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FULL LENGTH PAPER

Robust linear semi-infinite programming duality under uncertainty

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Abstract In this paper, we propose a duality theory for semi-infinite linear programming problems under uncertainty in the constraint functions, the objective function, or both, within the framework of robust optimization. We present robust duality by establishing strong duality between the robust counterpart of an uncertain semi-infinite linear program and the optimistic counterpart of its uncertain Lagrangian dual. We show that robust duality holds whenever a robust moment cone is closed and convex. We then establish that the closed-convex robust moment cone condition in the case of constraint-wise uncertainty is in fact necessary and sufficient for robust duality. In other words, the robust moment cone is closed and convex if and only if robust duality holds for every linear objective function of the program. In the case of uncertain problems with affinely parameterized data uncertainty, we establish that robust duality is easily satisfied under a Slater type constraint qualification. Consequently, we derive robust forms of the Farkas lemma for systems of uncertain semi-infinite linear inequalities.

Keywords Robust optimization · Semi-infinite linear programming · Parameter uncertainty · Robust duality · Convex programming

Mathematics Subject Classification (2010) 90C34 · 90C05 · 90C31

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1 Introduction

Duality theory has played a key role in the study of semi-infinite programming [13,15,16,23] which traditionally assumes perfect information (that is, accurate values for the input quantities or system parameters), despite the reality that such precise knowledge is rarely available in practice for real-world optimization problems. The data of real-world optimization problems are often uncertain (that is, they are not known exactly) due to estimation errors, prediction errors or lack of information [3–6].

Robust optimization [2] provides a deterministic framework for studying mathematical programming problems under data uncertainty. It is based on a description of uncertainty via sets, as opposed to probability distributions which are generally used in stochastic approaches [7,25]. A successful treatment of the robust optimization approach to linear programming problems as well as convex optimization problems under data uncertainty has been given by Ben-Tal and Nemirovski [3–5], and El Ghaoui [12].

The present work was motivated by the recent development of robust duality theory [1,21] for convex programming problems in the face of data uncertainty. To set the context of this work, consider a standard form of *linear semi-infinite programming* (SIP in brief) problem in the absence of data uncertainty:

$$\begin{array}{lll}(SP) & \inf & \langle c, x\rangle \\ & \text{s.t.} & \langle a_t, x\rangle \geq b_t,\ \forall t \in T,\end{array}$$

where T is an arbitrary (possible infinite) index set, $c,\ a_t \in \mathbb{R}^n$, and $b_t \in \mathbb{R}$, $t \in T$. The primal *linear SIP* problem in the face of *input-parameter uncertainty* in the constraints can be captured by the parameterized linear SIP model problem

$$\begin{array}{lll}(USP) & \inf & \langle c, x\rangle \\ & \text{s.t.} & \langle a_t, x\rangle \geq b_t,\ \forall\, t \in T,\end{array}$$

where the parameters a_t and b_t are uncertain, and the couple (a_t, b_t) belongs to an uncertainty set $\mathcal{U}_t \subset \mathbb{R}^{n+1}$ for all $t \in T$.

As an illustration of the model, consider the uncertain linear SIP problem:

$$\inf_{(x_1,x_2)\in\mathbb{R}^2} \{x_1 : a_t^1 x_1 + a_t^2 x_2 \geq b_t,\ \ t \in T\}$$

where the data a_t^1, a_t^2 are uncertain, and for each $t \in T := [0, 1]$, $a_t^1 \in [-1-2t, -1+2t]$, $a_t^2 \in [1/(2+t), 1/(2-t)]$ and $b_t \equiv -1$. Then, this uncertain problem can be captured by our parameterized model as

$$\begin{array}{ll}\inf & x_1 \\ \text{s.t.} & a_t^1 x_1 + a_t^2 x_2 \geq b_t,\ \ \forall t \in T,\end{array}$$

where $u_t := (a_t^1, a_t^2, b_t) \in \mathbb{R}^3$ is the uncertain parameter and $u_t \in \mathcal{U}_t := \mathcal{V}_t \times \mathcal{W}_t$ with $\mathcal{V}_t := [-1-2t, -1+2t] \times [1/(2+t), 1/(2-t)]$ and $\mathcal{W}_t = \{-1\}$.

Let $u_t = (a_t, b_t)$, for $t \in T$. The *uncertain set-valued mapping* $\mathcal{U} : T \rightrightarrows \mathbb{R}^{n+1}$, is defined as $\mathcal{U}(t) := \mathcal{U}_t$ for all $t \in T$. We represent by $u_t := (a_t, b_t) \in \mathcal{U}_t$ an element of $\mathcal{U}_t$ or a variable ranging on $\mathcal{U}_t$. So, $\operatorname{gph} \mathcal{U} = \{(t, u_t) : u_t \in \mathcal{U}_t, t \in T\}$ and $u \in \mathcal{U}$ means that u is a *selection* of $\mathcal{U}$, i.e., that $u : T \to \mathbb{R}^{n+1}$ and $u_t \in \mathcal{U}_t$ for all $t \in T$ (u can be also represented as $(u_t)_{t \in T}$). In stochastic programming [7,25] each set $\mathcal{U}_t$ is equipped with a probability distribution and each selection of $\mathcal{U}$ determines a *scenario* for (SP).

The *robust counterpart* of (USP) [1,3,5] is

$$
\begin{aligned}
(RSP)\ \inf \quad & \langle c, x\rangle \\
\text{s.t.} \quad & \langle a_t, x\rangle \geq b_t,\ \forall (a_t, b_t) \in \mathcal{U}_t,\ \forall t \in T,
\end{aligned}
$$

where the uncertain constraint is enforced for all realizations of the uncertainties within the uncertainty set $\mathcal{U}_t$. The robust feasible set F is defined by

$$
\begin{aligned}
F &:= \{x : \langle a_t, x\rangle \geq b_t,\ \forall (a_t, b_t) \in \mathcal{U}_t,\ \forall t \in T\} \\
&= \{x : \langle a_t, x\rangle \geq b_t,\ \forall (t, u_t) \in \operatorname{gph} \mathcal{U}\}.
\end{aligned}
$$

Therefore, the robust counterpart of (USP) can simply be written as

$$
\begin{aligned}
(RSP)\ \inf \quad & \langle c, x\rangle \\
\text{s.t.} \quad & \langle a_t, x\rangle \geq b_t,\ \forall (t, u_t) \in \operatorname{gph} \mathcal{U},
\end{aligned}
\tag{1}
$$

where $u_t = (a_t, b_t)$ for all $t \in T$. Obviously, F is a closed convex set, and we assume throughout this paper that it is nonempty.

