Math. Program., Ser. A (2014) 145:483–508
DOI 10.1007/s10107-013-0654-z

FULL LENGTH PAPER

Lattice-free sets, multi-branch split disjunctions, and mixed-integer programming

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Received: 15 February 2012 / Accepted: 26 February 2013 / Published online: 28 March 2013

Abstract In this paper we study the relationship between valid inequalities for mixed-integer sets, lattice-free sets associated with these inequalities and the multi-branch split cuts introduced by Li and Richard (Discret Optim 5:724–734, 2008). By analyzing n-dimensional lattice-free sets, we prove that for every integer n there exists a positive integer t such that every facet-defining inequality of the convex hull of a mixed-integer polyhedral set with n integer variables is a t-branch split cut. We use this result to give a finite cutting-plane algorithm to solve mixed-integer programs. We also show that the minimum value t, for which all facets of polyhedral mixed-integer sets with n integer variables can be generated as t-branch split cuts, grows exponentially with n. In particular, when $n = 3$, we observe that not all facet-defining inequalities are 6-branch split cuts.

Mathematics Subject Classification 90C11 · 90C60

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1 Introduction

Starting with the work of Andersen et al. [2], there has been renewed interest in lattice-free sets as a way to generate cutting planes for mixed-integer programs (MIPs). In [2], the authors study lattice-free sets in $\mathbb{R}^2$ and show that all facet-defining inequalities for the so-called two-row continuous group relaxation (defined by two equations, two integer variables and an arbitrary number of non-negative continuous variables) can be generated by these sets. Later, Dash et al. [11] showed that facets of the convex hull of any mixed-integer set with two integer variables can be obtained by *crooked-cross cuts*, a family of disjunctive cuts related to the *multi-branch split* cuts studied earlier by Li and Richard [26].

Currently, an area of active research is the *classification* of all maximal lattice-free sets in $\mathbb{R}^3$ (see [4]) and in higher dimensions. This project is motivated by the fact that any valid inequality for a polyhedral mixed-integer set in $\mathbb{Z}^n \times \mathbb{R}^l$ can be obtained using a lattice-free set in $\mathbb{R}^n$ and therefore classifying all lattice-free sets in $\mathbb{R}^n$ leads to a characterization of all facet-defining inequalities of the convex hull of mixed-integer sets in $\mathbb{Z}^n \times \mathbb{R}^l$. This classification project, however, seems difficult even in $\mathbb{R}^3$ and there has been very limited progress in higher dimensions.

In this paper we study the connection between valid inequalities for mixed-integer sets in $\mathbb{Z}^n \times \mathbb{R}^l$ and multi-branch split cuts. We show that any lattice-free set in $\mathbb{R}^n$ is contained in a t-branch split set for some finite integer t, or equivalently, in the union of t split sets. Furthermore, using ideas from Lenstra's algorithm [25] for integer programming in fixed dimension, we obtain t as a function of n alone and not of the data defining the mixed-integer set. This result leads to a characterization of all facet-defining inequalities of the convex hull of mixed-integer sets in $\mathbb{Z}^n \times \mathbb{R}^l$ without using an explicit classification of maximal lattice-free sets in $\mathbb{R}^n$. In addition, this result also leads to a finite cutting-plane algorithm for solving mixed-integer programs that only generates multi-branch split cuts.

The question of finite termination of pure cutting-plane algorithms has received some attention recently. Gomory [17] presented the first finite cutting-plane algorithm to solve *pure* integer programs. He later presented a cutting-plane algorithm for MIPs which uses the Gomory mixed-integer (GMI) cut [18], but proved finite termination only when the optimal objective value is known to be integral a priori (and an integer variable representing the objective function value is explicitly added to the constraint system). Later, Cook et al. [10] presented a very simple MIP which cannot be solved with split cuts alone. As GMI cuts are split cuts, their results imply that Gomory's algorithm will not terminate on this example. This result was recently extended by Dash and Günlük [13] to show that for a certain MIP with n integer variables and one continuous variable, no cutting-plane algorithm generating only $(n-1)$-branch split cuts will terminate in finite time (more precisely, the case when $n = 3$ was shown by Li and Richard [26] and Dash and Günlük extend this to arbitrary n). Cook, Kannan and Schrijver also note that if the data is rational it is possible to discretize continuous variables in an MIP (by treating such variables as integer variables after scaling by an appropriate constant) and solve the resulting pure integer program by Gomory's cutting-plane algorithm.

For MIPs with a bounded LP relaxation, Adams and Sherali [1] presented a hierarchy of relaxations which yield the convex hull of integer solutions in finitely many steps. Jörg [20,21] gave an algorithm which generates disjunctive cuts and solves any such MIP in finite time. Subsequently, Chen et al. [9] also gave a disjunctive cutting-plane algorithm to solve such MIPs in finite time. The case when the LP relaxation is not necessarily bounded was addressed recently by Del Pia and Weismantel [15] who show that the integral lattice-free closure of a polyhedron is again a polyhedron and the integer hull can be obtained by applying the closure operator finitely many times (though they do not show how to obtain this closure algorithmically). Our result in this paper gives the first finite cutting-plane algorithm for general MIPs which does not explicitly use the encoding complexity of the input data nor discretizes the continuous variables.

In this paper we also construct a family of lattice-free sets in $\mathbb{R}^n$ which cannot be covered by multi-branch split sets unless one uses at least $3 \cdot 2^{n-2}$ split sets. Using this construction, we present mixed-integer sets in $\mathbb{Z}^n \times \mathbb{R}$ which have facet-defining inequalities that are not t-branch split cuts unless $t \geq 3 \cdot 2^{n-2}$. For example, when $n = 3$, we show that 6-branch split cuts are not sufficient to obtain the integer hull, but 21-branch split cuts are. In order to obtain this result, we show that the lattice-width of a lattice-free, convex set in $\mathbb{R}^3$ is at most 4.2439; in $\mathbb{R}^4$ we show a corresponding bound of 6.8481.

2 Preliminaries

In this paper we work with polyhedral mixed-integer sets of the form

$$P = P^{LP} \cap (\mathbb{Z}^n \times \mathbb{R}^l) \quad \text{where} \quad P^{LP} = \{(x, y) \in \mathbb{R}^{n+l} \ : \ Ax + Gy \leq b\}, \quad (1)$$

and A, G and b have m rows and rational components. We call P^{LP} the continuous relaxation of P.

2.1 Disjunctive and lattice-free cuts

Disjunctive programming was introduced by Balas [5] and has proved to be a very important tool for generating valid inequalities for mixed-integer sets. We next review the main ideas that are relevant for this paper. Let $D_k = \{(x, y) \in \mathbb{R}^{n+l} \ : \ A^k x \leq b^k\}$ be finitely many polyhedral sets indexed by $k \in K$ and let $D = \cup_{k \in K} D_k$. We call D a *disjunction* for the mixed-integer lattice $\mathbb{Z}^n \times \mathbb{R}^l$ if

$$\mathbb{Z}^n \times \mathbb{R}^l \subset D$$

and we call each D_k an *atom* of the disjunction D (when the mixed-integer lattice is clear from the context, we omit it). By definition $D = D^x \times \mathbb{R}^l$ where $D^x \subseteq \mathbb{R}^n$ is the projection of D in the space of the integer components, and consequently, the condition above is same as requiring $\mathbb{Z}^n \subset D^x$. Notice that verifying $\mathbb{Z}^n \subset D^x$ is the same as checking if $\mathbb{Z}^n \setminus D^x = \emptyset$, which, in general, is not an easy task. In the next section, we will discuss simple disjunctions for which validity of the disjunction can be verified trivially.

A linear inequality is called a *disjunctive cut* for P derived from the disjunction D if it is valid for $P^{LP} \cap D_k$ for all $k \in K$; therefore

$$P \subseteq \operatorname{conv}\left(P^{LP} \cap D\right) = \operatorname{conv}\left(\bigcup_{k \in K} (P^{LP} \cap D_k)\right),$$

where $\operatorname{conv}\left(P^{LP} \cap D\right)$ is the set of points in P^{LP} that satisfy all disjunctive cuts derived from D.

Disjunctive cuts can also be seen as *lattice-free cuts*. Given a set $B \subset \mathbb{R}^n$, we call B *strictly lattice-free* if $B \cap \mathbb{Z}^n = \emptyset$, and we say that B is *lattice-free* if $\operatorname{int}(B) \cap \mathbb{Z}^n = \emptyset$, where $\operatorname{int}(B)$ stands for the points in the interior of B. Thus a lattice-free set may have integral points on its boundary. If B is strictly lattice-free, we define $P(B)$ as

$$P(B) = \operatorname{conv}\left(P^{LP} \backslash (B \times \mathbb{R}^l)\right) \Rightarrow P \subseteq P(B),$$

and any inequality valid for $P(B)$ is called a *lattice-free* cut derived from the set B. Note that the definition of a lattice-free cut above is different from that in most recent papers starting with [2] where convex sets which have strictly lattice-free interior are called lattice-free sets and cuts derived from these sets are called lattice-free cuts. We will see below (Observation 2.2) that these two families of cuts for a given polyhedron are equivalent.

A disjunctive cut derived from the disjunction $D^x \times \mathbb{R}^l$ is a lattice-free cut derived from the set $\mathbb{R}^n \backslash D^x$. Consequently, all disjunctive cuts are lattice-free cuts. As we discuss below, it is also possible to show that valid inequalities obtainable as lattice-free cuts from strictly lattice-free, convex sets are disjunctive cuts. Therefore all lattice-free cuts are disjunctive cuts. Before establishing the equivalence between lattice-free and disjunctive cuts we first make an important observation which we use throughout the paper.

