmatrix},\ A_2 = \begin{pmatrix} J_k - I_k & 0_{k\times\ell} \\ 0_{\ell\times k} & 0_{\ell\times\ell} \end{pmatrix},\ A_3 = \begin{pmatrix} 0_{k\times k} & J_{k\times\ell} \\ 0_{\ell\times k} & 0_{\ell\times\ell} \end{pmatrix},$$
$$A_4 = \begin{pmatrix} 0_{k\times k} & 0_{k\times\ell} \\ J_{\ell\times k} & 0_{\ell\times\ell} \end{pmatrix},\ A_5 = \begin{pmatrix} 0_{k\times k} & 0_{k\times\ell} \\ 0_{\ell\times k} & I_{\ell\times\ell} \end{pmatrix},\ A_6 = \begin{pmatrix} 0_{k\times k} & 0_{k\times\ell} \\ 0_{\ell\times k} & J_\ell - I_\ell. \end{pmatrix},$$

and its complex span is isomorphic (as a $*$-algebra) to $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^{2\times 2}$. The associated $*$-isomorphism ϕ satisfies:

$$\phi(A_1) = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 1 & 0 \\ & & 0 & 0 \end{pmatrix},\quad \phi(A_2) = \begin{pmatrix} -1 & & & \\ & 0 & & \\ & & k-1 & 0 \\ & & 0 & 0 \end{pmatrix},$$
$$\phi(A_3) = \sqrt{k\ell} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 1 \\ & & 0 & 0 \end{pmatrix},\quad \phi(A_4) = \sqrt{k\ell} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 0 \\ & & 1 & 0 \end{pmatrix},$$
$$\phi(A_5) = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & 0 \\ & & 0 & 1 \end{pmatrix},\quad \phi(A_6) = \begin{pmatrix} 0 & & & \\ & -1 & & \\ & & 0 & 0 \\ & & 0 & \ell-1 \end{pmatrix}.$$

One may verify that ϕ is indeed a $*$-isomorphism, by showing that the multiplication tables of $\{A_1, \ldots, A_6\}$ and $\{\phi(A_1), \ldots, \phi(A_6)\}$ are the same, and that $\phi(A_i^T) = \phi(A_i)^T$ for all $i = 1, \ldots, 6$.

Example 2 Consider the following coherent configuration associated with the complete graph $K_{m-1,m,\ldots,m}$ (i.e. k-partition of cardinality given by indices); where each matrix contains k^2 blocks (dimensions of the blocks are given only for the first matrix, they can be further deduced from the context):

$$A_1 = \begin{pmatrix} I_{m-1} & 0_{m-1\times m} & 0_{m-1\times m} & \cdots & 0_{m-1\times m} \\ 0_{m\times m-1} & 0_{m\times m} & 0_{m\times m} & \cdots & 0_{m\times m} \\ 0_{m\times m-1} & 0_{m\times m} & 0_{m\times m} & \cdots & 0_{m\times m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{m\times m-1} & 0_{m\times m} & 0_{m\times m} & \cdots & 0_{m\times m} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} J-I & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \; A_3 = \begin{pmatrix} 0 & J & J & \cdots & J \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ J & 0 & 0 & \cdots & 0 \\ J & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & 0 & 0 & \cdots & 0 \end{pmatrix}, \; A_5 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \end{pmatrix},$$

$$A_6 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & J-I & 0 & \cdots & 0 \\ 0 & 0 & J-I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J-I \end{pmatrix}, \; A_7 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & J & \cdots & J \\ 0 & J & 0 & \cdots & J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & J & J & \cdots & 0 \end{pmatrix},$$

and its complex span is isomorphic to $\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}^{2\times 2}$. The associated $*$-isomorphism ϕ satisfies:

$$\phi(A_1) = \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & 0 \\ & & & 0 & 0 \end{pmatrix}, \quad \phi(A_2) = \begin{pmatrix} -1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & m-2 & 0 \\ & & & 0 & 0 \end{pmatrix},$$

$$\phi(A_3) = \sqrt{(k-1)m(m-1)} \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & 1 \\ & & & 0 & 0 \end{pmatrix},$$

$$\phi(A_4) = \sqrt{(k-1)m(m-1)} \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & 0 \\ & & & 1 & 0 \end{pmatrix},$$

$$\phi(A_5) = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & 0 \\ & & & 0 & 1 \end{pmatrix}, \quad \phi(A_6) = \begin{pmatrix} 0 & & & & \\ & -1 & & & \\ & & m-1 & & \\ & & & 0 & 0 \\ & & & 0 & m-1 \end{pmatrix},$$

$$\phi(A_7) = m \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & -1 & & \\ & & & 0 & 0 \\ & & & 0 & k-2 \end{pmatrix}.$$

5 The new SDP bound for max k-section

The new SDP bound will be derived from the more general SDP relaxation for QAP that is given in (10). Due to the different structures of the coherent configurations of $K_{m-1,m}$ and $K_{m-1,m,\ldots,m}$, we treat the maximum bisection problem separately from maximum k-section ($k > 2$) problem.

5.1 New SDP relaxation for maximum bisection

We now describe the new SDP relaxation of max bisection where the variables in the relaxation $X_1, \ldots, X_6$ correspond to the matrices $A_1, \ldots, A_6$ respectively from Example 1.