The robust counterpart (*RSP*) provides us with a worst-case solution for the uncertain SIP and the value of the robust counterpart, inf(*RSP*), represents the "**primal worst value**". Now, for each fixed selection $u = (a_t, b_t)_{t \in T} \in \mathcal{U}$, the *Lagrangian dual* of (*USP*) is

$$
(DP) \quad \sup_{\lambda \in \mathbb{R}_+^{(T)}} \left\{ \sum_{t \in T} \lambda_t b_t : -c + \sum_{t \in T} \lambda_t a_t = 0_n \right\},
$$

where $\mathbb{R}_+^{(T)}$ denotes the set of mappings $\lambda : T \to \mathbb{R}_+$ (also denoted by $(\lambda_t)_{t \in T}$) such that $\lambda_t = 0$ except for finitely many indexes), and $\sup \emptyset = -\infty$ by convention.

The *optimistic counterpart* [1,21] *of the Lagrangian dual* of (*DP*) is given by

$$
(ODP) \quad \sup_{\substack{u=(a_t, b_t)_{t \in T} \in \mathcal{U} \\ \lambda \in \mathbb{R}_+^{(T)}}} \left\{ \sum_{t \in T} \lambda_t b_t : -c + \sum_{t \in T} \lambda_t a_t = 0_n \right\}.
$$

The solution of the optimistic counterpart (*RSP*) gives us a best-case solution of the uncertain Lagrangian dual problem and the value of the optimistic counterpart, sup(*ODP*), represents the "**dual best value**".

The purpose of this paper is to present a *robust duality* theory for linear (SIP) in the face of data uncertainty by examining strong duality between the robust counterpart (RSP) and the optimistic counterpart (ODP). Robust duality means that

$$\inf\{\langle c, x\rangle : x \in F\} = \max_{\substack{u=(a_t,b_t)_{t\in T}\in\mathcal{U} \\ \lambda\in\mathbb{R}_+^{(T)}}} \left\{ \sum_{t\in T} \lambda_t b_t : -c + \sum_{t\in T} \lambda_t a_t = 0_n \right\}$$

and it describes the relation **"primal worst equals dual best"** with the dual attainment in (*ODP*). Thus, $\inf(RSP) = \max(ODP)$.

In this paper, we make the following key contributions: Firstly, we establish that robust duality holds for (*USP*) whenever the robust moment cone,

$$M := \bigcup_{u=(a_t,b_t)_{t\in T}\in\mathcal{U}} \operatorname{co}\operatorname{cone}\{(a_t, b_t), t \in T; (0_n, -1)\},$$

is closed and convex. We further show that the closed-convex robust moment cone condition in the case of constraint uncertainty is in fact necessary and sufficient for robust duality. In other words, the robust moment cone is closed and convex if and only if robust duality holds for every linear objective function of the program.

On the other hand, the robust counterpart (*RSP*) can also be viewed as an ordinary linear semi-infinite programming problem. Thus, its *standard* (or *Haar*) *dual problem* of (*RSP*) is given by:

$$\begin{aligned} (DRSP)\ \sup_{\lambda\in\mathbb{R}_+^{(\operatorname{gph}\mathcal{U})}} & \sum\nolimits_{(t,u_t)\in\operatorname{gph}\mathcal{U}} \lambda_{(t,u_t)} b_{(t,u_t)} \\ \text{s.t.} \quad & -c + \sum\nolimits_{(t,u_t)\in\operatorname{gph}\mathcal{U}} \lambda_{(t,u_t)} a_{(t,u_t)} = 0_n. \end{aligned} \tag{2}$$

By construction, $\inf(RSP) \geq \sup(DRSP) \geq \sup(ODP)$. We also derive strong duality between the robust counterpart (RSP) and its standard (or Haar) dual problem [15] in terms of a robust characteristic cone, illustrating the link between the Haar dual and the optimistic dual.

Secondly, for the important case of affinely parametrized data uncertainty [2], we show that the convexity of the robust moment cone always holds and that it is closed under a robust Slater constraint qualification together with suitable topological requirements on the index set and the uncertainty set of the problem.

Thirdly, we derive robust forms of Farkas' lemma [10,11,17] for systems of uncertain semi-infinite linear inequalities in terms of the robust moment cone and the robust characteristic cones.

The organization of the paper is as follows. Section 2 provides robust duality theorems under geometric conditions in terms of robust moment and characteristic cones. Section 3 shows that these cone conditions are satisfied if and only if robust duality holds for every linear objective function. Section 4 presents robust versions of the Farkas lemma for uncertain semi-infinite linear inequalities. Section 5 provides some conclusions of the work.

2 Robust duality

Let us introduce the necessary notation. We denote by $\|\cdot\|$ and $\mathbb{B}_n$ the Euclidean norm and the open unit ball in $\mathbb{R}^n$. By 0_n we represent the null vector of $\mathbb{R}^n$. For a set $C \subset \mathbb{R}^n$, we define its convex hull $\operatorname{co} C$ and conical hull $\operatorname{cone} C$ as $\operatorname{co} C = \{\sum_{i=1}^m \lambda_i c_i : \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1, c_i \in C, m \in \mathbb{N}\}$ and $\operatorname{cone} C = \bigcup_{\lambda \geq 0} \lambda C$, respectively. The topological closure of C is $\operatorname{cl} C$. Given $h\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, such that $h \neq +\infty$**,** the epigraph of h is

$$\operatorname{epi} h := \{(x, r) \in \mathbb{R}^{n+1} : h(x) \leq r\}$$

and the *conjugate function* of h is $h^*\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that

$$h^*(v) := \sup\{\langle v, x\rangle - h(x) : x \in \operatorname{dom} h\}.$$

The *indicator* and the *support functions* of C are denoted respectively by δ_C and δ_C^*.

Let $C \subset \mathbb{R}^n$ be a convex set and $h\colon C \to \mathbb{R}$. Identifying h with its extension to $\mathbb{R}^n$ by defining $h(x) := +\infty$ for any $x \notin C$, h is called *convex* when $\operatorname{epi} h$ is convex, *concave* when $-h$ is convex, and *affine* when it is both convex and concave on C. We define the *robust moment cone* of (RSP) as

$$M = \bigcup_{u=(a_t,b_t)_{t\in T}\in\mathcal{U}} \operatorname{co}\operatorname{cone}\{(a_t, b_t), t \in T; (0_n, -1)\}. \tag{3}$$

Note first that if $\inf(RSP) = -\infty$, then (ODP) has no feasible solution, i.e.

$$c \notin \bigcup_{u=(a_t,b_t)_{t\in T}\in\mathcal{U}} \operatorname{cocone}\{a_t,\ t \in T\},$$

where $\operatorname{cocone}\{a_t, t \in T\}$ denotes the convex cone generated by $\{a_t, t \in T\}$. In this case, $\inf(RSP) = \sup(ODP) = -\infty$

Theorem 1 *If the robust moment cone M is closed and convex and* $\inf(RSP) \neq -\infty$, *then*

$$\inf(RSP) = \max(ODP).$$

Proof Let $\alpha := \inf(RSP) \in \mathbb{R}$. Then,

$$\left[\langle a_t, x\rangle \geq b_t,\ \forall\, (t, u_t) \in \operatorname{gph}\mathcal{U}\right] \Rightarrow \langle c, x\rangle - \alpha \geq 0, \tag{4}$$

where $u_t = (a_t, b_t)$. Define $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(x) = \sup_{(t,u_t)\in\operatorname{gph}\mathcal{U}} \{-\langle a_t, x\rangle + b_t\}.$$