Observation 2.1 *Let $D = D^x \times \mathbb{R}^l \subset \mathbb{R}^{n+l}$ be a disjunction and let $B \subset \mathbb{R}^n$ be a strictly lattice-free set. If $D^x \cap B = \emptyset$, then* $\operatorname{conv}\left(P^{LP} \cap D\right) \subseteq P(B)$. *In other words, when $D^x \cap B = \emptyset$, any lattice-free cut for P derived from B can be obtained as a disjunctive cut derived from D.*

Let $c^T x + d^T y \geq \gamma$ be a given rational valid inequality for P and let $\emptyset \neq V \subset \mathbb{R}^n$ be the points in P^{LP} that violate this inequality. In other words,

$$V = \{(x, y) \in P^{LP} \; : \; c^T x + d^T y < \gamma\}. \tag{2}$$

Furthermore, let $V^x \subset \mathbb{R}^n$ denote the projection of the set V in the space of the integer variables and note that $V^x \cap \mathbb{Z}^n = \emptyset$. It is known that V^x is defined by a finite collection of strict and non-strict rational inequalities, see [12]. Jörg [21] observes that the set V^x is contained in the interior of a polyhedral lattice-free set. In other words, there is a rational polyhedral set $B = \{x \in \mathbb{R}^n \; : \; \pi_i^T x \leq \gamma_i, \; i \in K\}$, where $\pi_i \in \mathbb{Z}^n$

and $\gamma_i \in \mathbb{Z}$ for all $i \in K$, such that $\text{int}(B) \cap \mathbb{Z}^n = \emptyset$ and

$$V^x \subseteq \text{int}(B) = \{x \in \mathbb{R}^n \ : \ \pi_i^T x < \gamma_i, \ i \in K\}. \tag{3}$$

Therefore $c^T x + d^T y \geq \gamma$ is valid for $P(\text{int}(B)) \subseteq P(V^x)$. Based on this observation, Jörg then argues that

$$D \ = \ \bigcup_{i \in K} \{(x, y) \in \mathbb{R}^{n+l} \ : \pi_i^T x \geq \gamma_i\} \tag{4}$$

is a disjunction for $\mathbb{Z}^n \times \mathbb{R}^l$ and the cut $c^T x + d^T y \geq \gamma$ is a disjunctive cut derived from D. Therefore, any valid inequality for P, and in particular, any facet-defining inequality for P is a disjunctive cut derived from *some* disjunction D and a lattice-free cut derived from some lattice-free, convex set. We emphasize that this approach is not prescriptive in the sense that the disjunction D is defined using the valid inequality $c^T x + d^T y \geq \gamma$ and not the other way around.

The discussion above also establishes the equivalence between lattice-free cuts obtained from convex sets as described in [2] and the seemingly more general lattice-free cuts obtained from sets that are not necessarily convex. When $c^T x + d^T y \geq \gamma$ is a lattice-free cut obtained from a possibly non-convex set B', the inclusion in (3) leads to the following observation.

Observation 2.2 *Let $B' \subset \mathbb{R}^n$ be a strictly lattice-free set (which is possibly non-convex). Any rational cut for P derived from B' can be obtained as a cut derived from a strictly lattice-free, convex set B.*

We would like to emphasize that the observation above does not claim the existence of a single convex set B that can produce all the cuts that B' can produce. For example, $\mathbb{R}^n \setminus \mathbb{Z}^n$ is a strictly lattice-free set that can yield all valid inequalities for P, and clearly there does not exist a convex set that can do the same.

2.2 Multi-branch split disjunctions

We next discuss simple disjunctions D for which it is easy to verify that $\mathbb{Z}^n \times \mathbb{R}^l \subset D$. The building block of these disjunctions is a *split disjunction* (see [10]) which is a disjunction that can be defined with two atoms D_1, D_2, where

$$D_1 = \{(x, y) \in \mathbb{R}^{n+l} \ : \ \pi^T x \leq \gamma\} \quad \text{and} \quad D_2 = \{(x, y) \in \mathbb{R}^{n+l} \ : \ \pi^T x \geq \gamma + 1\}$$

for some $\pi \in \mathbb{Z}^n, \gamma \in \mathbb{Z}$. We denote this disjunction as $D(\pi, \gamma)$, and define the associated *split set* as

$$S(\pi, \gamma) = \mathbb{R}^n \setminus D(\pi, \gamma) = \{(x, y) \in \mathbb{R}^{n+l} \ : \ \gamma < \pi^T x < \gamma + 1\}.$$

We will denote the topological closure of $S(\pi, \gamma)$ by $\bar{S}(\pi, \gamma)$ and call it a *closed* split set. If $x \in \mathbb{Z}^n$ then $\pi^T x \in \mathbb{Z}$ implying that $\pi^T x$ either satisfies $\pi^T x \leq \gamma$ or $\pi^T x \geq \gamma + 1$ and therefore $D(\pi, \gamma)$ is a valid disjunction for $\mathbb{Z}^n \times \mathbb{R}^l$.

Li and Richard [26] define a generalization of split disjunctions they call *t-branch split disjunctions*. Let $\pi_i \in \mathbb{Z}^n$ and $\gamma_i \in \mathbb{Z}$ for $i = 1, \ldots, t$. Then,

$$D(\pi_1, \ldots, \pi_t, \gamma_1, \ldots, \gamma_t) = \bigcup_{S \subseteq \{1,\ldots,t\}} \{(x, y) \in \mathbb{R}^{n+l} \ : \pi_i^T x \leq \gamma_i \text{ if } i \in S,\ \pi_i^T x \geq \gamma_i + 1 \quad \text{if } i \notin S\} \tag{5}$$

is called a t-branch split disjunction. A split disjunction is simply a 1-branch split disjunction. Further,

$$D(\pi_1, \ldots, \pi_t, \gamma_1, \ldots, \gamma_t) = \mathbb{R}^{n+l} \setminus \bigcup_{i=1,\ldots,t} S(\pi_i, \gamma_i).$$

In other words, the complement of $D(\pi_1, \ldots, \pi_t, \gamma_1, \ldots, \gamma_t)$ equals the union of t split sets and is thus lattice-free. Therefore $D(\pi_1, \ldots, \pi_t, \gamma_1, \ldots, \gamma_t)$ defines a valid disjunction. On the other hand, verifying that a set of the form in (4) is a valid disjunction requires solving an integer program.

A t-branch split disjunction can be specified using 2^t atoms of the form (5). We refer to disjunctive cuts derived from t-branch split disjunctions as *t-branch split cuts*.

3 Valid inequalities as t-branch split cuts

In this section we show that any valid inequality for P is a t-branch split cut for some t. Recently Chen et al. [9] showed this result for bounded P using t-branch split disjunctions defined by unit vectors. In addition the number t in their result depends on the data defining P. Next, we show that it is not necessary to require P to be bounded and, further, we also derive a bound on t that depends only on the number of integer variables defining P.

To show that a given valid inequality is a t-branch split cut for some t, we will consider the strictly lattice-free set V^x, defined after Eq. (2), on the space of integer variables, and *cover* it by t split sets: we say that a set B is covered by a collection of split sets if B is contained in their union.

Given a closed, bounded, convex set $B \subset \mathbb{R}^n$ and a vector $c \in \mathbb{Z}^n$, let the *width of B along the direction c*, denoted by $w(B, c)$, be defined as

$$w(B, c) = \max\{c^T x : x \in B\} - \min\{c^T x : x \in B\}. \tag{6}$$

The *lattice width* of B, denoted here as $w(B)$, is defined as

$$w(B) = \min_{c \in \mathbb{Z}^n \setminus \{\mathbf{0}\}} w(B, c).$$

If the set is not closed, we define its lattice width to be the lattice width of its topological closure. We call a closed, full-dimensional, bounded convex set a *convex body*. Any strictly lattice-free, bounded, convex set B is contained in a strictly lattice-free, convex body B' and the lattice width of B is bounded from above by the lattice width of B'.

Lenstra [25] gave a polynomial-time algorithm to solve the feasibility version of integer programs in fixed dimension. Given a polyhedron, Lenstra's algorithm either finds an integral point contained in the polyhedron or certifies that no such point exists. A central component of this algorithm is the use of algorithmic versions of Khinchine's *flatness* theorem [23]. The flatness theorem states that there exists a function $f : \mathbb{Z}_+ \to \mathbb{R}_+$ such that for any strictly lattice-free, bounded, convex set $B \subset \mathbb{R}^n$,

$$w(B) \leq f(n). \tag{7}$$

Notice that the function f only depends on the dimension of B and not on the encoding complexity of the data defining B. In [25], Lenstra uses this result to construct a finite enumeration tree to solve the integer feasibility problem. The number of nodes in the tree is bounded from above by a function of n which again is independent of the encoding complexity of the data defining B. Modifying Lenstra's idea slightly, we later show that every strictly lattice-free, convex body in $\mathbb{R}^n$ can be covered by the union of t split sets, where t is bounded from above by the maximum number of enumeration nodes used in Lenstra's algorithm.

Lenstra showed that (7) holds with $f(n) = 2^{n^2}$, which was later improved to $f(n) = c_0(n+1)n/2$ by Kannan and Lovász [22] for some constant c_0. This bound was subsequently improved by Banaszczyk, Litvak, Pajor, and Szarek to $O(n^{3/2})$ and by Rudelson [29] to $O(n^{4/3} \log^c n)$ for some constant c. The constant c_0 used by Kannan and Lovász [22] is $c_0 = \max\{1, 4/c_1\}$ where c_1 is another constant defined by Bourgain and Milman [8]. Independent of the value of c_1, the constant $c_0 \geq 1$ and therefore the upper bound defined by Kannan and Lovász on the lattice width is at least 3 for $\mathbb{R}^2$ and at least 6 for $\mathbb{R}^3$. When $B \subset \mathbb{R}^2$, Hurkens [19] proved that (7) holds with $f(2) = 1 + 2/\sqrt{3} \approx 2.1547$, and showed that this bound is tight. More precisely he showed the following result:

Theorem 3.1 *[19] If $B \subset \mathbb{R}^2$ is lattice-free, then $w(B) \leq 1 + \frac{2}{\sqrt{3}}$. Furthermore, there exists lattice-free $B \subset \mathbb{R}^2$ with $w(B) = 1 + \frac{2}{\sqrt{3}}$ and any such B is a triangle with vertices q_1, q_2, q_3 such that (let $q_4 := q_1$)*

$$\frac{1}{\sqrt{3}} q_i + \left(1 - \frac{1}{\sqrt{3}}\right) q_{i+1} \in \mathbb{Z}^2, \quad \textit{for } i = 1, 2, 3.$$

Letting $b_i = \frac{1}{\sqrt{3}} q_i + \left(1 - \frac{1}{\sqrt{3}}\right) q_{i+1}$ in the theorem above and taking $b_1 = (0,0)^T$, $b_2 = (0,1)^T$, and $b_3 = (1,0)^T$, one obtains a triangle $T \subset \mathbb{R}^2$ with $w(T) = 1 + \frac{2}{\sqrt{3}}$. The three vertices q_1, q_2, q_3 of this triangle are given by the columns of the following matrix:

$$\frac{1}{3}\begin{pmatrix} 2 & -1-\sqrt{3} & 2+\sqrt{3} \\ -1-\sqrt{3} & 2+\sqrt{3} & 2 \end{pmatrix}.$$

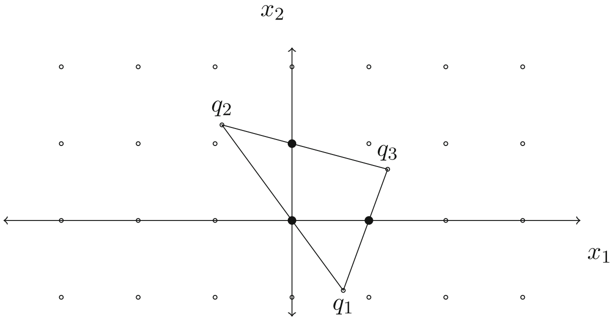


Fig. 1 The lattice-free triangle T in $\mathbb{R}^2$ with lattice-width $1 + 2/\sqrt{3}$

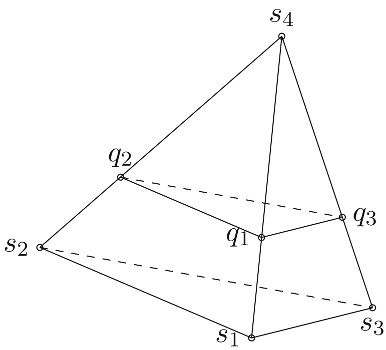


Fig. 2 A lattice-free tetrahedron in $\mathbb{R}^3$ with lattice width $2 + 2/\sqrt{3}$

Furthermore, T is a so-called type 3 maximal lattice-free triangle [2] that contains the lattice points b_1, b_2 and b_3 in the relative interior of its sides. As shown in Figure 1, T does not contain any other lattice points.