Letting $n = |V| = 2m$,

$$w = [W_{12} \ \ldots \ W_{1n}]^T$$

and

$$\bar{W} = \begin{pmatrix} W_{22} & \ldots & W_{2n} \\ \vdots & & \vdots \\ W_{n2} & & W_{nn} \end{pmatrix}, \tag{17}$$

the new relaxation takes the following form.

$$SDP_{new} := \max \ \text{trace}\,(\text{diag}(w)X_5) + \frac{1}{2}\text{trace}\left(\bar{W}(X_3 + X_4)\right) \tag{18}$$

subject to

$$X_1 + X_5 = I_{n-1}$$
$$\sum_{t=1}^{6} \text{trace}(JX_t) = (n-1)^2$$

$$\operatorname{trace}(X_1) = m - 1$$
$$\operatorname{trace}(X_5) = m$$
$$\operatorname{trace}(X_2 + X_3 + X_4 + X_6) = 0$$
$$X_3 = X_4^T$$
$$\begin{pmatrix} \frac{1}{m-1}(X_1 + X_2) & \frac{1}{\sqrt{m(m-1)}} X_3 \\ \frac{1}{\sqrt{m(m-1)}} X_4 & \frac{1}{m}(X_5 + X_6) \end{pmatrix} \succeq 0$$
$$X_1 - \frac{1}{m-2} X_2 \succeq 0$$
$$X_5 - \frac{1}{m-1} X_6 \succeq 0$$
$$X_i \geq 0 \quad (i = 1, \ldots, 6).$$

Note that the matrix variables X_i are all of order $n - 1$.

With reference to Example 1, the reader may verify that a feasible point of the new relaxation is given by $X_i = A_i$ $(i = 1, \ldots, 6)$ if $k = m - 1$ and $l = m$ in Example 1.

In what follows we show that the bound SDP_{new} in (18) coincides with the SDP bound for QAP from (10). The proof is via symmetry reduction, in the spirit of work by Schrijver [22,23] (see also Gatermann and Parrilo [9]). It was proven in [4] that we may restrict the variable Y from (10) to lie in the matrix $*$-algebra:

$$\mathcal{A}_{\operatorname{aut}(B(\beta))} \otimes \mathcal{A}_{\operatorname{aut}(A(\alpha))}, \tag{19}$$

where

$$\mathcal{A}_{\mathcal{G}} := \{X \in \mathbb{R}^{n \times n} \ : \ XP = PX, \quad \forall P \in \mathcal{G}\},$$

and $\mathcal{G}$ is the automorphism group of the corresponding matrix. If a matrix, say B, is the adjacency matrix of a graph, then $\mathcal{A}_{\operatorname{aut}(B)}$ is a coherent configuration that contains B.

For our purpose, recall that B is the usual adjacency matrix of $K_{m,m}$, namely

$$B = \begin{pmatrix} 0 & J_m \\ J_m & 0 \end{pmatrix},$$

and $A := \frac{1}{2}W$. We fix $r = s = 1$. Hence $\alpha = \{2, \ldots, n\}$, $\beta = \{2, \ldots, n\}$ and subsequently:

$$B(\beta) = \begin{pmatrix} 0_{m-1 \times m-1} & J_{m-1 \times m} \\ J_{m \times m-1} & 0_{m \times m} \end{pmatrix}, \quad A(\alpha) = \frac{1}{2}\bar{W}, \quad \bar{c} = aw^T,$$

where $a^T = [0_{1 \times m-1} \ e^T]$ with $e \in R^m$ the all-ones vector as before.

Therefore, we can assume $Y \in \mathcal{A}_{\operatorname{aut}(K_{m-1,m})} \otimes \mathcal{A}_{\operatorname{aut}(\bar{W})}$ and since there is no symmetry assumption on the weight matrix $\bar{W}$ we have $Y \in \mathcal{A}_{\operatorname{aut}(K_{m-1,m})} \otimes \mathbb{R}^{n-1 \times n-1}$.

Revisiting Example 1 one can see that $\{A_t : t = 1, \dots, 6\}$ form a basis of $\mathcal{A}_{\mathrm{aut}(K_{m-1,m})}$. Let $\{E_{ij} : i, j = 1, \dots, n-1\}$ denote the standard basis of $\mathbb{R}^{n-1 \times n-1}$.

One can recover the basis of $\mathcal{A}_{\mathrm{aut}(K_{m-1,m})} \otimes \mathbb{R}^{n-1 \times n-1}$ as $\{A_t \otimes E_{ij} : i, j = 1, \dots, n-1$ and $t = 1, \dots, 6\}$ (for details see [4]). Thus,

$$Y = \sum_{t=1}^{6} \sum_{i,j=1}^{n-1} y_{ij}^t A_t \otimes E_{ij},$$

for some real numbers y_{ij}^t. Further, if we denote $Y_t := \sum_{i,j=1}^{n-1} y_{ij}^t E_{ij}$, we can write:

$$Y = \sum_{t=1}^{6} A_t \otimes Y_t. \tag{20}$$

Notice that since Y is symmetric and A_t, $t = 1, \dots, 6$ have distinct support, $Y_{t^*} = Y_t^T$ whenever $A_{t^*} = A_t^T$, for $t, t^* \in \{1, \dots, 6\}$.

We now substitute (20) in (10).

Since the A_t are 0-1 matrices with distinct support, $Y \geq 0$ is equivalent to $Y_t \geq 0$ for $t = 1, \dots, 6$.