Then, g is convex, $F = \{x : g(x) \leq 0\}$ and (4) is equivalent to

$$g(x) \leq 0 \Rightarrow \langle c, x\rangle - \alpha \geq 0.$$

Let $f(x) := \langle c, x\rangle + \delta_F(x)$. This implies that $f(x) \geq \alpha$ for all $x \in \mathbb{R}^n$. So,

$$\begin{aligned}(0_n, -\alpha) \in \operatorname{epi} f^* &= \operatorname{cl}\left(\operatorname{epi}(\langle c, .\rangle)^* + \operatorname{epi}\delta_F^*\right)\\ &= \operatorname{epi}(\langle c, .\rangle)^* + \operatorname{epi}\delta_F^*\\ &= \big(\{c\} \times \mathbb{R}_+\big) + \operatorname{epi}\delta_F^*.\end{aligned}$$

On the other hand,

$$\operatorname{epi}\delta_F^* = \operatorname{epi}\left(\sup_{\lambda\geq 0}(\lambda g)\right)^* = \operatorname{clco}\left(\bigcup_{\lambda\geq 0}\operatorname{epi}(\lambda g)^*\right)$$

(see, e.g., [9] and [24]) and

$$\begin{aligned}\operatorname{clco}\left(\bigcup_{\lambda\geq 0}\operatorname{epi}(\lambda g)^*\right) &= -\operatorname{cl}\operatorname{co}\left(\bigcup_{\lambda\geq 0}\ \bigcup_{(t,u_t)\in\operatorname{gph}\mathcal{U}}\left[\lambda(a_t, b_t) + \{0_n\} \times \mathbb{R}_-\right]\right)\\ &= -\operatorname{cl}\operatorname{co} M.\end{aligned} \tag{5}$$

As M is closed and convex by our assumption, we have $\operatorname{epi}\delta_F^* = -M$, and so, $(0_n, -\alpha) \in (\{c\} \times \mathbb{R}_+) - M$. This implies that there exists $\widehat{u} = (\widehat{a}_t, \widehat{b}_t)_{t\in T} \in \mathcal{U}$ and $\widehat{\lambda} \in \mathbb{R}_+^{(T)}$ such that

$$c = \sum_{t\in T}\widehat{\lambda}_t\widehat{a}_t \ \text{ and } \ -\alpha \geq -\sum_{t\in T}\widehat{\lambda}_t\widehat{b}_t.$$

So, we see that $\alpha \leq \sum_{t\in T}\widehat{\lambda}_t\widehat{b}_t \leq \max(ODP)$. Thus the conclusion follows from the weak duality, and we also conclude that $(\widehat{u}, \widehat{\lambda})$ is optimal for (ODP). □

For the sake of self-containment, we have provided a short and direct proof for Theorem 1 using convex analysis. Recall, on the other hand, that the Haar dual of (RSP) was given in (2) by

$$\begin{aligned}(DRSP)\ \sup\nolimits_{\lambda\in\mathbb{R}_+^{(\operatorname{gph}\mathcal{U})}} &\sum\nolimits_{(t,u_t)\in\operatorname{gph}\mathcal{U}}\lambda_{(t,u_t)}b_{(t,u_t)}\\ \text{s.t.}\qquad &-c + \sum\nolimits_{(t,u_t)\in\operatorname{gph}\mathcal{U}}\lambda_{(t,u_t)}a_{(t,u_t)} = 0_n.\end{aligned}$$

Now consider the so-called *characteristic cone* of (RSP)

$$K = \operatorname{co}\operatorname{cone}\{(a_t, b_t), (t, u_t) \in \operatorname{gph}\mathcal{U}; (0_n, -1)\}, \tag{6}$$

where $u_t = (a_t, b_t)$ for all $t \in T$. Then the ordinary dual problem ($DRSP$) is equivalent to $\sup\{\gamma : (c, \gamma) \in K\}$ in the sense that both problems have the same optimal value and are simultaneously solvable or not. It is worth observing that $K = \operatorname{co} M$. In fact, $\operatorname{co} M \subset K$ because $M \subset K$ trivially and K is convex. To show the reverse inclusion, take an arbitrary generator of K different of $(0_n, -1) \in M$, say $u_s = (a_s, b_s)$, with $(s, u_s) \in \operatorname{gph}\mathcal{U}$. As

$$(a_s, b_s) \in \operatorname{co}\operatorname{cone}\{(a_t, b_t), t \in T; (0_n, -1)\},$$

we have $(a_s, b_s) \in M$. Thus, $K \subset \operatorname{co} M$ and so $K = \operatorname{co} M$.

From the linear SIP strong duality theorem, $\inf(RSP) = \max(DRSP)$ whenever K is closed (see, e.g., [15, Chapter 8]). If M is closed and convex, then $K = \operatorname{co} M = M$ is closed and so

$$\begin{aligned} \inf(RSP) &= \max(DRSP) \\ &= \max\{\gamma : (c, \gamma) \in K\} \\ &= \max\{\gamma : (c, \gamma) \in M\} \\ &= \max(ODP). \end{aligned}$$

Thus, we have obtained an alternative proof of Theorem 1 appealing to linear SIP machinery.

The next two examples show that M can be neither convex nor closed. Robust duality fails in the first example due to the existence of an infinite duality gap between (RSP) and (ODP) and also in the second example, where (ODP) is not solvable.

Example 1 Consider the simple uncertain linear SIP problem

$$\begin{aligned} (SP)\ \inf\ & -x_1 - x_2 \\ \text{s.t. } & \langle a_t, x\rangle \geq b_t,\ \forall t \in T, \end{aligned}$$

where $T = [0, 1]$, a_0 is uncertain on the set

$$\mathcal{V}_0 = \{(\cos\alpha, \sin\alpha) : \alpha \in [0, 2\pi] \cap \mathbb{Q}\}$$

(a dense subset of the circle $\{x : \|x\| = 1\}$) whereas the remaining data are deterministic: $b_0 = -1$ and $(a_t, b_t) = (0_2, -1)$ for $t \in]0, 1]$. This uncertain problem can be modeled as

$$\begin{aligned} (USP)\ \inf\ & -x_1 - x_2 \\ \text{s.t. } & \langle a_t, x\rangle \geq b_t, \forall t \in T, \end{aligned}$$

with uncertain mapping $\mathcal{U}$ such that $\mathcal{U}_0 = \mathcal{V}_0 \times \{-1\}$ and $\mathcal{U}_t = \{(0_2, -1)\}$ for all $t \in]0, 1]$. Observe that there exists a one-to-one correspondence between the selections of $\mathcal{U}$ and the elements of $\mathcal{V}_0$ because the unique uncertain constraint is