3.1 Width of lattice-free sets in $\mathbb{R}^3$

Recently, Averkov et al. [4] obtained a complete list (up to unimodular transformations) of maximal lattice-free polytopes in $\mathbb{R}^3$ with integer vertices. One can verify that the lattice width of such bodies does not exceed three in $\mathbb{R}^3$. However, in $\mathbb{R}^3$, one can construct lattice-free bodies with lattice width slightly greater than 3. Recall the vectors $q_1, q_2, q_3 \in \mathbb{R}^2$ which define the vertices of the triangle T. Consider the tetrahedron H with vertices $s_1, \ldots, s_4$, where $s_4 = (0, 0, 2 + 2/\sqrt{3})$, and s_1, s_2, s_3 are points on the plane $\{x : x_3 = 0\}$ such that the points $(q_i, 1) \in \mathbb{R}^3$ lie on the line segment from s_i to s_4. By definition, $H \cap \{x : x_3 = 1\}$ is congruent to T. See Fig. 2. H has lattice width $2 + 2/\sqrt{3} \approx 3.1547$. To see this note that if $c \in \mathbb{Z}^3 \backslash \{0\}$ such that $c_1 = c_2 = 0$, then $c_3 \neq 0$ and $w(H, c) \geq \max\{x_3 : x \in H\} - \min\{x_3 : x \in H\} = 2 + 2/\sqrt{3}$. On the other hand, if $(c_1, c_2) \neq (0, 0)$, then $w(H, c) \geq w(H \cap \{x : x_3 = 0\}, c) \geq 2 + 2/\sqrt{3}$. This is because the triangles $H \cap \{x : x_3 = 0\}$ and $H \cap \{x : x_3 = 1\}$ are homothetic triangles, and the ratio of their respective lattice widths (when treated as convex bodies in $\mathbb{R}^2$) is $(2 + 2/\sqrt{3})/(1 + 2/\sqrt{3})$, which is the ratio of their distance from s_4.

We do not know of any result analogous to Hurkens' result which gives the best possible upper bound on the lattice width in $\mathbb{R}^3$.

Using the best known value for c_1, and refining the result of Kannan and Lovász slightly, we next give an upper bound in $\mathbb{R}^3$ on the lattice width of strictly lattice-free, convex bodies. We need a few definitions to give the result and its proof. In [8], Bourgain and Milman show that if $K \subset \mathbb{R}^n$ is a convex body symmetric about the origin and K^* is its polar body, i.e., $K^* = \{y \in \mathbb{R}^n : y^T x \leq 1 \; \forall x \in K\}$, then

$$\mathrm{vol}(K)\mathrm{vol}(K^*) \geq \left(\frac{c_1}{n}\right)^n \tag{8}$$

where $\mathrm{vol}(K)$ denotes the n-dimensional volume (Lebesgue measure) of K and $c_1 > 0$ is a universal constant that does not depend on n.

If S and T are subsets of $\mathbb{R}^n$, and δ is a positive real number, then let $S + T = \{s + t : s \in S, t \in T\}$, and let $\delta S = \{\delta s : s \in S\}$. $S - T$ is similarly defined. For a convex body B in $\mathbb{R}^n$, let $\mu_j(B)$ be defined as

$$\mu_j(B) = \inf\{t \in \mathbb{R}_+ : tB + \mathbb{Z}^n \text{ intersects every } (n-j)\text{-dimensional affine subspace of } \mathbb{R}^n\},$$

where inf is short for infimum. Then $\mu_n(B)$ is the infimum of all t such that $tB + \mathbb{Z}^n = \mathbb{R}^n$, and is called the *covering radius* of B. Therefore $\mu_n(B) \geq 1$ if B is lattice-free and convex and $\mu_n(B) > 1$ if B is a strictly lattice-free, convex body. Let

$$\lambda_1(B) = \inf\{t \in \mathbb{R} : t(B - B) \text{ contains a nonzero integer vector}\}.$$

Theorem 3.2 *If $B \subset \mathbb{R}^3$ is lattice-free, then $w(B) \leq 1 + 2/\sqrt{3} + (90/\pi^2)^{\frac{1}{3}} \approx 4.2439$.*

Proof We first define functions $\phi_0, \phi_1 : \mathbb{Z}_+ \to \mathbb{R}_+$ that we will use instead of the universal constants c_0 and c_1. Let B_n stand for the unit ball in $\mathbb{R}^n$ and define

$$\phi_1(n) = n\left(\frac{2^n (n!)^2}{(2n)!}\mathrm{vol}(B_n)^2\right)^{\frac{1}{n}}$$

and let $\phi_0(n) = 4/\phi_1(n)$. Subsequently, we will refer to c_0 as the least upper bound on $\phi_0(n)$ for all n, and c_1 as the largest lower bound on $\phi_1(n)$ for all n.

In [24], Kuperberg gave the best-known value for c_1 and showed that if K is a convex body symmetric about the origin, then

$$\mathrm{vol}(K)\mathrm{vol}(K^*) \geq \frac{2^n (n!)^2}{(2n)!}\mathrm{vol}(B_n)^2.$$

Using our notation, this can be rewritten as

$$\mathrm{vol}(K)\mathrm{vol}(K^*) \geq \left(\frac{\phi_1(n)}{n}\right)^n \tag{9}$$

which is identical to (8) except the universal constant c_1 is now replaced with the function $\phi_1(n)$.

In [22], Kannan and Lovász show that $\lambda_1(B)w(B) \leq 4/(\mathrm{vol}(B-B)\mathrm{vol}((B-B)^*))^{1/n}$ which implies that

$$w(B) \leq \frac{4n}{\lambda_1(B)\phi_1(n)} = \frac{n\phi_0(n)}{\lambda_1(B)}$$

by (9). In addition (in Lemma 2.3) they also show that $\mu_1(B) = 1/w(B)$ and therefore, substituting out $w(B)$ from the inequality above, we obtain $\lambda_1(B) \leq n\phi_0(n)\mu_1(B)$.

Now combining $\mu_n(B) \leq \mu_{n-1}(B) + \lambda_1(B)$ [22, Lemma 2.5] with the fact that $\mu_2(B) \leq (1+2/\sqrt{3})\mu_1(B)$ (see [22, p 587]) we obtain

$$\mu_3(B) \leq \mu_2(B) + \lambda_1(B) \leq (1+2/\sqrt{3})\mu_1(B) + \lambda_1(B) \leq (1+2/\sqrt{3}+3\phi_0(3))\mu_1(B).$$

As $1 \leq \mu_3(B)$ for a lattice-free body in $\mathbb{R}^3$, we have

$$\frac{1}{\mu_1(B)} = w(B) \leq 1 + 2/\sqrt{3} + 3\phi_0(3).$$

Substituting $\phi_0(3) = (10/3\pi^2)^{\frac{1}{3}} \approx 0.6964$ we obtain the desired value. □

We can similarly refine the lattice-width bound in [22] in higher dimensions. Lemma 2.6 in [22] asserts that $\mu_{j+1}(B) \leq \mu_j(B) + (j+1)c_0\mu_1(B)$ for $j = 1, \ldots, n-1$. Adding up these inequalities, one obtains that if $B \subset \mathbb{R}^n$, then $\mu_n(B) \leq (1 + c_0 \sum_{i=2}^n i)\mu_1(B)$. Therefore [22, Theorem 2.7],

$$\text{if } c_0 \geq 1,\ \mu_n(B) \leq c_0 n(n+1)/(2w(B)) \text{ as } \mu_1(B) = 1/w(B).$$

(We noted before that c_0 is chosen to be $\max\{1, 4/c_1\}$ in [22, p. 581].) Looking at the proofs of Lemma 2.6 and Lemma 2.5 in [22], and the fact that $\lambda_1(B) \leq \phi_0(n)n\mu_1(B)$, one can replace [22, Lemma 2.6] by

$$\mu_{j+1}(B) \leq \mu_j(B) + (j+1)\phi_0(j+1)\mu_1(B) \text{ for } j = 1, \ldots, n-1$$
$$\Rightarrow \mu_n(B) \leq (1 + 2/\sqrt{3} + \sum_{i=3}^{n} i\phi_0(i))/w(B),$$

and therefore $w(B) \leq 1 + 2/\sqrt{3} + \sum_{i=3}^n i\phi_0(i)$ for lattice-free bodies in $\mathbb{R}^n$. As $\phi_0(4) \approx 0.6510$, we can conclude that if $B \subset \mathbb{R}^4$ and B is convex and lattice-free, then $w(B) \leq 6.8481$.

3.2 Lattice-free sets and integer programming

We next review some basic properties of *unimodular* matrices, i.e., integral, square matrices with determinant ± 1. If U is an $n \times n$ unimodular matrix, and $v \in \mathbb{R}^n$, the affine transformation $\sigma(x) = Ux + v$ is a one-to-one, invertible, mapping of $\mathbb{R}^n$ to $\mathbb{R}^n$ with $\sigma^{-1}(x) = U^{-1}(x - v)$ and this transformation preserves volumes (see [3, Thm 15.13]). If U is an unimodular matrix, then so is U^{-1}; if in addition $v \in \mathbb{Z}^n$, then the function $\sigma(x)$ is a one-to-one, invertible, mapping of $\mathbb{Z}^n$ to $\mathbb{Z}^n$. Further, if $a \in \mathbb{Z}^n, b \in \mathbb{Z}$, the set $\{x \in \mathbb{R}^n : a^T x = b\}$ is mapped to the set

$$\{x' \in \mathbb{R}^n : a^T U^{-1}(x' - v) = b\} \equiv \{x' \in \mathbb{R}^n : a^T U^{-1} x' = b + a^T U^{-1} v\}$$

and $a^T U^{-1} \in \mathbb{Z}^n$. Therefore, given a split set $S(a, b)$, $\sigma(S(a, b))$ and $\sigma^{-1}(S(a, b))$ are both split sets. If $a \in \mathbb{Z}^n$ and the g.c.d. of the coefficients of a is one, then there is a unimodular matrix U such that $a^T U = (0, \ldots, 0, 1)$ and $a^T x = b$ has an integral solution for any integer b (see [30, Corollary 4.1c]). Note that the previous statement implies that a^T is the last row of U^{-1}.