The positive semidefinite constraint from (10) becomes:

$$\sum_{t=1}^{6} A_t \otimes Y_t \succeq 0. \tag{21}$$

If U is the unitary matrix from Theorem 2, then (21) is equivalent to:

$$(U^* \otimes I_{n-1}) \left(\sum_{t=1}^{6} A_t \otimes Y_t \right) (U \otimes I_{n-1}) \succeq 0,$$

and using (1) one obtains

$$\sum_{t=1}^{6} U^* A_t U \otimes Y_t \succeq 0.$$

After eliminating identical blocks from $U^* A_t U$, we reduce the matrix size of the SDP constraint in (10) from $(n-1)^2$ to $4(n-1)$, and write it in the form:

$$\sum_{t=1}^{6} \phi(A_t) \otimes Y_t \succeq 0,$$

where ϕ is the *-isomorphism from Example 1. Defining

$$X_t := \|A_t\|^2 Y_t, \quad t = 1, \dots, 6, \tag{22}$$

where $\|A\|$ is the Frobenius norm of matrix A, we have

$$\sum_{t=1}^{6} \frac{\phi(A_t)}{\|A_t\|^2} \otimes X_t \succeq 0.$$

Thus:

$$\begin{aligned}
&\frac{1}{m-1}\begin{pmatrix} X_1 & & & \\ & 0 & & \\ & & X_1 & 0 \\ & & 0 & 0 \end{pmatrix} + \frac{1}{(m-1)(m-2)}\begin{pmatrix} -X_2 & & & \\ & 0 & & \\ & & (m-2)X_2 & 0 \\ & & 0 & 0 \end{pmatrix} \\
&\quad + \frac{\sqrt{m(m-1)}}{m(m-1)}\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & X_3 \\ & & 0 & 0 \end{pmatrix} + \frac{\sqrt{m(m-1)}}{m(m-1)}\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 0 \\ & & X_4 & 0 \end{pmatrix} \\
&\quad + \frac{1}{m}\begin{pmatrix} 0 & & & \\ & X_5 & & \\ & & 0 & 0 \\ & & 0 & X_5 \end{pmatrix} + \frac{1}{m(m-1)}\begin{pmatrix} 0 & & & \\ & -X_6 & & \\ & & 0 & 0 \\ & & 0 & (m-1)X_6 \end{pmatrix} \succeq 0.
\end{aligned}$$

Simplifying the last expression yields:

$$\begin{pmatrix} \frac{1}{m-1}(X_1 - \frac{1}{m-2}X_2) & & & \\ & \frac{1}{m}(X_5 - \frac{1}{m-1}X_6) & & \\ & & \frac{1}{m-1}(X_1 + X_2) & \frac{1}{\sqrt{m(m-1)}}X_3 \\ & & \frac{1}{\sqrt{m(m-1)}}X_4 & \frac{1}{m}(X_5 + X_6) \end{pmatrix} \succeq 0.$$

We now consider the linear constraints. Using (20), the properties of the Kronecker product (1) and (2), and the fact that only A_1 and A_5 have nonzero traces, one has:

$$\begin{aligned}
\operatorname{trace}(I \otimes E_{jj})Y &= \operatorname{trace}\left((I \otimes E_{jj})\left(\sum_{t=1}^{6} A_t \otimes Y_t\right)\right) \\
&= \sum_{t=1}^{6} \operatorname{trace}(A_t \otimes E_{jj}Y_t) = \sum_{t=1}^{6} \operatorname{trace}(A_t)\operatorname{trace}(E_{jj}Y_t) \\
&= \operatorname{trace}(E_{jj}\|A_1\|^2 Y_1) + \operatorname{trace}(E_{jj}\|A_5\|^2 Y_5) \\
&= \operatorname{trace}(E_{jj}(X_1 + X_5)).
\end{aligned}$$

This yields:

$$\operatorname{trace}(E_{jj}(X_1 + X_5)) = 1, \quad j = 1, \ldots, n-1,$$

hence $X_1 + X_5 = I_{n-1}$. Indeed, the diagonal blocks of the matrix Y are diagonal matrices (see Lemma 1), and therefore X_1 and X_5 are diagonal matrices too, by (20) and (22).

Continuing in the same vein,

$$\text{trace}(E_{jj} \otimes I)Y = 1, \quad j = 1, \dots, n-1$$

reduces to

$$\sum_{t=1}^{6} \text{trace}(E_{jj} A_t)\text{trace}(Y_t) = 1, \;\; j = 1, \dots, n-1.$$

If we note that only $\text{trace}(E_{jj} A_1)$ or $\text{trace}(E_{jj} A_5)$ can be nonzero — and this can not happen for the same fixed value of j — we obtain:

$$\text{trace}(Y_1) = 1 \quad \text{and} \quad \text{trace}(Y_5) = 1.$$

Multiplying these two equations with the squared norms of A_1 and A_5 respectively one obtains two more linear equalities from (18), namely

$$\text{trace}(X_1) = m - 1 \quad \text{and} \quad \text{trace}(X_5) = m.$$

Furthermore:

$$\begin{aligned}
\text{trace}(J_{(n-1)^2}\mathrm{Y}) &= \text{trace}(J_{n-1} \otimes J_{n-1})\Big(\sum_{t=1}^{6} A_t \otimes Y_t\Big) \\
&= \sum_{t=1}^{6} \text{trace}(J_{n-1} A_t)\text{trace}(J_{n-1} Y_t) = \sum_{t=1}^{6} \text{trace}(J_{n-1} \|A_t\|^2 Y_t) \\
&= \sum_{t=1}^{6} \text{trace}(J_{n-1} X_t).
\end{aligned}$$