$\langle a_0, x\rangle \geq b_0$. So, the robust counterpart and the robust moment cone are

$$\begin{aligned}(RSP)\ \inf\ & -x_1 - x_2 \\ \text{s.t.}\ & -\langle v_0, x\rangle \geq -1, \forall v_0 \in \mathcal{V}_0, \\ & \langle 0_2, x\rangle \geq -1, \forall t \in]0,1],\end{aligned}$$

and

$$M = \bigcup_{v_0 \in \mathcal{V}_0} \operatorname{co}\operatorname{cone}\{(-v_0, -1), (0_2, -1)\},$$

respectively. Thus M is the union of countable many 2-dimensional convex cones having a common edge on the vertical axis. Obviously, M is neither convex nor closed. As $F = \operatorname{cl}\mathbb{B}$, the unique optimal solution is $\left(\sqrt{2}/2, \sqrt{2}/2\right)$ and $\min(RSP) = -\sqrt{2}$. Concerning the robust dual problem (ODP), it is inconsistent because $(1,1) \notin \operatorname{cocone}\{v_0, 0_2\} = \mathbb{R}_+\{v_0\}$ whichever $v_0 \in \mathcal{V}_0$ we consider ($\cos\alpha \neq \sin\alpha$ for any $\alpha \in [0, 2\pi] \cap \mathbb{Q}$), so that $\sup(ODP) = -\infty$ by convention, i.e., there is an infinite duality gap.

Example 2 Let us replace the set $\mathcal{V}_0$ in Example 1 by the set obtained eliminating from the square $[-1,1]^2$ the relative interior of its edges. Then the robust moment cone

$$M = \operatorname{cone}\{\left(]-1,1[^2 \cup \{(1,1), (-1,1), (-1,-1), (1,-1)\}\right) \times \{-1\}\}$$

is neither closed nor convex and $F = \operatorname{co}\{\pm(1,0), \pm(0,1)\}$, so that $\min(RSP) = -1 = \sup(ODP)$, i.e., there is no duality gap but (ODP) is not solvable.

In Example 1, the characteristic cone

$$\begin{aligned}K &= \operatorname{co}\operatorname{cone}\{(-v_0, -1), v_0 \in \mathcal{V}_0; (0_2, -1)\} \\ &= \left\{x \in \mathbb{R}^3 : x_3 < -\sqrt{x_1^2 + x_2^2}\right\} \cup \operatorname{cone}\{\mathcal{V}_0 \times \{-1\}\}\end{aligned}$$

is not closed, $(DRSP)$ is not solvable and

$$\inf(RSP) = -\sqrt{2} = \sup(DRSP) > \sup(ODP) = -\infty.$$

(The equality $\inf(RSP) = \sup(DRSP)$ comes from [15, Theorem 8.1(v)].)

In Example 2, the characteristic cone

$$K = \operatorname{co}\operatorname{cone}\{(1,1,-1), (-1,1,-1), (-1,-1,-1), (1,-1,-1)\}$$

is finitely generated and so it is closed, even though M is neither closed nor convex. Moreover,

$$\inf(RSP) = -1 = \max(DRSP) = \sup(ODP).$$

3 Robust moment cones: convexity and closure

We say that (*RSP*) satisfies the *convexity condition* if for every $t \in T$,

$$\mathcal{U}_t = \{(a_t(z_t), b_t(z_t)) : z_t \in Z_t\},$$

where Z_t is a convex set in $\mathbb{R}^q$ for some $q \in \mathbb{N}$, $a_t(\cdot)$ is affine, i.e., $a_t = (a_t^1, \ldots, a_t^n)$ and each $a_t^j(\cdot)$ is an affine function, $j = 1, \ldots, n$, and $b_t(\cdot)$ is concave.

Proposition 1 *Suppose that (RSP) satisfies the convexity condition. Then the robust moment cone M is convex.*

Proof Let $a^1, a^2 \in M$ and $\mu \in [0, 1]$. Let $a := \mu a^1 + (1-\mu)a^2$ and denote $a = (y, \gamma) \in \mathbb{R}^n \times \mathbb{R}$, with $a^1 = (y^1, \gamma_1) \in \mathbb{R}^n \times \mathbb{R}$ and $a^2 = (y^2, \gamma_2) \in \mathbb{R}^n \times \mathbb{R}$. Then, there exist $z_t^1 \in Z_t$ for all $t \in T$, $\lambda^1 = (\lambda_t^1)_{t\in T} \in \mathbb{R}_+^{(T)}$, and $\alpha_1 \leq 0$ such that

$$(y^1, \gamma_1) = \sum_{t\in T} \lambda_t^1 (a_t(z_t^1), b_t(z_t^1)) + (0_n, \alpha_1).$$

Similarly, there exist $z_t^2 \in Z_t$ for all $t \in T$, $\lambda^2 = (\lambda_t^2)_{t\in T} \in \mathbb{R}_+^{(T)}$ and $\alpha_2 \leq 0$ such that

$$(y^2, \gamma_2) = \sum_{t\in T} \lambda_t^2 (a_t(z_t^2), b_t(z_t^2)) + (0_n, \alpha_2).$$

Then, we have

$$\mu y^1 + (1-\mu)y^2 = \sum_{t\in T} \left(\mu\lambda_t^1 a_t(z_t^1) + (1-\mu)\lambda_t^2 a_t(z_t^2)\right)$$

and

$$\mu\gamma_1 + (1-\mu)\gamma_2 = \sum_{t\in T} \left(\mu\lambda_t^1 b_t(z_t^1) + (1-\mu)\lambda_t^2 b_t(z_t^2)\right) + \mu\alpha_1 + (1-\mu)\alpha_2.$$

We associate with $t \in T$ the scalar $\lambda_t := \mu\lambda_t^1 + (1-\mu)\lambda_t^2$ and the vector

$$z_t := \begin{cases} z_t^1, & \text{if } \lambda_t = 0, \\ \frac{\mu\lambda_t^1}{\lambda_t} z_t^1 + \frac{(1-\mu)\lambda_t^2}{\lambda_t} z_t^2, & \text{if } \lambda_t > 0. \end{cases}$$

Then $z_t \in Z_t$ for all $t \in T$, and

$$\mu\lambda_t^1 z_t^1 + (1-\mu)\lambda_t^2 z_t^2 = \lambda_t z_t.$$

By our convexity assumption, we see that, for each $t \in T$,

$$\mu\lambda_t^1 a_t(z_t^1) + (1-\mu)\lambda_t^2 a_t(z_t^2) = \lambda_t a_t(z_t), \tag{7}$$

and

$$\mu\lambda_t^1 b_t(z_t^1) + (1-\mu)\lambda_t^2 b_t(z_t^2) \leq \lambda_t b_t(z_t). \tag{8}$$

Then, there exists $\rho \leq 0$ such that

$$\begin{aligned} a &= (y, \gamma) = \mu(y^1, \gamma_1) + (1-\mu)(y^2, \gamma_2) \\ &= (\mu y^1 + (1-\mu)y^2, \mu\gamma_1 + (1-\mu)\gamma_2) \\ &= \left(\sum_{t\in T} \lambda_t a_t(z_t), \sum_{t\in T} \lambda_t b_t(z_t)\right) + (0_n, \mu\alpha_1 + (1-\mu)\alpha_2 + \rho) \\ &= \sum_{t\in T} \lambda_t (a_t(z_t), b_t(z_t)) + (-\mu\alpha_1 - (1-\mu)\alpha_2 - \rho)(0_n, -1), \end{aligned}$$

and this implies that $a = (y, \gamma) \in M$. □

In particular, the robust moment cone M is convex in the important affinely data parametrization case [2], i.e.

$$\mathcal{U}_t = \left\{ (a_t, b_t) = (a_t^0, b_t^0) + \sum_{j=1}^{q} z_t^j \left(a_t^j, b_t^j\right) : z_t = (z_t^1, \ldots, z_t^q) \in Z_t \right\},$$

where Z_t is closed and convex for each $t \in T$. In this case, $\mathcal{U}$ is convex-valued and the convexity condition holds with $b_t(\cdot)$ being also affine.