In the remainder of the section, we do not use any specific bound on the lattice width, but just use $f(n)$ to stand for a function which gives an upper bound on the lattice width of strictly lattice-free bodies in $\mathbb{R}^n$. For any positive integer n, we define the functions

$$\bar{f}_n = 1 + \lceil f(n) \rceil \quad \text{and} \quad h(n) = \bar{f}_n + \bar{f}_n \bar{f}_{n-1} + \bar{f}_n \bar{f}_{n-1} \bar{f}_{n-2} + \cdots + \Pi_{i=1}^n \bar{f}_i.$$

Lemma 3.3 *Any strictly lattice-free, bounded, convex set $B \subset \mathbb{R}^n$ is contained in the union of some $h(n)$ split sets.*

Proof The result is trivially true when $n = 1$ as $w(B) \leq 1$ in that case and $h(1)$ can be assumed to be 2. Assume it is true for all dimensions up to $n - 1$ and consider a strictly lattice-free, bounded, convex set $B \subset \mathbb{R}^n$. By Khinchine's flatness result, there is a nonzero vector $a \in \mathbb{Z}^n$ such that $u - l \leq f(n)$ where $u = \max\{a^T x : x \in B\}$ and $l = \min\{a^T x : x \in B\}$. Therefore $B \subseteq \{x \in \mathbb{R}^n : \lfloor l \rfloor \leq a^T x \leq \lceil u \rceil\}$. We can assume the g.c.d. of the coefficients of a is one, otherwise $a = k\bar{a}$ for some positive integer k and $\bar{a} \in \mathbb{Z}^n$, and $\max\{\bar{a}^T x : x \in B\} - \min\{\bar{a}^T x : x \in B\} = (u - l)/k \leq f(n)/k$.

Let $\mathcal{L}$ be the collection of the split sets $S(a, b)$ for $b \in V = \{\lfloor l \rfloor, \ldots, \lceil u \rceil - 1\}$ and notice that

$$B \setminus \bigcup_{b \in V} S(a, b) = \bigcup_{b \in \bar{V}} \{x \in B : a^T x = b\}$$

where $\bar{V} = \{\lceil l \rceil, \ldots, \lfloor u \rfloor\}$. Each one of the $|\bar{V}|$ sets in the right hand side of this expression is strictly lattice-free, and has dimension at most $n - 1$. As the g.c.d. of the coefficients of a is one, $a^T x = b$ has an integral solution (say v^b) for any $b \in \mathbb{Z}$, and there is a unimodular matrix U with a^T as its last row. Then, under the affine transformation $x \rightarrow Ux - Uv^b$ (with inverse transformation $x \rightarrow U^{-1}x + v^b$), there is a one-to-one mapping of $\{x \in \mathbb{R}^n : a^T x = b\}$ to the set $\mathbb{R}^{n-1} \times \{0\}$, and of the integer points in the respective sets. Therefore, for any $b \in \mathbb{Z}$, the set $\{x \in B : a^T x = b\}$ is mapped to a strictly lattice-free set $B' \times \{0\}$ with $B' \subset \mathbb{R}^{n-1}$. By the induction hypothesis, B' can be covered by $h(n - 1)$ split sets, and so can $B' \times \{0\}$ (by split sets $S(a^i, b^i)$ where $a^i_n = 0$). Applying the affine transformation $x \rightarrow U^{-1}x + v^b$ to the split sets $S(a^i, b^i)$, we get $h(n-1)$ split sets covering $\{x \in B : a^T x = b\}$. Add each of these split sets to $\mathcal{L}$.

Then $\mathcal{L}$ has size at most $|V| + |\bar{V}|h(n-1)$. Notice that

$$u - l \leq f(n) \Longrightarrow \lceil u \rceil - \lceil l \rceil \leq \lceil f(n) \rceil,$$

and $|V|, |\bar{V}| \leq \lceil u \rceil - \lceil l \rceil + 1$. Consequently, $|V|, |\bar{V}| \leq \bar{f}_n$ and the set $\mathcal{L}$ has size at most $\bar{f}_n(1 + h(n-1)) = h(n)$ and the desired bound follows. □

The previous result obviously also holds for the interior of any (maximal) lattice-free, bounded, convex set in $\mathbb{R}^n$. If the strictly lattice-free, convex set is unbounded, additional conditions are needed for Lemma 3.3 to hold; the conditions we choose may not be the least restrictive but suffice for our purpose. Lovász [27] showed that any maximal lattice-free, convex set is a polyhedron. Furthermore, if such a set is unbounded, then it is either an irrational hyperplane or it is full-dimensional and it can be expressed as $Q + L$ where Q is a polytope and L a rational linear space. In the latter case, $Q + L$ is called a *cylinder over the polytope* Q. Also see Basu et. al. [7] for a more recent and complete proof of Lovász's result.

Lemma 3.4 *Let $B \subset \mathbb{R}^n$ be a strictly lattice-free, unbounded, convex set. If B is contained in the interior of a maximal lattice-free, convex set in $\mathbb{R}^n$, then it is contained in the union of some $h(n)$ split sets.*

Proof Let B' be a maximal lattice-free, convex set containing B in its interior; then B' is full-dimensional. Therefore B' is not an irrational hyperplane. By Lovász's result, $B' = Q + L$, where L is rational, $\dim(L) = r$ for some $0 < r < n$, and Q is a lattice-free polytope contained in $L^\perp$, the orthogonal complement of L, and has dimension $n - r$. Furthermore, B is contained in $\text{int}(Q) + L$. As L and $L^\perp$ are rational, we can define an $n \times n$ unimodular matrix U such that $UL^\perp = \mathbb{R}^{n-r} \times \{0\}^r$. Therefore, every point in the set $UQ = \{Ux : x \in Q\}$ has its last r components zero. Further UQ is lattice-free, and Lemma 3.3 theorem gives $h(n-r)$ split sets in $\mathbb{R}^{n-r}$ whose union covers the projection of $\text{int}(UQ)$ on the first $n-r$ components. Let $S(\pi_i, \gamma_i)$ be the ith split set in the above union. Let $S(\pi'_i, \gamma_i)$ be the corresponding split set in $\mathbb{R}^n$ which is defined as follows: π'_i is obtained by appending r zeros to π_i and then multiplying by U^{-1}. It is easy to see that $\text{int}(Q) + L$ and therefore B is covered by $\cup_{i=1}^{h(n-r)} S(\pi'_i, \gamma_i)$. As $h(n-r) \leq h(n)$, the result follows. □

Applying Lemma 3.4 with the bound of Rudelson [29] on $f(n)$, it is easy to obtain an exponential upper bound of $O(n!^{4/3})$ on the number of split sets needed to cover a strictly lattice-free, convex set. We will give a smaller exponential lower bound in the next section.

Recall the mixed-integer set P defined in (1). Let $c^T x + d^T y \geq \gamma$ be a non-trivial rational valid inequality for conv (P), i.e., $c^T x + d^T y \geq \gamma$ is not valid for P^{LP}, but is valid for conv (P). Let $V \subset \mathbb{R}^{n+l}$ be defined as in (2), and let V^x be defined as the projection of V on the space of the integer variables. V^x is strictly lattice-free, and is non-empty as $c^T x + d^T y \geq \gamma$ is not valid for P^{LP}. As we discussed earlier, Jörg [21] showed that V^x is contained in the interior of a rational, lattice-free polyhedron

$B \subset \mathbb{R}^n$, and thus in the interior of a maximal lattice-free, convex set. Depending on whether V^x is bounded or unbounded, we can use either of the previous two lemmas to obtain the following result.

Theorem 3.5 *Any facet-defining inequality for* conv (P) *is a* $h(n)$*-branch split cut.*

We observed earlier that Jörg's results already express every facet-defining inequality as a disjunctive cut. The previous theorem gives an alternative expression of every facet-defining inequality as a disjunctive cut. We next obtain a finite disjunctive cutting-plane algorithm for arbitrary MIPs based on Theorem 3.5. This algorithm is however of purely theoretical interest, and is highly impractical.

Theorem 3.6 *The mixed-integer program* $\max\{c^T x + d^T y : (x, y) \in P\}$ *can be solved in finite time via a pure cutting-plane algorithm which generates only* $h(n)$*-branch split cuts.*

Proof Let $t = h(n)$. We will represent any t-branch split disjunction $D(\pi_1, \ldots, \pi_t, \gamma_1, \ldots, \gamma_t)$ by a vector v in $\mathbb{Z}^{(n+1)t}$; the components of $\pi_1, \ldots, \pi_t$ are arranged as the first nt components of v, and $\gamma_1, \ldots, \gamma_t$ form the last t components. Let $\Omega = \mathbb{Z}^{(n+1)t}$. As Ω is a countable set, by definition the vectors in Ω can be arranged in a sequence $\{\Omega_i\}$, say by increasing norm. Further let D_i be the t-branch split disjunction defined by Ω_i. For any facet-defining inequality of conv (P), there exists a (finite) integer k such that the inequality is a t-branch split cut defined by the disjunction D_k. Let k^* be the largest index of a disjunction associated with facet-defining inequalities. Now consider the following algorithm which does not compute or use the value of k^*. Let P_0 denote the continuous relaxation of P.

Repeat the following two steps for $i = 1, 2, \ldots$

1. Compute $P_i = P_{i-1} \cap \text{conv}\,(P_0 \cap D_i)$.
2. If the basic optimal solution of $\max\{c^T x + d^T y : (x, y) \in P_i\}$ is integral, terminate.

As P_i is a relaxation of P, an integral optimal solution over P_i is also an optimal solution over P. Further, as $P_{k^*} = \text{conv}\,(P)$, the algorithm must terminate for some $i \leq k^*$. □

If one wants to check validity of a given inequality, the termination criterion in the above algorithm can be modified to check it. Finally, if one wants to compute conv (P), then the termination criterion can be changed to verifying that all vertices of P_i are integral.