This yields the following equality constraint from (18):

$$\sum_{t=1}^{6} \text{trace}(J X_t) = (n-1)^2.$$

There is only one equality constraint left to verify. To this end let us denote by $S = \{2, 3, 4, 6\}$ and notice the following:

$$\text{trace}(J - I)A_t = \begin{cases} 0 & \text{if } t \in \{1, 5\} \\ \|A_t\|^2 & \text{if } t \in S. \end{cases}$$

We get:

$$\begin{aligned}
\operatorname{trace}((J-I)\otimes I)Y &= \operatorname{trace}((J-I)\otimes I\sum_{t=1}^{6} A_t \otimes Y_t) \\
&= \sum_{t=1}^{6} \operatorname{trace}((J-I)\otimes I)(A_t\otimes Y_t) \\
&= \sum_{t=1}^{6} \operatorname{trace}(J-I)A_t \operatorname{trace}(Y_t) = \sum_{t\in S} \operatorname{trace}(\|A_t\|^2 Y_t) \\
&= \sum_{t\in S} \operatorname{trace}(X_t).
\end{aligned}$$

Also $\operatorname{trace}(A_t) = 0$ if $t \in S$, and $X_1 + X_5 = I$, hence:

$$\begin{aligned}
\operatorname{trace}(I\otimes(J-I))Y &= \operatorname{trace}(I\otimes(J-I)\sum_{t=1}^{6} A_t\otimes Y_t) \\
&= \sum_{t=1}^{6} \operatorname{trace}(I\otimes(J-I))(A_t\otimes Y_t) \\
&= \sum_{t=1}^{6} \operatorname{trace}(A_t)\operatorname{trace}((J-I)Y_t) \\
&= \operatorname{trace}(J-I)\|A_1\|^2 Y_1 + \operatorname{trace}(J-I)\|A_5\|^2 Y_5 \\
&= \operatorname{trace}(J-I)X_1 + \operatorname{trace}(J-I)X_5 \\
&= \operatorname{trace}(J-I)I = 0.
\end{aligned}$$

We can now derive the last constraint in (18) immediately, since

$$\operatorname{trace}(I\otimes(J-I)+(J-I)\otimes I)Y = 0$$

is equivalent to

$$\sum_{t\in S} \operatorname{trace}(X_t) = 0.$$

The last step is to obtain the objective function. Recalling the vectors and matrices from (20) and the equality (3), we may write:

$$\begin{aligned}
\operatorname{trace}(\operatorname{Diag}(\bar{c}))Y &= \operatorname{trace}(\operatorname{Diag}(a)\otimes\operatorname{Diag}(w))Y \\
&= \sum_{t=1}^{6} \operatorname{trace}(\operatorname{Diag}(a)A_t)\operatorname{trace}(\operatorname{Diag}(w)Y_t) \\
&= \operatorname{trace}(\operatorname{Diag}(w)X_5),
\end{aligned}$$

where, for the last step, we used the fact that

$$\operatorname{trace}(\operatorname{Diag}(a)A_t) = \begin{cases} 0 & \text{if } t \in \{1,2,3,4,6\} \\ \|A_5\|^2 & \text{if } t = 5. \end{cases}$$

The first term of the objective function becomes

$$\begin{aligned}\operatorname{trace}(B(\beta)\otimes A(\alpha))Y &= \frac{1}{2}\operatorname{trace}(B(\beta)\otimes \bar{W})Y \\ &= \frac{1}{2}\sum_{t=1}^{6}\operatorname{trace}(B(\beta)A_t)\operatorname{trace}(\bar{W}Y_t) \\ &= \frac{1}{2}\operatorname{trace}(\bar{W}(X_3+X_4)),\end{aligned}$$

where, for the last step, we used the fact that

$$\operatorname{trace}(B(\beta)A_t) = \begin{cases} 0 & \text{if } t \in \{1,2,5,6\} \\ \|A_t\|^2 & \text{if } t \in \{3,4\}. \end{cases}$$

Therefore we have proved the following.

Theorem 3 *The bound SDP_{new} from (18) coincides with the SDP bound (10) for the QAP formulation of maximum bisection.*

5.2 SDP relaxation for max k-section

We now describe a new SDP relaxation of max $k - equipartition$, $k \geq 3$, where the variables in the relaxation $X_1, \ldots, X_7$ correspond to the matrices $A_1, \ldots, A_7$ respectively in Example 2.

Letting $n = |V| = km$, the new relaxation takes the following form.

$$SDP_{new} := \max \ \operatorname{trace}(\operatorname{diag}(w)X_5) + \frac{1}{2}\operatorname{trace}\bar{W}(X_3+X_4+X_7) \tag{23}$$

subject to

$$\begin{aligned} X_1 + X_5 &= I_{n-1} \\ \sum_{t=1}^{7}\operatorname{trace}(JX_t) &= (n-1)^2 \\ \operatorname{trace}(X_1) &= m-1 \\ \operatorname{trace}(X_5) &= (k-1)m \\ \operatorname{trace}(X_2+X_3+X_4+X_6+X_7) &= 0 \\ X_3 &= X_4^T \end{aligned}$$

$$\begin{pmatrix} \frac{1}{m-1}(X_1+X_2) & \frac{1}{\sqrt{(k-1)m(m-1)}}X_3 \\ \frac{1}{\sqrt{(k-1)m(m-1)}}X_4 & \frac{1}{(k-1)m}(X_5+X_6+X_7) \end{pmatrix} \succeq 0$$

$$X_1 - \frac{1}{m-2}X_2 \succeq 0$$

$$X_5 - \frac{1}{m-1}X_6 \succeq 0$$

$$X_5 + X_6 - \frac{1}{k-2}X_7 \succeq 0$$

$$X_i \geq 0 \quad (i=1,\ldots,7).$$

Note that the matrix variables X_i are all of order $n-1$.