It is easy to see from Example 1 that M may not be convex when the convexity condition fails. In fact, in this example, $\mathcal{V}_0$ cannot be the image of certain convex set by an affine mapping (because $\mathcal{V}_0$ is not even connected).

The set-valued mapping $\mathcal{U} : T \rightrightarrows \mathbb{R}^{n+1}$, with (T, d) being a metric space, is said to be (Hausdorff) *upper semicontinuous* at $t \in T$ if for any $\epsilon > 0$ there exists $\eta > 0$ such that

$$\mathcal{U}_s \subset \mathcal{U}_t + \epsilon\mathbb{B}_{n+1} \; \forall s \in T \text{ with } d(s, t) \leq \eta.$$

In particular, $\mathcal{U}$ is *uniformly upper semicontinuous* on T if for any $\epsilon > 0$ there exists $\eta > 0$ such that

$$\mathcal{U}_s \subset \mathcal{U}_t + \epsilon\mathbb{B}_{n+1} \; \forall s, t \in T \text{ with } d(s, t) \leq \eta.$$

If, additionally, T is compact, then there exists a finite set $\{t_1, \ldots, t_m\} \subset T$ such that $d(t, \{t_1, \ldots, t_m\}) < \eta$ for all $t \in T$. Then, $\mathcal{U}_s \subset \bigcup_{i=1,\ldots,m} \left(\mathcal{U}_{t_i} + \epsilon\mathbb{B}_{n+1}\right) \forall s \in T$, so that $\text{gph}\,\mathcal{U}$ is bounded whenever $\mathcal{U}$ is compact-valued, i.e., for each $t \in T, \mathcal{U}_t$ is a compact set.

We say that (*RSP*) satisfies the *Slater condition* when there exists $x_0 \in \mathbb{R}^n$ (called *Slater point*) such that $\langle a_t, x_0 \rangle > b_t$ for all $(t, u_t) \in \text{gph}\,\mathcal{U}$, i.e. when there exists a strict solution of the constraint system of (*RSP*).

Proposition 2 *Suppose that the following three assumptions hold:*

(i) *T is a compact metric space;*
(ii) *$\mathcal{U}$ is compact-valued and uniformly upper semicontinuous on T;*
(iii) *(RSP) satisfies the Slater condition.*

Then, the robust moment cone M is closed.

Proof Let

$$(z^k, r_k) \in M = \bigcup_{u \in \mathcal{U}} \text{co cone}\{(a_t, b_t), t \in T; (0_n, -1)\}, \; k = 1, 2, \ldots,$$

such that $(z^k, r_k) \to (z, r)$. Then, for each k, there exists $u^k \in \mathcal{U}$, with $u_t^k = (a_t^k, b_t^k) \in \mathcal{U}_t$ for all $t \in T$, such that

$$(z^k, r_k) \in \text{co cone}\{(a_t^k, b_t^k), t \in T; (0_n, -1)\}.$$

From the Carathéodory theorem, we can find $\lambda_i^k \geq 0, i = 1, \ldots, n+1, \mu_k \geq 0$, $\{t_1^k, \ldots, t_{n+1}^k\} \subset T$ and $(a_{t_i^k}, b_{t_i^k}) \in \mathcal{U}_{t_i^k}, i = 1, \ldots, n+1$, such that

$$(z^k, r_k) = \sum_{i=1}^{n+1} \lambda_i^k (a_{t_i^k}, b_{t_i^k}) + \mu_k (0_n, -1). \tag{9}$$

As T is compact, we may assume that $t_i^k \to t_i \in T, i = 1, \ldots, n+1$.

Fix $i = 1, \ldots, n+1$. Since we are assuming that $\mathcal{U}$ is uniformly upper semicontinuous and so, for any $\epsilon > 0$, there exists $\eta > 0$ such that

$$\mathcal{U}_t \subseteq \mathcal{U}_{t_i} + \epsilon \mathbb{B}_{n+1}, \quad \text{for all } t \text{ such that } d(t, t_i) \leq \eta.$$

It follows that

$$d((a_{t_i^k}, b_{t_i^k}), \mathcal{U}_{t_i}) \to 0 \quad \text{as } k \to \infty.$$

Since $\mathcal{U}_{t_i}$ is compact, we may assume the existence of $(a_{t_i}, b_{t_i}) \in \mathcal{U}_{t_i}$ such that

$$(a_{t_i^k}, b_{t_i^k}) \to (a_{t_i}, b_{t_i}) \quad \text{as } k \to \infty. \tag{10}$$

Now, we show that $l_k := \sum_{i=1}^{n+1} \lambda_i^k + \mu^k$ is bounded. Granting this, by passing to subsequence if necessary, we may assume that

$$\lambda_i^k \to \lambda_i \in \mathbb{R}_+ \quad \text{and} \quad \mu_k \to \mu \in \mathbb{R}_+,$$

as each λ_i^k and μ_k are non-negative. Then, passing to the limit in (9) we have

$$(z, r) = \sum_{i=1}^{n+1} \lambda_i (a_{t_i}, b_{t_i}) + \mu(0_n, -1) \in M.$$

Therefore, the robust moment cone M is closed.