One drawback of the proof of finiteness in Theorem 3.6 is that it gives no information on how long the algorithm will run for any given P. We next prove that the number of generated disjunctions is bounded by a function of the encoding size of $P^{LP} = \{(x, y) \in \mathbb{R}^{n+l} : Ax + Gy \leq b\}$. The encoding size of an integer is the number of bits in its binary encoding plus one. The encoding size of a rational number is the sum of the encoding sizes of its numerator and denominator, and the encoding size of a rational vector is the sum of encoding sizes of its components. The encoding size of a hyperplane is the sum of encoding sizes of its coefficients (including the zero coefficients). A polyhedron is said to have facet-complexity ϕ if it can be described by a list of inequalities such that the corresponding hyperplanes have encoding size at most ϕ (the inequalities are then said to have encoding size at most ϕ).

Theorem 3.7 *If P in Theorem 3.6 is defined by m linear constraints, each with encoding size ϕ, then there exists a function $g : \mathbb{Z}^2 \to \mathbb{Z}$ such that the maximum number of $h(n)$-branch split disjunctions enumerated by the algorithm in Theorem 3.6 is at most $g(m, \phi)$.*

Proof We say that an expression or a set "has bounded complexity" to mean that its encoding size is bounded from above by a function of m and ϕ, and note that $n+l \leq \phi$ by definition. As P^{LP} has facet-complexity at most ϕ, the facet-complexity of conv (P) is at most $\phi_1 = 24(n+l)^5\phi$ (see [30, Corollary 17.1a]). In other words, for every facet of conv (P), there is an inequality $c^T x + d^T y \geq \gamma$ defining the facet with encoding size at most ϕ_1. Consider the set V in (2) defined as the points of P^{LP} cut off by $c^T x + d^T y \geq \gamma$, and remember that V^x denotes the projection of V onto the space of integer variables. It is shown in [12,21] that $V^x = \{x \in \mathbb{R}^n : A_1 x \leq b_1, A_2 < b_2\}$ for some rational matrices A_1, A_2 and rational vectors b. Further, for any inequality that appears in the description of this set, if a is the vector of coefficients of the x variables and β is the right hand side, then

$$a^T = \bar{\lambda} A + \bar{\mu} c, \quad \beta = \bar{\lambda} b + \bar{\mu}\gamma, \tag{10}$$

where $(\bar{\lambda}, \bar{\mu})$ is an extreme direction of the cone

$$\mathcal{C} = \{(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R} : \lambda G + \mu d = 0, \lambda \geq 0, \mu \geq 0\}.$$

Notice that the facet complexity of $\mathcal{C}$ is bounded by the encoding size of a column of $\begin{bmatrix} G \\ d \end{bmatrix}$. Therefore, the facet complexity of $\mathcal{C}$ is at most $\phi_2 = m\phi + \phi_1$ and consequently, the extreme points and directions of $\mathcal{C}$ have encoding size at most $4(m+1)^2\phi_2$ ([30, Theorem 10.2]). As each extreme direction $(\bar{\lambda}, \bar{\mu})$ that appears in (10) has bounded complexity we conclude that every inequality defining V^x has bounded complexity. It is shown in [12] that V^x is contained in the interior of the rational, full-dimensional, lattice-free polyhedron $B = \{x \in \mathbb{R}^n : A_1 x \leq b_1 + \mathbf{1}, A_2 x \leq b_2\}$ which again has bounded facet complexity (say at most ϕ_3). Therefore, it is possible to write $B = Q_B + C_B$ for some $Q_B = \text{conv}\left(v_1, \ldots, v_q\right)$ and $C_B = \text{cone}\,(r_1, \ldots, r_s)$ where q and s are finite and $q \geq 1$. Furthermore the vectors $v_1, \ldots, v_q$ and $r_1, \ldots, r_s$ are rational and have encoding size at most $4n^2\phi_3$ (see [30, Theorem 10.2]).

We next show that there exists an integral vector $z \in \mathbb{R}^n$ with bounded complexity such that the width of B along the direction z equals the lattice width of B. For $k = 1, \ldots, n$, consider the integer program IP_k which, as we show below, gives the minimum $w(B, z)$ along all vectors z with $z_k \geq 1$:

$$\begin{aligned} \min \quad & t \\ s.t. \quad & t \geq z^T(v_i - v_j) \quad \forall i,j \in \{1,\dots,q\}, \\ & r_j^T z = 0, \quad \forall j \in \{1,\dots,s\}, \\ & z_k \geq 1, \quad z \in \mathbb{Z}^n, \\ & t \geq 0. \end{aligned}$$

First consider a feasible integral solution $(\bar{t}, \bar{z})$ for IP_k. Then, as $r_j^T \bar{z} = 0$ for all $j \in \{1,\dots,s\}$, we have $\max\{\bar{z}^T x : x \in B\} = \max\{\bar{z}^T v_i : i \in \{1,\dots,q\}\}$ and $\min\{\bar{z}^T x : x \in B\} = \min\{\bar{z}^T v_i : i \in \{1,\dots,q\}\}$. Consequently, $\bar{t} \geq w(B,\bar{z}) = \max\{\bar{z}^T(v_i - v_j) : i,j \in \{1,\dots,q\}\}$. Therefore, if (t^k, z^k) is an optimal solution of IP_k, we have $w(B, z^k) = t^k$.

As B is a full-dimensional lattice-free polyhedron $w(B)$ is bounded; in other words, there exists a non-zero integral vector z^* such that $w(B) = w(B, z^*) \leq f(n)$. As $w(B)$ is bounded, $r_j^T z^* = 0$ for all $j \in \{1,\dots,s\}$. In addition, if $z_k^* \neq 0$, then either $(w(B), z^*)$ or $(w(B), -z^*)$ is feasible for IP_k. Therefore, if p is such that $t^p \leq t^k$ for all $k = 1,\dots,n$ for which IP_k is feasible, then $w(B) = t^p = w(B, z^p)$.

Finally, note that IP_p has an optimal solution with bounded complexity as the encoding sizes of $v_1,\dots,v_q$ and $r_1,\dots,r_s$ are all bounded above by $4n^2\phi_3$, and the numbers q and s also have bounded complexity. Therefore there exists a vector z with bounded complexity such that $w(B,z) = w(B)$. This implies that B can be covered by split sets of the form $\{x \in \mathbb{R}^n : \tau < z^T x < \tau + 1\}$ and split sets which cover the sets $\{x \in B : z^T x = \tau\}$ for $\tau \in [\lfloor \min_j\{z^T v_j\}\rfloor, \lceil \max_j\{z^T v_j\}\rceil]$. The numbers τ in the sets above have bounded complexity. We can assume, via induction, that $\{x \in B : z^T x = \tau\}$ can be covered by $h(n-1)$ split sets with encoding size bounded above by a function of the facet complexity of $\{x \in B : z^T x = \tau\}$ and the number of constraints defining it. As these two numbers in turn have bounded complexity, we can conclude that there is a $h(n)$-branch disjunction of bounded complexity which implies the inequality $c^T x + d^T y \geq \gamma$. □

4 Covering lattice-free sets with split sets

In this section, we construct a lattice-free, bounded, convex set in $\mathbb{R}^n$ such that its interior cannot be covered by fewer than $\Omega(2^n)$ split sets. Note that the upper bound $h(n)$ on the number of split sets needed to cover such a set is significantly larger.

Recall that $S(a,b) = \{x \in \mathbb{R}^n : b < a^T x < b + 1\}$ is an open set. Given an integer vector $a \in \mathbb{Z}^n$, we refer to the collection of split sets $\{S(a,b) : b \in \mathbb{Z}\}$ as the collection of split sets defined by a. We refer to a as the defining vector of these split sets, and denote this fact using a function $d.v.(\cdot)$ where $d.v.(S(a,b)) = a$. We denote the Euclidean norm of a by $||a||$.

Definition 4.1 Let $K \subset \mathbb{R}^n$ be a compact set and let $\varepsilon > 0$ be given. We define

$$\mathcal{L}(K,\varepsilon) = \{a \in \mathbb{Z}^n \ : \ \mathrm{vol}(K \cap S(a,b)) > \varepsilon \quad \text{for some } b \in \mathbb{Z}\}.$$

Note that $\mathcal{L}(K,\varepsilon)$ can be empty, for example if ε is greater than the volume of K.

Lemma 4.2 *For any compact set $K \subset \mathbb{R}^n$ and any number $\varepsilon > 0$, the set $\mathcal{L}(K, \varepsilon)$ is finite.*

Proof Let $l \in \mathbb{R}$ be an upper bound on the $(n-1)$-dimensional volume of the intersection of a hyperplane with K. For any vector $a \in \mathbb{Z}^n$, the distance between two parallel hyperplanes of the form $\{x : a^T x = b\}$ and $\{x : a^T x = b + 1\}$ is $1/||a||$. Therefore, if $||a|| > l/\varepsilon$, the volume of the intersection of a split set $S(a, b)$ (for some $b \in \mathbb{Z}$) with K is at most $l/||a|| < \varepsilon$. Therefore $\mathcal{L}(K, \varepsilon)$ is a subset of $\{a \in \mathbb{Z}^n : ||a|| \leq l/\varepsilon\}$ and is a finite set. □

For example, if K is the set of points in $\mathbb{R}^2$ that lie inside a given triangle, then the length of the longest side of this triangle can be used as the number l in the proof of the previous lemma.

Lemma 4.3 *There exists a rational, lattice-free triangle $T_0 \subset \mathbb{R}^2$ and an $\varepsilon > 0$ such that $T_0 \backslash (S_1 \cup S_2)$ has area at least ε for any pair of split sets $S_1, S_2 \subset \mathbb{R}^2$.*

Proof Let T be the lattice-free triangle defined in the previous section with lattice width $1 + 2/\sqrt{3} \approx 2.1547$. Remember that vertices of T are irrational points. By slightly rotating each side of T about the integer point in its relative interior, we can obtain a rational, maximal, lattice-free triangle T_0 with lattice width arbitrarily close to $1 + 2/\sqrt{3}$, say equal to 2.15. Therefore, the interior of T_0 is not contained in the union of two split sets defined by linearly independent vectors [11,19], or by linearly dependent vectors (as $w(T_0) > 2$). (One can combine this observation with results in [19] to complete the proof, however we give a self-contained proof below.)

The intersection of the split set $\{x \in \mathbb{R}^n : 0 < x_1 < 1\}$ with T_0 has area at least 1/2, and therefore $(1, 0) \in \mathcal{L}(T_0, 1/2) \neq \emptyset$. As $\mathcal{L}(T_0, 1/2)$ is finite and T_0 is bounded, there are finitely many split sets defined by vectors in $\mathcal{L}(T_0, 1/2)$ such that their intersection with T_0 has an area of $1/2$ or more; find the one with maximum area of intersection with T_0. Let $0 < \varepsilon_1 \leq area(T_0) - 1/2$ be the area left uncovered by this split set. Split sets defined by vectors not contained in $\mathcal{L}(T_0, 1/2)$ cover an area of T_0 less than $1/2$ and consequently, the minimum area of T_0 left uncovered by any split set is at least ε_1.