With reference to Example 2, the reader may verify that a feasible point of the new relaxation is given by $X_i = A_i$ $(i=1,\ldots,7)$.

The bound in (23) coincides with the SDP bound for QAP in (10). The derivation is similar to the max bisection case, by using the isomorphism in Example 2, and we therefore omit the proof, and only state the result.

Theorem 4 *For any given integer $k>2$, the upper bound in* (23) *on the weight of a maximum k-section for a given graph coincides with the SDP bound* (10) *when applied to the QAP formulation* (8) *of maximum k-section.*

6 Comparison between SDP bounds of max k-equipartition

In this section we aim to prove that the new SDP relaxation defined in (18) and (23) dominates the relaxation $k-GP_{R2}$ in (5), for any $k \geq 2$. The proof is slightly different for $k=2$ and $k \geq 3$. We will only deal with the proof for $k \geq 3$, the proof for $k=2$ being similar, but simpler. To this end, one needs some valid implied equalities for the feasible region of (23). This result will follow as a consequence of Lemma 1.

Let us consider the following structure of the matrix variable Y of (10):

$$Y := \begin{pmatrix} Y^{11} & \cdots & Y^{1(n-1)} \\ \vdots & \ddots & \vdots \\ Y^{(n-1)(1)} & \cdots & Y^{(n-1)(n-1)} \end{pmatrix}, \tag{24}$$

where $Y^{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$. Following the same argument as for the case $k=2$ and using Example 2, one can verify that

$$Y = \sum_{t=1}^{7} A_t \otimes Y_t.$$

Therefore,

$$Y^{ij} = \sum_{t=1}^{7} (A_t)_{ij} Y_t, \tag{25}$$

where A_t are the matrices from Example 2.

From Lemma 1 one has:

$$e^T Y^{ij} = \mathrm{diag}(Y^{jj})^T, \quad i, j = 1, \ldots, n-1. \tag{26}$$

Multiplying this relation with the all ones vector to the right, one obtains:

$$\mathrm{trace}(JY^{ij}) = \mathrm{trace}(Y^{jj}), \quad i, j = 1, \ldots, n-1,$$

and furthermore

$$\mathrm{trace}(JY^{ij}) = 1, \quad i, j = 1, \ldots, n-1.$$

If we substitute $i = 1$ and $j = m$ in (25), then $Y^{1m} = Y_3$; or if $i = m$ and $j = 1$ then $Y^{m1} = Y_4$. Continuing in the same vein, for suitable choices of i and j, one obtains

$$\mathrm{trace}(JY_t) = 1, \quad t = 1, \ldots, 7,$$

which is equivalent to

$$\mathrm{trace}(JX_t) = \|A_t\|^2, \quad t = 1, \ldots, 7,$$

and furthermore

$$\mathrm{trace}(JX_t) = \mathrm{trace}(JA_t), \quad t = 1, \ldots, 7. \tag{27}$$

Lemma 2 *Assume the matrices $X_1, \ldots, X_7$ are feasible for the new SDP relaxation* (23)*. Then:*

$$\sum_{t=1}^{7} X_t = J, \tag{28}$$

$$\begin{aligned}
X_1 + X_2 + X_4 &= e\mathrm{diag}(X_1)^T, \\
X_3 + X_5 + X_6 + X_7 &= e\mathrm{diag}(X_5)^T, && (29)\\
e^T X_2 &= (m-2)\mathrm{diag}(X_1)^T, \\
e^T X_3 &= (m-1)\mathrm{diag}(X_5)^T, && (30)\\
e^T X_4 &= m(k-1)\mathrm{diag}(X_1)^T, && (31)\\
e^T X_6 &= (m-1)\mathrm{diag}(X_5)^T, \\
e^T X_7 &= (k-2)m\mathrm{diag}(X_5)^T. && (32)
\end{aligned}$$

Proof We will give the proof of (28), (29) and (30). The rest of the equalities can be derived in a similar way.

From $\sum_{i=1}^{n-1} \text{diag}(Y^{ii}) = e$ and $\sum_{i=1}^{n-1} Y^{ij} = e\text{diag}(Y^{jj})^T \quad (j = 1, \ldots, n-1)$ (see 14 and 16), one obtains $\sum_{i,j=1}^{n-1} Y^{ij} = J$, and further using (25) and the fact that $A_t, t = 1, \ldots, 7$ form a coherent configuration, we get:

$$\sum_{t=1}^{7} \|A_t\|^2 Y_t = J,$$

which yields: $\sum_{t=1}^{7} X_t = J$, and (28) is proved. In order to prove (29) we again use $\sum_{i=1}^{n-1} Y^{ij} = e\text{diag}(Y^{jj})^T$, $j = 1, \ldots, n-1$. If we let $j = m$ then:

$$(m-1)Y_3 + Y_5 + (m-1)Y_6 + (k-2)mY_7 = e\text{diag}(Y_5)^T,$$

and using the norms of $A_t, t \in \{3, 5, 6, 7\}$

$$X_3 + X_5 + X_6 + X_7 = e\text{diag}(X_5)^T.$$

For the proof of (30) we use (25) and (26). If we let $i = 1$ and $j = m$

$$e^T Y_3 = \text{diag}(Y_5)^T.$$

Again using the norms of A_3 and A_5 one obtains the desired equality. □

We can now prove the main theorem of this section.