To show the boundedness of l_k, we proceed by the method of contradiction and assume without loss of generality that $l_k := \sum_{i=1}^{n+1} \lambda_i^k + \mu^k \to +\infty$. By passing to subsequence if necessary, we may assume that $\frac{\lambda_i^k}{l_k} \to \overline{\lambda}_i \in \mathbb{R}_+$, $\frac{\mu^k}{l_k} \to \overline{\mu} \in \mathbb{R}_+$ and

$$\sum_{i=1}^{n+1} \overline{\lambda}_i + \overline{\mu} = 1. \tag{11}$$

Dividing by l_k both members of (9) and passing to the limit, we obtain that

$$(0_n, 0) = \sum_{i=1}^{n+1} \overline{\lambda}_i (a_{t_i}, b_{t_i}) + \overline{\mu}(0_n, -1).$$

So, we have $\sum_{i=1}^{n+1} \overline{\lambda}_i a_{t_i} = 0_n$ and $\sum_{i=1}^{n+1} \overline{\lambda}_i b_{t_i} = \overline{\mu}$ and so, taking a Slater point x_0, we have

$$\sum_{i=1}^{n+1} \overline{\lambda}_i \left(\langle a_{t_i}, x_0 \rangle - b_{t_i} \right) = -\overline{\mu} \leq 0.$$

On the other hand, since $(a_{t_i}, b_{t_i}) \in \mathcal{U}_{t_i}$, assumption (iii) implies that $\langle a_{t_i}, x_0 \rangle - b_{t_i} > 0$ for all $i = 1, \ldots, n+1$. Note that $(\overline{\lambda}_1, \ldots, \overline{\lambda}_{n+1}) \neq 0_{n+1}$ (otherwise, $\overline{\mu} = \sum_{i=1}^{n+1} \overline{\lambda}_i r_i = 0$ and so, $(\overline{\lambda}_1, \ldots, \overline{\lambda}_{n+1}, \overline{\mu}) = 0_{n+2}$ which contradicts (11)). This implies that

$$\sum_{i=1}^{n+1} \overline{\lambda}_i \left(\langle a_{t_i}, x_0 \rangle - b_{t_i} \right) > 0.$$

This is a contradiction and so, $\{l_k\}$ is a bounded sequence. □

Example 1 violates assumptions (ii) in Proposition 2 because $\mathcal{U}_0$ is neither compact nor convex and $\mathcal{U}$ is not upper semicontinuous at 0. In fact, if we consider the sequence $(t_k, u_{t_k}) = (1/k, (0_2, -1))$ which converges to $(0, (0_2, -1))$, we have $(a_{t_k}, b_{t_k}) = (0_2, -1)$ which does not converge to (a_0, b_0) as $0_2 \notin \mathcal{V}_0$.

As an immediate consequence of the previous results, we obtain the following sufficient condition for robust duality. In the special case when $|T| < +\infty$, this result collapses to the robust strong duality result for linear programming problems in [1].

Corollary 1 *Suppose that the following assumptions hold:*

(i) *T is a compact metric space;*
(ii) *$\mathcal{U}$ is compact-convex-valued and uniformly upper semicontinuous on T;*
(iii) *(RSP) satisfies the convexity and the Slater conditions.*

Then, the robust duality holds, i.e. $\inf(RSP) = \max(ODP)$.

Proof The conclusion follows from Theorem 1, Propositions 1 and 2. □

We now present an example verifying Corollary 1.

Example 3 Let $T = [0, 1]$ and consider the following uncertain linear SIP problem:

$$\begin{array}{ll} \inf_{x\in\mathbb{R}} & x \\ \text{s.t.} & a_t x \geq b_t, \ t \in T, \end{array}$$

where the data a_t, b_t are uncertain, $a_t \in [-1-2t, -1+2t]$ and $b_t \equiv -1$. This uncertain problem can be captured by our model as

$$\begin{array}{l} \inf \ x \\ \text{s.t. } a_t x \geq b_t, \ t \in T, \end{array}$$

where $(a_t, b_t) \in \mathcal{U}_t = [-1-2t, -1+2t] \times \{-1\}$. Setting $u_t = (a_t, b_t), t \in T$, the robust counterpart is

$$\begin{array}{l} \inf \ x \\ \text{s.t. } a_t x \geq b_t, \ \forall (t, u_t) \in \operatorname{gph} \mathcal{U}. \end{array}$$

It can be verified that the feasible set is $[-1, 1/3]$. So, the optimal value of the robust counterpart is -1. The optimistic counterpart of the dual problem is

$$(ODP) \qquad \sup_{\lambda\in\mathbb{R}_+^{(T)},\ u=(a_t,b_t)_{t\in T}\in\mathcal{U}} \left\{ -\sum_{t\in T} \lambda_t : -1 + \sum_{t\in T} \lambda_t(-1+2a_t) = 0 \right\}.$$

Let $\overline{\lambda} \in \mathbb{R}_+^{(T)}$ be such that $\overline{\lambda}_1 = 1$ and $\overline{\lambda}_t = 0$ for all $t \in T\backslash\{1\}$. Let $u = (a_t, -1)_{t\in T} \in \mathcal{U}$ be such that $a_1 = 1$. Then $-1 + \sum_{t\in T} \overline{\lambda}_t(-1+2a_t) = -1 + \overline{\lambda}_1(-1+2a_1) = 0$ and $-\sum_{t\in T} \overline{\lambda}_t = -1$. So, $\max(ODP) = -1$ and robust strong duality holds. In fact, $M = \operatorname{cocone}\{(-3,-1), (-1,-1)\}$ is closed and convex. Finally, one can see that all the conditions in the preceding corollary are satisfied.

The following theorem shows that our assumption is indeed a characterization for robust strong duality in the sense that "the convexity and closedness of the robust moment cone" hold if and only if the robust strong duality holds *for each linear objective function* of (*RSP*).

Theorem 2 *The following statements are equivalent to each other:*

(i) *For all* $c \in \mathbb{R}^n$,

$$\inf\{\langle c, x\rangle : x \in F\} = \max_{\lambda\in\mathbb{R}_+^{(T)},\ u=(a_t,b_t)_{t\in T}\in\mathcal{U}} \left\{\sum_{t\in T}\lambda_t b_t : -c + \sum_{t\in T}\lambda_t a_t = 0_n\right\}.$$

(ii) *The robust moment cone* M *is closed and convex.*

Proof [(ii) $\Rightarrow$ (i)] It follows by Theorem 1.

[(i) $\Rightarrow$ (ii)] We proceed by contradiction and let $(c_0, r_0) \in (\text{clco}M) \setminus M$. So, (5) implies that $(-c_0, -r_0) \in \text{epi}\delta_F^*$ where $F := \{x : g(x) \le 0\}$ and

$$g(x) = \sup_{(t,u_t)\in\text{gph}\,\mathcal{U}} \{-\langle a_t, x\rangle + b_t\}.$$

Thus, we have $\delta_F^*(-c_0) \le -r_0$. So, for every $x \in F$, $\langle c_0, x\rangle \ge r_0$. It now follows that

$$r_0 \le \inf\{\langle c_0, x\rangle : \langle a_t, x\rangle \ge b_t,\ \forall\,(t, u_t) \in \text{ gph}\,\mathcal{U}\}.$$