Now consider $\mathcal{L}(T_0, \varepsilon_1/2)$. Let $\varepsilon_2 > 0$ be the area of T_0 left uncovered by any two split sets with defining vectors from $\mathcal{L}(T_0, \varepsilon_1/2)$. As the number of pairs of such split sets is finite, ε_2 exists. Let S_1 and S_2 be two arbitrary split sets. If their defining vectors belong to $\mathcal{L}(T_0, \varepsilon_1/2)$, then the area of T_0 not covered by these split sets is at least ε_2. If $d.v.(S_2) \notin \mathcal{L}(T_0, \varepsilon_1/2)$, notice that S_1 does not cover a portion of T_0 with area of at least ε_1, and S_2 covers a portion of T_0 with area at most $\varepsilon_1/2$. Consequently, S_1 and S_2 leave an area of $\min\{\varepsilon_1/2, \varepsilon_2\} > 0$ of T_0 uncovered. □

The triangle T_0 in the previous lemma contains lattice points on its boundary; the phrase "lattice-free" in the previous lemma can be replaced by "strictly lattice-free" if we shrink the triangle T_0 slightly so that the lattice-width remains strictly greater than two, but no integer points lie on the boundary. We denote the collection of all split sets in $\mathbb{R}^n$ by $\mathcal{S}^n$.

Definition 4.4 The set $A \subset \mathbb{R}^n$ is weakly covered by the split sets $S_1, \ldots, S_j \in \mathcal{S}^n$ if the volume of $A \backslash (S_1 \cup \ldots \cup S_j)$ is zero.

Recall that any convex, bounded, lattice-free set in $\mathbb{R}^2$ can be weakly covered by three split sets, but the triangle in the previous lemma cannot be weakly covered by two split sets.

Lemma 4.5 *Let l, m with $l \geq m$ be given positive integers and $K \subset \mathbb{R}^n$ be a compact set that cannot be weakly covered by any $m-1$ split sets in $\mathcal{S}^n$. Then there exists a finite set $\Sigma \subset \mathbb{Z}^n$ such that whenever K is weakly covered by $S_1, \dots, S_l \in \mathcal{S}^n$, then the defining vectors of at least m of these split sets are contained in Σ.*

Proof We use induction with respect to m and construct a family of sets $\Sigma(K, l, m)$ satisfying the desired property.

If $m = 1$ then at least one split set in a weak covering of K by l split sets must cover a volume of $\frac{\text{vol}K}{l}$ of K and therefore we choose

$$\Sigma(K, l, 1) = \mathcal{L}\left(K, \frac{\text{vol}K}{l}\right),$$

which is finite by Lemma 4.2.

For some $m \geq 1$, assume the result has been proved for all compact sets that cannot be weakly covered by $m-1$ split sets. Let K be a compact set that cannot be weakly covered by m split sets. Let σ be a collection of $l \geq m+1$ split sets weakly covering K. Let $S_0 \in \sigma$ be a split set whose intersection with K has the greatest volume of all split sets in σ. Then $\text{vol}(K \cap S_0) \geq \frac{\text{vol}K}{l}$ and therefore $d.v.(S_0) \in \mathcal{L}(K, \text{vol}K/l)$. The set $K \backslash S_0$ is a compact set which is weakly covered by $\sigma \backslash \{S_0\}$.

Further, $K \backslash S_0$ cannot be weakly covered by $m-1$ split sets, otherwise K can be weakly covered by m split sets. By induction, there exists a finite set $\Sigma(K \backslash S_0, l-1, m)$ such that at least m of the split sets in $\sigma \backslash \{S_0\}$ have their defining vectors in $\Sigma(K \backslash S_0, l-1, m)$. We take

$$\Sigma(K, l, m+1) = \mathcal{L}\left(K, \frac{\text{vol}K}{l}\right) \cup \bigcup_{S \in \mathcal{S}^n : \text{vol}(K \cap S) \geq \frac{\text{vol}K}{l}} \Sigma(K \backslash S, l-1, m)$$

which is a finite union of finite families of sets. □

We next make an important observation on how the set $\Sigma(K, l, m)$ changes under unimodular transformations. Given an $n \times n$ matrix M and a set $S \subseteq \mathbb{R}^n$, we define $MS = \{Ms : s \in S\}$. Recall from Section 3.2 that a linear transformation defined by a unimodular matrix maps any split set to a split set and does not alter the volume of a bounded set. In particular, given a bounded set A and a split set $S \in \mathcal{S}^n$, the volume of $A \cap S$ is the same as that of $MA \cap MS$ if M is an $n \times n$ unimodular matrix.

Remark 4.6 From the proof of Lemma 4.5 it also follows that $\Sigma(K, l, m)$ is equal to the union of finitely many sets of the form $\mathcal{L}(K', \varepsilon')$, where K' is obtained by subtracting up to $m-1$ split sets in $\mathcal{S}^n$ from K.

Furthermore, given any $n \times n$ unimodular matrix N and $\varepsilon > 0$, as $\mathcal{L}(NK, \varepsilon) = N\mathcal{L}(K, \varepsilon)$, it can easily be shown (by induction on m) that $\Sigma(NK, l, m) = N\Sigma(K, l, m)$ for any integer $l > 0$.

Lemma 4.7 *Given any two finite sets of vectors $V, W \subset \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, there exists a unimodular matrix M such that $MV \cap W = \emptyset$.*

Proof Let $q = \max_{v \in V \cup W} ||v||_\infty$ and let

$$M = \begin{pmatrix} 1 & \mu \\ \mu & \mu^2 + 1 \end{pmatrix} \quad \text{where } \mu = 3q.$$

Observe that M is integral and has determinant 1. Let $v = (v_1, v_2)^T \in V$. To prove that $Mv \notin W$, we first show that the first component of Mv, denoted by α, is nonzero. Note that $\alpha = v_1 + \mu v_2$. If $|v_2| \geq 1$, then α is a nonzero integer as $|v_1| < \mu$. If $v_2 = 0$, then α equals v_1 which is nonzero as every vector in V is nonzero.

The second row of Mv equals $\mu(v_1 + \mu v_2) + v_2 = \mu\alpha + v_2$. As $|\alpha| \geq 1$, $|v_2| \leq q$ and $\mu = 3q$, we have $|\mu\alpha + v_2| \geq 3q - q > q$. Thus $||Mv||_\infty > q$ and therefore Mv cannot belong to W. □

Lemma 4.8 *Let $l \geq 3, k \geq 1$ be integers. There exist rational, lattice-free triangles $T_0, \ldots, T_{k-1} \subset \mathbb{R}^2$ with the following properties: $(i) T_i$ cannot be weakly covered by fewer than three split sets for any $i = 0, \ldots, k-1$, (ii) the sets $\Sigma(T_0, l, 3), \ldots, \Sigma(T_{k-1}, l, 3)$ are pairwise-disjoint.*

Proof Let T_0 be the lattice-free triangle constructed in Lemma 4.3. For any 2×2 unimodular matrix N, clearly NT_0 is lattice-free. We will now construct unimodular matrices $N_0, \ldots, N_{k-1}$, where N_0 is the identity matrix, such that the triangles $T_i = N_i T_0$ for $i = 0, \ldots, k-1$, have the desired property. Consider any $k > 0$, and assume we have constructed $N_0, \ldots, N_{k-1}$. Let $V = \Sigma(T_0, l, 3)$, and let $W = \cup_{i=0}^{k-1} \Sigma(N_i T_0, l, 3)$. By Lemma 4.7, we can construct a unimodular matrix N_k such that $N_k V$ has no elements in common with W. By Remark 4.6, $N_k V = \Sigma(N_k T_0, l, 3)$. The result follows by induction on k. □

For any $n \geq 3$, applying Lemma 4.8 with $l = 3 \times 2^{n-2} - 1$ and $k = 2^{n-2}$ we obtain triangles T_i for $i \in \{0, \ldots, 2^{n-2} - 1\}$ such that no triangle T_i can be weakly covered by fewer than three split sets and for any pair of indices $i \neq j$ the intersection $\Sigma(T_i, 3 \times 2^{n-2} - 1, 3) \cap \Sigma(T_j, 3 \times 2^{n-2} - 1, 3)$ is empty.

For an integer $\Delta \in \{0, \ldots, 2^{n-2} - 1\}$, let δ_l stand for the lth bit in the binary expansion of Δ in $n-2$ bits. In other words, $\Delta = \sum_{l=1}^{n-2} \delta_l 2^{l-1}$ with each $\delta_l \in \{0, 1\}$. For each $\Delta \in \{0, \ldots, 2^{n-2} - 1\}$, we define the corresponding 2-dimensional affine subspace

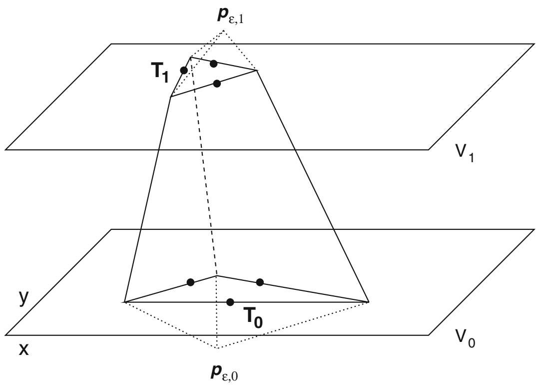
$$V_\Delta := \{(\delta_1, \ldots, \delta_{n-2}, x, y) | x, y \in \mathbb{R}\}$$

and a triangle in this two-dimensional affine subspace

$$\mathbf{T}_\Delta := \{(\delta_1, \ldots, \delta_{n-2}, x, y) | (x, y) \in T_\Delta\}.$$

Let $\text{cent}(T_\Delta)$ stand for the *centroid* of the triangle T_Δ, i.e., if the vertices of T_Δ are $u, v, w \in \mathbb{R}^2$, then $\text{cent}(T_\Delta) = (u + v + w)/3$. For any positive number ε, and any

Fig. 3 $\mathbf{T}^\varepsilon$ when $n = 3$



$\Delta \in \{0, \dots, 2^{n-2} - 1\}$, we define the point

$$p_{\varepsilon,\Delta} := (\delta_1, \dots, \delta_{n-2}, \mathrm{cent}(T_\Delta)) + ((2\delta_1 - 1)\varepsilon, \dots, (2\delta_{n-2} - 1)\varepsilon, 0, 0).$$

For example, when $\delta_i = 0$ for $i = 1, \dots, n-2$ (i.e., $\Delta = 0$) then $p_{\varepsilon,\Delta} = (-\varepsilon, \dots, -\varepsilon, \bar{x}, \bar{y})$ where $(\bar{x}, \bar{y})$ is the centroid of T_0, and similarly, when $\delta_i = 1$ for $i = 1, \dots, n-2$, then $p = (1+\varepsilon, \dots, 1+\varepsilon, x', y')$ where (x', y') is the centroid of $T_{2^{n-2}-1}$. Finally, we define the polytope $\mathbf{T}^\varepsilon$ as

$$\mathbf{T}^\varepsilon := \mathrm{conv}\left(\bigcup_{\Delta=0}^{2^{n-2}-1} (\mathbf{T}_\Delta \cup \{p_{\varepsilon,\Delta}\})\right).$$

In Figure 3, we depict $\mathbf{T}^\varepsilon$ when $n = 3$; here $V_0 = \{(0, x, y) | x, y \in \mathbb{R}\}$ and $V_1 = \{(1, x, y) | x, y \in \mathbb{R}\}$. The filled circles represent integer points on the boundaries of the triangles $\mathbf{T}_0 \subset V_0$ and $\mathbf{T}_1 \subset V_1$.