Theorem 5 *For any fixed integer $k \geq 2$, the new SDP relaxation* (23) *dominates the relaxation $k - GP_{R2}$ from* (5).

Proof We will show that for any feasible point of the new SDP relaxation one can construct a feasible point of $k - GP_{R2}$ with the same objective value.

Assume $X_1, \ldots, X_7$ form a feasible point for (23). The dimension of the all one vector, denoted e, can be deduced from the context and is either $n-1$ or n.

Define:

$$\widetilde{X} := \begin{pmatrix} 1 & e^T - \text{diag}(X_5)^T \\ e - \text{diag}(X_5) & J - (X_3 + X_4 + X_7) \end{pmatrix}. \tag{33}$$

The traces of X_3, X_4 and X_7 are zero therefore $\text{diag}(\widetilde{X}) = e$.

We have $X_i \geq 0, i = 1, \ldots, 7$ and $\sum_{t=1}^{7} X_t = J$ (from (28)), therefore $J - (X_3 + X_4 + X_7) \geq 0$ and further $\widetilde{X} \geq 0$.

Recall that $n = km$; using also (30), (31) and (32) one has:

$$\begin{aligned}\widetilde{X}e &= \begin{pmatrix} 1 & e^T - \mathrm{diag}(X_5)^T \\ e - \mathrm{diag}(X_5) & J - (X_3 + X_4 + X_7) \end{pmatrix} \begin{pmatrix} 1 \\ e \end{pmatrix} \\ &= \begin{pmatrix} 1 + e^T e - \mathrm{trace}(X_5) \\ e - \mathrm{diag}(X_5) + Je - (X_4^T + X_3^T + X_7^T)e \end{pmatrix} \\ &= \begin{pmatrix} 1 + (n-1) - (k-1)m \\ ne - (k-1)m(\mathrm{diag}(X_1) + \mathrm{diag}(X_5)) \end{pmatrix} \\ &= \begin{pmatrix} m \\ kme - (k-1)me \end{pmatrix} = me.\end{aligned}$$

To prove that $\widetilde{X} \succeq 0$ we make use of (29) and write $X_3 = e\mathrm{diag}(X_5)^T - (X_5 + X_6 + X_7)$. Since also $X_3 = X_4^T$ we have:

$$\widetilde{X} = \begin{pmatrix} 1 & e^T - \mathrm{diag}(X_5)^T \\ e - \mathrm{diag}(X_5) & J - e\mathrm{diag}(X_5)^T - \mathrm{diag}(X_5)e^T + 2(X_5 + X_6) + X_7 \end{pmatrix}.$$

This matrix is positive semidefinite (psd) whenever the Schur complement (denoted further by S) of $J - e\mathrm{diag}(X_5)^T - \mathrm{diag}(X_5)e^T + 2(X_5 + X_6) + X_7$ is psd.

We have

$$\begin{aligned}S &= 2(X_5 + X_6) + X_7 - \mathrm{diag}(X_5)\mathrm{diag}(X_5)^T \\ &= (X_5 + X_6) + (X_5 + X_6 + X_7) - \mathrm{diag}(X_5)\mathrm{diag}(X_5)^T.\end{aligned}$$

S is psd as the sum of two psd matrices. To see this first notice that $X_5 + X_6 + X_7 \succeq 0$ because:

$$\begin{pmatrix} \frac{1}{m-1}(X_1 + X_2) & \frac{1}{\sqrt{(k-1)m(m-1)}} X_3 \\ \frac{1}{\sqrt{(k-1)m(m-1)}} X_4 & \frac{1}{(k-1)m}(X_5 + X_6 + X_7) \end{pmatrix} \succeq 0.$$

Further, summing $\frac{1}{k-2}(X_5 + X_6 + X_7) \succeq 0$ and $X_5 + X_6 - \frac{1}{k-2} X_7 \succeq 0$ one obtains that $X_5 + X_6 \succeq 0$.

Then $(X_5 + X_6 + X_7) - \mathrm{diag}(X_5)\mathrm{diag}(X_5)^T$ can be seen as the Schur complement of $X_5 + X_6 + X_7$, which is a submatrix of:

$$M = \begin{pmatrix} 1 & \mathrm{diag}(X_5)^T \\ \mathrm{diag}(X_5) & X_5 + X_6 + X_7 \end{pmatrix}.$$

To conclude that $\widetilde{X} \succeq 0$ we only have to prove that $M \succeq 0$. To this end, notice that since $\mathrm{diag}(X_6) = \mathrm{diag}(X_7) = 0$ the matrix M has a special structure:

$$M = \begin{pmatrix} 1 & \mathrm{diag}(N)^T \\ \mathrm{diag}(N) & N \end{pmatrix}.$$

It was proven in [11], §2.3 that such a matrix is positive semidefinite if and only if $N \succeq 0$ and $\text{trace}(JN) \geq \text{trace}(N)^2$.

In our case we saw earlier that $N \succeq 0$; and using (27):

$$\begin{aligned}\text{trace}(JN) &= \text{trace}(J(X_5 + X_6 + X_7)) = \text{trace}(J(A_5 + A_6 + A_7)) \\ &= (k-1)m + (k-1)m(m-1) + (k-1)(k-2)m^2 = (k-1)^2 m^2 \\ &= \text{trace}(X_5)^2 = \text{trace}(N)^2.\end{aligned}$$

Therefore $M \succeq 0$ and eventually $\widetilde{X} \succeq 0$.