Thus, the statement (i) gives us that

$$r_0 \le \max_{\lambda\in\mathbb{R}_+^{(T)},u\in\mathcal{U}} \left\{\sum_{t\in T}\lambda_t b_t : -c_0 + \sum_{t\in T}\lambda_t a_t = 0_n\right\},$$

and so, there exists $\overline{\lambda} \in \mathbb{R}_+^{(T)}$ and $(\overline{a}_t, \overline{b}_t) \in \mathcal{U}_t$ for all $t \in T$, with $-c_0 + \sum_{t\in T} \overline{\lambda}_t\,\overline{a}_t = 0_n$ and such that $r_0 \le \sum_{t\in T} \overline{\lambda}_t\,\overline{b}_t$. This shows that

$$(c_0, r_0) \in \text{co}\,\text{cone}\{(\overline{a}_t, \overline{b}_t), t \in T;\ (0_n, -1)\} \subset M,$$

which constitutes a contradiction. □

Now, we show that the more general case, i.e., linear SIP problems where uncertainty occurs in both objective function and in the constraints can also be handled by our approach. Indeed, this situation can be modeled as the parameterized linear SIP problem

$$\begin{array}{lll} (\widetilde{USP}) & \inf_{x\in\mathbb{R}^n} & \langle c, x\rangle \\ & \text{s.t.} & \langle a_t, x\rangle \ge b_t,\ \forall t \in T, \end{array} \tag{12}$$

where the data (c, a_t, b_t) are uncertain, $(a_t, b_t) \in \mathcal{U}_t \subset \mathbb{R}^{n+1}$ and $c \in Z$ where $Z \subset \mathbb{R}^n$. Fix a $s \notin T$ and define $\mathcal{U}_s = Z$. The problem $(\widetilde{USP})$ can be equivalently rewritten as

$$\begin{array}{ll} \inf_{(y,x)\in\mathbb{R}\times\mathbb{R}^n} & y \\ \text{s.t.} & \langle a_t, x\rangle \ge b_t,\ (a_t, b_t) \in \mathcal{U}_t,\ \forall t \in T, \\ & y - \langle c, x\rangle \ge 0,\ c \in \mathcal{U}_s. \end{array} \tag{13}$$

So, the *robust* (or *pessimistic*) *counterpart* of $(\widetilde{USP})$ can be formulated as

$$
\begin{array}{lll}
(\widetilde{RSP}) & \inf_{(y,x)\in\mathbb{R}\times\mathbb{R}^n} & y \\
& \text{s.t.} & 0\cdot y + \langle a_t, x\rangle \geq b_t,\ \forall\,(t,u_t)\in \operatorname{gph}\mathcal{U}, \\
& & y - \langle c, x\rangle \geq 0,\ \ \forall c\in \mathcal{U}_s,
\end{array}
\tag{14}
$$

whose decision space is $\mathbb{R}\times\mathbb{R}^n$. In other words, the robust counterpart $(\widetilde{RSP})$ is indeed a linear SIP problem with $n+1$ decision variables $y, x_1, \ldots, x_n$ and deterministic objective function as follows:

$$
\begin{array}{lll}
(\widetilde{RSP}) & \inf_{(y,x)\in\mathbb{R}\times\mathbb{R}^n} & \langle (1, 0_n), (y,x)\rangle \\
& \text{s.t.} & \langle \widetilde{a}_t, (y,x)\rangle \geq \widetilde{b}_t,\ \forall\,(t,\widetilde{u}_t)\in \operatorname{gph}\widetilde{\mathcal{U}},
\end{array}
\tag{15}
$$

where $x=(x_1,\ldots,x_n)$, $\widetilde{u}_t=(\widetilde{a}_t,\widetilde{b}_t)\in\widetilde{\mathcal{U}}_t$ for all $t\in\widetilde{T}:=T\cup\{s\}$ (as $s\notin T$), and $\widetilde{\mathcal{U}}:\widetilde{T}\rightrightarrows\mathbb{R}^{n+2}$ is the extension of $\mathcal{U}:T\rightrightarrows\mathbb{R}^{n+1}$ to $\widetilde{T}$ which results by defining, for each $t\in\widetilde{T}$,

$$
\widetilde{\mathcal{U}}_t := \begin{cases} \{1\}\times(-Z)\times\{0\}, & \text{if } t=s, \\ \{0\}\times\mathcal{U}_t, & \text{otherwise.} \end{cases}
$$

The constraint system of the *optimistic counterpart* for $(\widetilde{RSP})$, say $(\widetilde{ODP})$, is

$$
\sum_{t\in T}\lambda_t\begin{pmatrix}0\\ a_t\end{pmatrix}+\mu\begin{pmatrix}1\\ -c\end{pmatrix}=\begin{pmatrix}1\\ 0_n\end{pmatrix},
$$

where $(a_t,b_t)\in\mathcal{U}_t$, $t\in T$, $c\in Z=\mathcal{U}_s$, $\lambda\in\mathbb{R}_+^{(T)}$, and $\mu\in\mathbb{R}_+$. Eliminating $\mu=1$ we get

$$
(\widetilde{ODP}) \qquad \sup_{(a_t,b_t)_{t\in T}\in\mathcal{U},\, c\in Z,\, \lambda\in\mathbb{R}_+^{(T)}} \left\{\sum_{t\in T}\lambda_t b_t : -c+\sum_{t\in T}\lambda_t a_t = 0_n\right\}.
$$

Finally, the *robust moment cone* of $(\widetilde{RSP})$ is

$$
\begin{aligned}
\widetilde{M} &= \bigcup_{u\in\widetilde{\mathcal{U}}} \operatorname{co}\operatorname{cone}\{(\widetilde{a}_t,\widetilde{b}_t),\, t\in\widetilde{T};\ (0_{n+1},-1)\} \\
&= \bigcup_{\substack{u=(a_t,b_t)_{t\in T}\in\mathcal{U}\\ c\in Z}} \operatorname{co}\operatorname{cone}\{(0,a_t,b_t),\, t\in T;\ (1,-c,0),\ (0_n,0,-1)\}.
\end{aligned}
$$

□

Therefore, we have the following robust duality result for linear SIP problems where uncertainty occurs in both objective function and in the constraints.

Corollary 2 *Suppose that the following assumptions hold:*

(i) *T is a compact metric space;*
(ii) *$\mathcal{U}$ is compact-convex-valued and uniformly upper semicontinuous on T, and Z is a compact and convex subset of $\mathbb{R}^n$;*
(iii) *for every $t \in T$,*

$$\mathcal{U}_t = \{(a_t(z_t), b_t(z_t)) : z_t \in Z_t\} \text{ and } Z = \{c(v) : v \in W\}$$

where Z_t, W are closed and convex sets in $\mathbb{R}^q$ for some $q \in \mathbb{N}$, each component of $a_t(\cdot)$ and $c(\cdot)$ are affine, and $b_t(\cdot)$ is concave;
(iv) *There exists $x_0 \in \mathbb{R}^n$ such that $\langle a_t, x_0\rangle > b_t$ for all $(t, u_t) \in gph\,\mathcal{U}$.*

Then, robust duality holds, i.e. $\inf(\widetilde{RSP}) = \max(\widetilde{ODP})$.

Proof The conclusion follows from Theorem 2. □

4 Robust semi-infinite Farkas' lemma

In this Section, as consequences of robust duality results of previous Sections, we derive two forms of robust Farkas' lemma for a system of uncertain semi-infinite linear inequalities. Related results may be found in [8,18–21].