Let e_i stand for the unit vector in $\mathbb{R}^n$ with a one in the ith component and a 0 in all other components, and let $\mathbf{0}$ (respectively, $\mathbf{1}$) stand for the all-zero (resp., all-ones) vector in $\mathbb{R}^n$.

Lemma 4.9 *For any rational $\varepsilon > 0$, $\mathbf{T}^\varepsilon$ is a rational polytope and its integer hull is full-dimensional.*

Proof Let ε be a positive rational number. The triangles $\mathbf{T}_\Delta$ are rational polytopes for $\Delta \in \{0, \dots, 2^{n-2} - 1\}$, and the points $p_{\varepsilon,\Delta}$ are also rational for all Δ. Therefore $\mathbf{T}^\varepsilon$ is a rational polytope.

For the second part, recall that the triangle T_0 contains the integer points $(0, 0)$, $(1, 0)$ and $(0, 1)$. Furthermore, the point $(0, 0) \in T_i$ for all $i = 0, \dots, 2^{n-2} - 1$, as $T_i = N_i T_0$ for some unimodular matrix N_i. Therefore, $\mathbf{T}^\varepsilon$ contains the integer points $\mathbf{0}, e_1, \dots, e_n$ as e_{n-1}, e_n and $\mathbf{0}$ belong to $\mathbf{T}_0$ and $e_i \in \mathbf{T}_{2^{i-1}}$ for $i = 1, \dots, n-2$. The set of points $\{\mathbf{0}, e_1, \dots, e_n\}$ has affine dimension n and therefore the integer hull of $\mathbf{T}^\varepsilon$ is full-dimensional. □

Theorem 4.10 *There exists a rational $\varepsilon > 0$ such that*

(i) *the relative interior of* $\mathbf{T}_\Delta$ *is contained in the interior of* $\mathbf{T}^\varepsilon$*, for* $0 \leq \Delta < 2^{n-2}$*,*
(ii) $\mathbf{T}^\varepsilon \cap V_\Delta = \mathbf{T}_\Delta$ *for* $0 \leq \Delta < 2^{n-2}$*, and*
(iii) $\mathbf{T}^\varepsilon$ *is lattice-free.*

Proof (i) Let p be a point in the relative interior of $\mathbf{T}_\Delta$ for some $\Delta \in \{0, \ldots, 2^{n-2}-1\}$; then $p = (\delta_1, \ldots, \delta_{n-2}, x_p, y_p)$ where $(x_p, y_p) \in \text{int}(T_\Delta)$. We will show that p strictly satisfies every facet-defining inequality of $\mathbf{T}^\varepsilon$, and as $\mathbf{T}^\varepsilon$ is full-dimensional by Lemma 4.9, this will imply that p is contained in the interior of $\mathbf{T}^\varepsilon$. Let $a^T x \leq b$ be an inequality defining a facet of $\mathbf{T}^\varepsilon$; it is uniquely defined up to multiplication by a scalar. Let $a = (a_1, \ldots, a_n)$.

Assume that $a^T p = b$. Clearly, there exists some $\sigma > 0$ such that the points $(x_p, y_p) \pm \sigma(1, 0)$ and $(x_p, y_p) \pm \sigma(0, 1)$ are all contained in T_0. Therefore $p \pm \sigma e_{n-1} \in \mathbf{T}^\varepsilon$ and $p \pm \sigma e_n \in \mathbf{T}^\varepsilon$ which implies that $a_{n-1} = a_n = 0$. For $i = 1, \ldots, n-2$, the point

$$q^i = (\delta_1, \ldots, \delta_{i-1}, 1 - \delta_i, \delta_{i+1}, \delta_{n-2}, 0, 0) \in \mathbf{T}^\varepsilon \Rightarrow a^T q^i \leq b,$$

and therefore $a_i \geq 0$ if $\delta_i = 1$ and $a_i \leq 0$ if $\delta_i = 0$ for $i = 1, \ldots, n-2$. But this sign pattern of the coefficients of a implies that $a^T p_{\varepsilon,\Delta} > b$ unless $a = \mathbf{0}$, but then $a^T x \leq b$ is not a facet-defining inequality, a contradiction. Therefore, $a^T p < b$.

(ii) As $\mathbf{T}^\varepsilon$ contains the convex hull of sets of the form $\mathbf{T}_\Delta$, for any $\Delta \in \{0, \ldots, 2^{n-2} - 1\}$ we have $\mathbf{T}^\varepsilon \cap V_\Delta \supseteq \mathbf{T}_\Delta$. We need to show the reverse inclusion. Once again, for convenience, we only prove the result for $\Delta = 0$; the proof for $\Delta \neq 0$ is similar. Let $d > 0$ be such that for all $\Delta \in \{0, \ldots, 2^{n-2} - 1\}$ the distance of $\text{cent}(T_\Delta)$ to the boundary of T_Δ is at least d. In addition, let B be the maximum distance between any two vertices of $\mathbf{T}^\varepsilon$. Clearly these numbers exist.

Now consider an arbitrary point $v \in \mathbf{T}^\varepsilon \cap V_0$. Clearly v can be expressed as a convex combination of the vertices of $\mathbf{T}^\varepsilon$ and note that these vertices come from the set of vertices of $\mathbf{T}_\Delta$ and from the points $p_{\varepsilon,\Delta}$ for varying Δ. Thus, for some index set U we have $v = \sum_{j \in U} \lambda_j v^j$ with $\sum_{j \in U} \lambda_j = 1, \lambda_j > 0$ and v_j is a vertex of $\mathbf{T}^\varepsilon$ for all $j \in U$.

Note that $\mathbf{T}^\varepsilon$ has at most four vertices which have all first $n-2$ components less than or equal to 0, namely, the three vertices of $\mathbf{T}_0$ which all have their first $n-2$ components equal to 0 and $p_{\varepsilon,0}$ which has all of the first $n-2$ components equal to $-\varepsilon$. Let $U' \subseteq U$ be the indices of the remaining vertices. Let v_i stand for the ith component of v, and let v_i^j stand for the ith component of v^j. For $j \in U'$ we have

$$\sum_{i=1}^{n-2} v_i^j \geq 1 - n\varepsilon,$$

as among the first $n-2$ components of such a v^j at least one component is at least 1 and the rest are at least $-\varepsilon$. When $\varepsilon > 0$ is chosen to be less than $1/n$, we have $1 - n\varepsilon > 0$ and therefore the only vertex of $\mathbf{T}^\varepsilon$ that has the sum of the first $n-2$ components strictly negative is $p_{\varepsilon,0}$. We now consider two cases:

Case 1 Assume $p_{\varepsilon,0} \notin \{v^j : j \in U\}$. As $v = \sum_{j \in U} \lambda_j v^j$, we have,

$$\sum_{i=1}^{n-2} v_i = 0 = \sum_{j \in U} \lambda_j \left(\sum_{i=1}^{n-2} v_i^j \right). \tag{11}$$

Using the fact that $\lambda > 0$ and $\sum_{i=1}^{n-2} v_i^j > 0$ for all $j \in U'$ we conclude that $U' = \emptyset$, which in turn implies that $v \in \mathbf{T}_0$.

Case 2 Now assume that $p_{\varepsilon,0} \in \{v^j : j \in U\}$ and let $v^0 = p_{\varepsilon,0}$ with the associated coefficient λ_0. Also let $w = \sum_{j \in U'} \lambda_j$. Note that the sum of the first $n-2$ components of $p_{\varepsilon,0}$ is greater than $-n\varepsilon$. Once again, from (11) we get

$$0 \geq -\lambda_0 n\varepsilon + (1 - n\varepsilon)w \Rightarrow \lambda^0 \geq (1/(n\varepsilon) - 1)w.$$

When ε is small enough, we have $1/(n\varepsilon) - 1 > B/d$, and therefore $\lambda^0/w > B/d$. We now rewrite $v = \sum_{j \in U} \lambda_j v^j$ as

$$v = \sum_{j \neq 0, j \notin U'} \lambda_j v^j + (\lambda_0 + w) \left(\frac{\lambda_0}{\lambda_0 + w} v^0 + \frac{w}{\lambda_0 + w} \left[\sum_{j \in U'} \frac{\lambda_j}{w} v^j \right] \right). \tag{12}$$

Consider the point $p' = \sum_{j \in U'} (\lambda_j / w) v^j$ given by the sum inside the square brackets in (12) and note that it is a convex combination of the vertices of $\mathbf{T}^\varepsilon$. All of these vertices are at a distance of at most B from any other point in $\mathbf{T}^\varepsilon$, and therefore so is p'. Let the last two coordinates of p' be (x', y'), and note that the last two coordinates of v^0 equal $\text{cent}(T_0)$ and the last two coordinates of v^j define a vertex of T_0 for all $j \neq 0$, $j \notin U'$. The vector (v_{n-1}, v_n) consisting of the last two coordinates of v is a convex combination of (x', y'), $\text{cent}(T_0)$ and vertices of T_0.

The expression inside the curved brackets in (12) gives a convex combination of v^0 and p' and the last two coordinates of this point equals

$$\frac{\lambda_0}{\lambda_0 + w} \text{cent}(T_0) + \frac{w}{\lambda_0 + w} (x', y').$$

Let $\bar{p} \in \mathbb{R}^2$ stand for the point above. As $\lambda^0/w > B/d$, the ratio of the distance between $\bar{p}$ and $\text{cent}(T_0)$ to the distance between $\bar{p}$ and (x', y') is less than d/B. As the distance of $\text{cent}(T_0)$ to the boundary of T_0 is at least d and the distance of (x', y') to the boundary of T_0 is at most B, $\bar{p}$ is contained in the interior of T_0. Consequently, the vector consisting of the last two coordinates of v is a convex combination of points in T_0 and is thus contained in T_0. Therefore $v \in \mathbf{T}_0$ and $\mathbf{T}^\varepsilon \cap V_0 \subseteq \mathbf{T}_0$.