To end the proof we still have to show that the objective values coincide.

Recall from (17) that:

$$W = \begin{pmatrix} 0 & w^T \\ w & \overline{W} \end{pmatrix}.$$

Then:

$$\begin{aligned}\frac{1}{2}\text{trace}(W(J - \widetilde{X})) &= \frac{1}{2}w^T(e - e + \text{diag}(X_5)) + \frac{1}{2}\text{trace}(w(e^T - e^T + \text{diag}(X_5)^T) \\ &\quad + \overline{W}(J - J + X_3 + X_4 + X_7)) \\ &= \frac{1}{2}(w^T \text{diag}(X_5) + \text{trace}(w\text{diag}(X_5)^T)) \\ &\quad + \frac{1}{2}\text{trace}(\overline{W}(X_3 + X_4 + X_7)) \\ &= w^T \text{diag}(X_5) + \frac{1}{2}\text{trace}(\overline{W}(X_3 + X_4 + X_7)) \\ &= \text{trace}(\text{diag}(w)X_5) + \frac{1}{2}\text{trace}(\overline{W}(X_3 + X_4 + X_7)).\end{aligned}$$

□

Using similar techniques, for $k = 2$ (i.e. bisection), and defining:

$$\widetilde{X} := \begin{pmatrix} 1 & e^T - \text{diag}(X_5)^T \\ e - \text{diag}(X_5) & J - (X_3 + X_4) \end{pmatrix},$$

one can prove the following.

Theorem 6 *The new SDP relaxation from* (18) *dominates the relaxation* $2 - GP_{R2}$ *from* (5).

7 The SDP bounds for symmetric graphs

The computation of the SDP bounds $k - GP_{R2}$ in (5), as well as that of the new SDP bounds (18) and (23), may be simplified for max-k-section in graphs that have suitable algebraic symmetry, by performing a further symmetry reduction; for details, see [4].

To illustrate this point, and for later use in the numerical examples in Sect. 8, we consider a special class of graphs with suitable symmetry, namely *strongly regular graphs*. The maximum k-section problem in strongly regular graphs is of some interest, since it is related to so-called *Hoffman colorings* of these graphs; see [13] for details and definitions.

For our purposes, a κ-regular graph $G = (V, E)$ with adjacency matrix A is called strongly regular, if the three matrices $\{I, A, J - A - I\}$ form a commutative coherent configuration (called an *association scheme*). In other words, the three matrices span a commutative 3-dimensional matrix algebra. The matrix A has exactly two distinct eigenvalues associated with eigenvectors orthogonal to e. These eigenvalues are called the *restricted eigenvalues*, and are usually denoted by r and s. A strongly regular graph is completely characterized by the values $(n = |V|, \kappa, r, s)$.

The following theorem shows that, for strongly regular graphs, the SDP bound $k - GP_{R2}$ in (5) has a closed form expression.

Theorem 7 *Let $G = (V, E)$ be a strongly regular graph with parameters $(n = |V|, \kappa, r, s)$ where r and s are the restricted eigenvalues, and κ is the valency. Let an integer $k > 0$ be given such that $m = n/k$ is integer. Define the value*

$$x_r = \begin{cases} \frac{(r+1)m - r - n + \kappa}{r(n-1)+\kappa} & \text{if } r(n-1) + \kappa > 0 \\ 0 & \text{else} \end{cases}$$

and define x_s similarly by replacing r by s in the last expression.

The bound $k - GP_{R2}$ in (5) on the maximum k-section of G is now given by:

$$|E|(1 - \max\{x_r, x_s\}).$$

Proof (sketch) The first observation is that there exists an optimal solution X to problem (5) in the algebra spanned by $\{I, A, J - A - I\}$. Thus we may assume that

$$X = I + x_1 A + x_2(J - A - I)$$

for some nonnegative scalar variables x_1 and x_2. Since the matrices $\{I, A, J - A - I\}$ may be simultaneously diagonalized, the constraint $X \succeq 0$ becomes a system of linear inequalities in the two variables x_1 and x_2:

$$\begin{aligned} 1 + \kappa x_1 + (n - \kappa - 1)x_2 &\geq 0 \\ 1 + r x_1 - (r + 1)x_2 &\geq 0 \\ 1 + s x_1 - (s + 1)x_2 &\geq 0. \end{aligned}$$

Proceeding along these lines, the SDP problem (5) is reduced to an LP problem in the two variables x_1 and x_2, and the closed form solution of this LP problem is readily obtained. □

Note that, for strongly regular graphs, the bound $k - GP_{R2}$ in (5) is completely determined by the eigenvalues of the graph. The new SDP bound (23) does not have a closed form expression in general for strongly regular graphs, but it is still possible to reduce its size, by using the symmetry reduction methodology described in [4].

8 Numerical comparison of bounds

In this section we present numerical results comparing the new SDP bounds (18) and (23) to the bound $k - GP_{R2}$ in (5), as well as the bounds LP-MET, SDP-MET and $k - GP_{R3}$ (see Sect. 2.1).

The matrices in our test problems have dimensions between 9 and 30 in order to be tractable for all the approaches that we are interested in. Computation was done on a dual core Pentium IV with 2 GB RAM, and we used the SDP solver SeDuMi 1.1R3 [24].

Note that the times reported for the new SDP bounds (18) and (23) include the time to solve several SDP relaxations: as explained in Sect. 3.1 there as many bounds as there are orbits of the automorphism group of W, e.g. for random W this number is n.