Corollary 3 (Robust Farkas' Lemma: Characterization I) *The following statements are equivalent to each other:*

(i) *For all $c \in \mathbb{R}^n$, the following statements are equivalent:*

$$\begin{aligned} &1)\ \left[\langle a_t, x\rangle \geq b_t,\ \forall (t, u_t) \in gph\,\mathcal{U}\right] \Rightarrow \langle c, x\rangle \geq r \\ &2)\ \exists \lambda = (\lambda_t)_{t\in T} \in \mathbb{R}_+^{(T)} \text{ and } (a_t, b_t) \in \mathcal{U}_t,\ t \in T, \\ &\quad \text{such that} \begin{cases} -c + \sum_{t\in T} \lambda_t a_t = 0_n, \\ \text{and } \sum_{t\in T} \lambda_t b_t \geq r. \end{cases} \end{aligned}$$

(ii) *The robust moment cone M is closed and convex.*

Proof Fix an arbitrary $c \in \mathbb{R}^n$. Consider the robust counterpart (RSP) and optimistic counterpart (ODP) given respectively by

$$\begin{aligned} (RSP)\ \inf\quad & \langle c, x\rangle \\ \text{s.t.}\quad & \langle a_t, x\rangle \geq b_t,\ \forall (t, u_t) \in \text{gph}\,\mathcal{U} \end{aligned}$$

and

$$(ODP) \max_{\lambda\in\mathbb{R}_+^{(T)},\ u=(a_t,b_t)_{t\in T}\in\mathcal{U}} \left\{ \sum_{t\in T} \lambda_t b_t : -c + \sum_{t\in T} \lambda_t a_t = 0_n \right\}.$$

Then, statement (i) is equivalent to $\inf(RSP) \geq r \ \Leftrightarrow\ \max(ODP) \geq r$ for every $c \in \mathbb{R}^n$ which is, in turn, equivalent to

$$\inf(RSP) = \max(ODP) \quad \text{for all } c \in \mathbb{R}^n.$$

So, the conclusion follows from Theorem 2. □

Next we compare the previous results with similar ones involving the characteristic cone K defined in (6) and the standard dual ($DRSP$) introduced in (2) instead of M and (ODP), respectively.

Recall that the assumptions in Propositions 1 and 2 guarantee that the robust moment cone M is convex and closed (and so, the characteristic cone K is also closed). In the following, we show that the assumptions in Proposition 2 alone ensure that a robust form of the Farkas lemma holds. This is achieved by first establishing that the characteristic cone K is closed.

Theorem 3 *Under the same assumptions as in Proposition 2, the following statements are equivalent:*

$$
\begin{array}{l}
1)\ \left[\langle a_t, x\rangle \geq b_t,\ \forall\, (t, u_t) \in gph\mathcal{U}\right] \Rightarrow \langle c, x\rangle \geq r.\\
2)\ \exists \lambda = (\lambda_t)_{t\in T} \in \mathbb{R}_+^{(gph\mathcal{U})}\\
\quad such\ that \begin{cases} -c + \sum_{(t,u_t)\in gph\mathcal{U}} \lambda_{(t,u_t)} a_{(t,u_t)} = 0_n,\\ and \sum_{(t,u_t)\in gph\mathcal{U}} \lambda_{(t,u_t)} b_{(t,u_t)} \geq r.\end{cases}
\end{array}
$$

Proof We first prove that the characteristic cone K of the robust linear SIP problem is closed. Condition (ii) in Proposition 2 guarantees the closedness of the index set gph$\mathcal{U}$. In fact, let $\{(t_r, a_r, b_r)\} \subset$ gph$\mathcal{U}$ be a sequence such that $(t_r, a_r, b_r) \to (t, a, b) \in T \times \mathbb{R}^{n+1}$. Then $(a_r, b_r) \in \mathcal{U}_{t_r}$ for all $r \in \mathbb{N}$, $t_r \to t$, and $(a_r, b_r) \to (a, b)$. Assume that $(t, a, b) \notin$ gph$\mathcal{U}$, i.e. that $(a, b) \notin \mathcal{U}_t$. Since $\mathcal{U}_t$ is closed, there exists $\epsilon > 0$ such that $(a, b) \notin \mathcal{U}_t + \epsilon\mathbb{B}_{n+1}$. Let $\eta > 0$ be such that $\mathcal{U}_s \subset \mathcal{U}_t + \frac{\epsilon}{2}\mathbb{B}_{n+1}$ for any $s \in T$ with $d(s, t) \leq \eta$. We have $d(t_r, t) \leq \eta$ and $d\left((a_r, b_r), (a, b)\right) < \frac{\epsilon}{2}$ for sufficiently large r, so that $(a_r, b_r) \in \mathcal{U}_{t_r} \subset \mathcal{U}_t + \frac{\epsilon}{2}\mathbb{B}_q$ and $(a, b) \in \mathcal{U}_t + \epsilon\mathbb{B}_q$ (contradiction). We have also seen that condition (ii) implies the boundedness of gph$\mathcal{U}$, which turns out to be compact. As we are assuming that the Slater condition holds, K is closed by [15, Theorem 5.3 (ii)]. Finally, the closedness of K and [15, (8.5)–(8.6)] imply $\inf(RSP) = \max(DRSP)$. As in the proof of Corollary 3, the equality $\inf(RSP) = \max(DRSP)$ guarantees the desired conclusion. □

Corollary 4 (Robust Farkas' Lemma: Characterization II) *The following statements are equivalent to each other:*

(i) *For all $c \in \mathbb{R}^n$, the following statements are equivalent:*

$$
\begin{array}{l}
1)\ \left[\langle a_t, x\rangle \geq b_t,\ \forall\, (t, u_t) \in gph\mathcal{U}\right] \Rightarrow \langle c, x\rangle \geq r.\\
2)\ \exists \lambda = (\lambda_t)_{t\in T} \in \mathbb{R}_+^{(gph\mathcal{U})}\\
\quad such\ that \begin{cases} -c + \sum_{(t,u_t)\in gph\mathcal{U}} \lambda_{(t,u_t)} a_{(t,u_t)} = 0_n,\\ and \sum_{(t,u_t)\in gph\mathcal{U}} \lambda_{(t,u_t)} b_{(t,u_t)} \geq r.\end{cases}
\end{array}
$$

(ii) *The characteristic cone K is closed.*

Proof It is straightforward consequence of [15, Theorem 8.4]. □

5 Conclusion

In this paper, we presented a duality theory for linear semi-infinite programming problems in the face of data uncertainty via robust optimization. We established robust duality for uncertain linear (SIP) by proving strong duality between the robust counterpart of an uncertain linear semi-infinite program and the optimistic counterpart of its uncertain Lagrangian dual. The duality theorems were given in terms of a robust moment cone. As a consequence, we also provided characterizations of robust versions of the Farkas lemma for infinite linear inequality systems under data uncertainty. Our theory suggests that a worst-case solution of an uncertain linear (SIP) can be obtained by finding a dual best solution. This provides a way of solving a robust optimization problem by finding a solution of its optimistic counterpart.

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