In Tables we deal with minimization (to compare with existing results for min bisection), and Table presents computational results and times for max 3-equipartition.

The instances denoted by R and a number are randomly generated, up to dimension 21, so that we could also solve them to optimality by exact enumeration. The instances cb.30.47 and cb.30.56 were taken from the PhD thesis of Ambruster [1]. The optimal value of this problem was reported in table C.50 of Appendix A on page 203 of the thesis.

The instances from Tables 1 and 3 are available on line at: http://lyrawww.uvt.nl/~cdobre/equipart_instances.rar.

Some observation on the results in Tables 1, 2, 3 and 4:

- The bounds LP-MET, SDP-MET and $k - GP_{R3}$ were computed using cutting plane schemes where the violated valid inequalities were added in a simple manner: we did not implement a sophisticated scheme where certain inequalities are subsequently removed as in [19]. As a result, the computational times should be seen in this light.
- The bound $2 - GP_{R3}$ is the strongest in all the examples.

Table 1 Bounds on optimal values of min bisection

Problem	n	New SDP	$2 - GP_{R2}$	$2 - GP_{R3}$	SDP-MET	LP-MET	OPT
R1	14	4,375.1	4,316.3	4,387	4, 387	4,387	4,387
R2	12	3,300	3,267.9	3,300	3,300	3,300	3,300
R3	16	538	531.4	538	538	538	538
R4	18	701.9	694.6	709	707.5	703	709
R5	20	773	767.3	773	773	773	773
cb.30.47	30	213	201.22	213	213	213	266
cb.30.56	30	302	291.82	302	302	302	379

Table 2 Computational times(s) for min bisection

Problem	n	New SDP	$2-GP_{R2}$	$2-GP_{R3}$	SDP-MET	LP-MET
R1	14	88	0.24	2.11	1.88	1.58
R2	12	33	0.20	1.69	1.67	1.60
R3	16	185	0.27	3.48	3.03	2.09
R4	18	356	0.31	4.60	4.56	3.21
R5	20	715	0.49	5.98	5.91	4.76
cb.30.47	30	10,447	2.57	35.88	146.71	22.96
cb.30.56	30	10,139	2.62	36.78	90.26	20.36

Table 3 Bounds on optimal values of max 3-section

Problem	n	New SDP	$3-GP_{R2}$	$3-GP_{R3}$	SDP-MET	LP-MET	OPT
R6	9	2,773	2,774.54	2,773	2,773	2,773	2,773
R7	12	5,255	5,265.58	5,255	5,255	5,255	5,255
R8	15	8,029.87	8,095.34	8,013.5	8,036	8,074	8,000
R9	18	11,460.04	11,526.20	11,459	11,459	11,489	11,459
R10	21	16,238.74	16,316.74	16,219.5	16,239.44	16,302.5	16,175

Table 4 Computational times(s) for max 3-section

Problem	n	New SDP	$3-GP_{R2}$	$3-GP_{R3}$	SDP-MET	LP-MET
R6	9	5.47	0.24	0.72	0.7	0.54
R7	12	39.28	0.23	1.45	1.29	0.74
R8	15	179.37	0.36	2.67	1.8	1.65
R9	18	676.49	0.45	4.1	2.8	2.78
R10	21	1,743.1	0.67	6.95	4.32	3.66

– There the bounds SD-MET and LP-MET do not dominate the new SDP bound, nor vice versa.

Since it is clear from these examples that the new SDP bounds are not cost effective for random instances, we also present results for more structured instances, namely strongly regular graphs.

8.1 Numerical results for strongly regular graphs

We consider the max k-section problem on a strongly regular graph, called the Higman-Sims graph [16], where $(n, \kappa, r, s) = (100, 22, 2, -8)$. Recall from Theorem 7, that the bound $k - GP_{R2}$ has a closed form expression in this case. The size of the new SDP problem (23) may also be reduced here by performing symmetry reduction. On the other hand, the bounds LP-MET, SDP-MET and $k - GP_{R3}$ involve valid inequalities

Table 5 Bounds on optimal values of max k-section on the Higman-Sims graph

k	n	New SDP	$2 - GP_{R2}$	$k - GP_{R3}$	SDP-MET	LP-MET
2	100	750	750	750	750	909.09
4	100	1,098	1,100	1,100	1,100	1,100
5	100	1,100	1,100	1,100	1,100	1,100

Table 6 Computational times (s) for max k-section on the Higman-Sims graph

k	n	New SDP	$k - GP_{R3}$	SDP-MET	LP-MET
2	100	0.5	1,293	1,226	2,828
4	100	0.9	4,351	1,449	4.64
5	100	0.9	47,376	1,287	4.55

that destroy the symmetry (see Sect. 2.1), and are therefore more expensive to compute here.

In Table 5, we show the different bounds for max k-section on the Higman-Sims graph, for $k = 2, 4, 5$. Note that, for $k = 4$, the new SDP bound (23) is the best.

Moreover, in Table 6 one may see that the time required to compute the new SDP bound is small here compared to the bounds LP-MET, SDP-MET and $k - GP_{R3}$, due to the possibility of symmetry reduction. Note that we do not give the time to compute the bound $k - GP_{R2}$ since here it is given by the expression in Theorem 7.

9 Conclusion

We have introduced a new SDP bound for the maximum k-section problem that is at least as good as an earlier bound (called $k - GP_{R2}$) due to Karisch and Rendl [18]. The new bound comes at a higher computational cost, but we demonstrated that it may still be computed for larger graphs that have suitable algebraic symmetry.

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