(iii) If $v \in \mathbb{Z}^n$ is an integer point in $\mathbf{T}^\varepsilon$, the first $n-2$ components of v must be 0–1, and thus $v \in V_\Delta$ for some $0 \leq \Delta < 2^{n-2}$. But, by construction, $\mathbf{T}^\varepsilon \cap V_\Delta = \mathbf{T}_\Delta$. Therefore $v \in \mathbf{T}_\Delta$ for some $0 \leq \Delta < 2^{n-2}$. Further, if v is contained in the interior of $\mathbf{T}^\varepsilon$, then it must be contained in the relative interior of $T \cap H$ for any affine subspace

H of dimension one or more. But v is not contained in the relative interior of $\mathbf{T}^{\varepsilon} \cap V_{\Delta}$. Thus $\mathbf{T}^{\varepsilon}$ is lattice-free. □

Let $\Pi : \mathbb{R}^n \to \mathbb{R}^2$ denote projection to the last two coordinates and consider a split set $S = S(a, b)$ in $\mathbb{R}^n$ where $a = (\alpha_1, \ldots \alpha_n) \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$. For any $\Delta \in \{0, \ldots, 2^{n-2} - 1\}$ we have

$$S \cap V_{\Delta} = \left\{ (\delta_1, \ldots, \delta_{n-2}, x, y) : b - \sum_{l=1}^{n-2} \delta_l \alpha_l < \alpha_{n-1} x + \alpha_n y < b + 1 - \sum_{l=1}^{n-2} \delta_l \alpha_l \right\}.$$

Notice that if $\alpha_{n-1} = \alpha_n = 0$, then $\alpha_{n-1} x + \alpha_n y = 0$, which implies that $S \cap V_{\Delta} = \emptyset$ as 0 cannot be strictly contained between two consecutive integers. On the other hand, if $(\alpha_{n-1}, \alpha_n) \neq (0, 0)$, then $S \cap V_{\Delta}$ is a nonempty split set and its defining vector is $(\alpha_{n-1}, \alpha_n)^T$. Therefore $\Pi(S \cap V_{\Delta})$ is either a split set in $\mathbb{R}^2$ with defining vector $(\alpha_{n-1}, \alpha_n)^T$, or it is an empty set.

We are now ready to show an exponential lower bound on the number of split sets that can cover a lattice-free set.

Theorem 4.11 (Lower bound result) *Let $\varepsilon > 0$ be such that $\mathbf{T}^{\varepsilon}$ satisfies the properties in Theorem 4.10. The interior of $\mathbf{T}^{\varepsilon}$ is not contained in the union of any $3 \times 2^{n-2} - 1$ split sets.*

Proof By contradiction. Suppose that $\text{int}(\mathbf{T}^{\varepsilon})$ is contained in the union of t split sets $S_1, \ldots, S_t$ in $\mathcal{S}^n$, where $t < 3 \times 2^{n-2}$. Then, Theorem 4.10(i) implies that the relative interiors of the triangles $\mathbf{T}_{\Delta}$ for $0 \leq \Delta < 2^{n-2}$ are also contained in $\cup_{i=1}^{t} S_t$; therefore the triangles $\mathbf{T}_{\Delta}$ are weakly covered by $\cup_{i=1}^{t} S_t$.

Let $\Delta \in \{0, \ldots, 2^{n-2} - 1\}$. Let $S_i^{\Delta} = \Pi(S_i \cap V_{\Delta})$. By the previous lemma, S_i^{Δ} is either a split set in $\mathbb{R}^2$ if the last two components of $d.v.(S_i)$ are nonzero, or the empty set otherwise. In the latter case, we define $d.v.(S_i^{\Delta}) = (0, 0)$ for convenience; then $d.v.(S_i^{\Delta}) = \Pi(d.v.(S_i))$ for any i, and therefore, $d.v.(S_i^{\Delta})$ does not depend on Δ. As $\text{int}(\mathbf{T}^{\varepsilon}) \subseteq S_1 \cup \cdots \cup S_t$, it follows that $\Pi(\text{int}(\mathbf{T}^{\varepsilon}) \cap V_{\Delta}) = \Pi(\text{int}(\mathbf{T}_{\Delta})) = \text{int}(T_{\Delta})$ is contained in $S_1^{\Delta} \cup \cdots \cup S_t^{\Delta}$. Therefore, there exist three sets, say $S_1^{\Delta}, S_2^{\Delta}, S_3^{\Delta}$, such that $d.v.(S_i^{\Delta}) \in \Sigma(T_{\Delta}, 3 \times 2^{n-2} - 1, 3)$ for $i = 1, 2, 3$.

For any $\Delta' \neq \Delta$, there are indices p, q, r such that $d.v.(S_i^{\Delta'}) \in \Sigma(T_{\Delta'}, 3 \times 2^{n-2} - 1, 3)$ for $i = p, q, r$. But as the sets $\Sigma(T_{\Delta}, 3 \times 2^{n-2} - 1, 3)$ and $\Sigma(T_{\Delta'}, 3 \times 2^{n-2} - 1, 3)$ are disjoint, and $d.v.(S_i^{\Delta}) = d.v.(S_i^{\Delta'})$ for $i = 1, \ldots, t$, it follows that $\{p, q, r\}$ is disjoint from $\{1, 2, 3\}$. Arguing similarly for indices not equal to Δ and Δ', we can conclude that $t \geq 3 \times 2^{n-2}$, a contradiction. □

Note that the first coordinate of any point in $\mathbf{T}^{\varepsilon}$ is contained in the interval $[-\varepsilon, 1+\varepsilon]$ and consequently $\mathbf{T}^{\varepsilon}$ has lattice-width at most $1+2\varepsilon < 2$. Therefore $\mathbf{T}^{\varepsilon}$ can be covered by three closed split sets, even though its interior can only be covered by an exponential number (in n) of split sets.

In $\mathbb{R}^3$, Theorem 4.11 yields a rational, lattice-free polytope which needs 6 split sets to cover its interior. We can improve this number by one (in general, we can improve the bound in Theorem 4.11 by one too).

Theorem 4.12 *In $\mathbb{R}^3$, there is a rational, lattice-free polytope such that its interior is not contained in the union of any 6 split sets.*

Proof Let T_0 and T_1 be triangles in $\mathbb{R}^2$, constructed using Lemma 4.8, such that each triangle cannot be weakly covered by fewer than three split sets and $\Sigma(T_0, 6, 3) \cap \Sigma(T_1, 6, 3) = \emptyset$. Let

$$V_i = \{(x, y, i) : x, y \in \mathbb{R}\} \quad \text{and} \quad \mathbf{T}_i = \{(x, y, i) : (x, y) \in T_i\} \quad \text{for } i = 0, 1.$$

Let $p_{\varepsilon,0} = (\text{cent}(T_0), -\varepsilon)$ and $p_{\varepsilon,1} = (\text{cent}(T_1), 1 + \varepsilon)$. We define $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$ to be the projection to the first two coordinates. In addition let $Q = \{(x, y) : 0 \leq x, y \leq 7\}$. Q has an area of 49, but the area of Q covered by any (translated) split set in $\mathbb{R}^2$ is at most 7, and hence at least 7 such (translated) split sets are needed to cover Q. Finally, let $\mathbf{T}' = \text{conv}\left(\mathbf{T}_0, \mathbf{T}_1, \{p_{\varepsilon,0}, p_{\varepsilon,1}\}, Q^{1/2}\right)$, where $Q^{1/2} = \{(x, y, z) : 0 \leq x, y \leq 7,\ z = 1/2\}$. We will show that $\mathbf{T}'$ has the desired property.

One can choose ε so that $\mathbf{T}'$ is a rational, lattice-free polytope, and the relative interiors of $\mathbf{T}_0$ and $\mathbf{T}_1$ are contained in the interior of $\mathbf{T}'$, as in the proof of Theorem 4.10, and we assume ε is chosen in such a manner. Assume that there exist $t \leq 6$ split sets $S_1, \ldots, S_t$ in $\mathbb{R}^3$ such that their union contains $\text{int}(\mathbf{T}')$. Let $d.v.(S_i) = (\alpha_1^i, \alpha_2^i, \alpha_3^i) \in \mathbb{Z}^3$. It is clear from the proof of Theorem 4.11 that there must be three split sets, say S_1, S_2, S_3, such that the defining vectors of the split sets $\Pi(S_i \cap V_0)$, i.e., (α_1^i, α_2^i), are contained in $\Sigma(T_0, 6, 3)$ for $i = 1, 2, 3$. Similarly, there must be three split sets S_p, S_q, S_r such that $(\alpha_1^i, \alpha_1^i) \in \Sigma(T_1, 6, 3)$ for $i = p, q, r$. As $\Sigma(T_0, 6, 3) \cap \Sigma(T_1, 6, 3) = \emptyset$, it follows that $\{1, 2, 3\} \cap \{p, q, r\} = \emptyset$, which implies that $\{p, q, r\} = \{4, 5, 6\}$ and $t = 6$. Further, the split sets S_i for $i = 1, \ldots, 6$ have the property that their defining vectors $(\alpha_1^i, \alpha_2^i, \alpha_3^i)$ satisfy $(\alpha_1^i, \alpha_2^i) \neq (0, 0)$, and therefore the intersection of these split sets with $\{(x, y, z) : z = 1/2\}$ are nonempty, two-dimensional translated split sets which cover Q. But Q cannot be covered by any six translated split sets in $\mathbb{R}^2$, a contradiction. □

Note that the integer hull of $\mathbf{T}'$ in the previous theorem has dimension 3. The next theorem connects the previous results with the inexpressibility of facet-defining inequalities of polyhedral sets as t-branch split cuts. (See [14, Lemma 2] for similar proof techniques for lattice-free cuts.)

Theorem 4.13 *Let t be a positive integer and $B \subset \mathbb{R}^n$ be a rational, full-dimensional, lattice-free polytope. Assume that the integer hull of B has dimension n and the interior of B cannot be covered by t split sets. Then there exists a mixed-integer set in $\mathbb{Z}^n \times \mathbb{R}$, defined by rational linear inequalities, that has a facet-defining inequality which cannot be expressed as a t-branch split cut.*

Proof Let B and t satisfy the conditions of the theorem. Let $\bar{x}$ be a point in the interior of B. Let B' be the polyhedron in $\mathbb{R}^{n+1}$ defined as

$$B' = \text{conv}\left((B \times \{-1\}) \cup (B \times \{0\}) \cup (\bar{x} \times \{1/2\})\right).$$

We define a mixed-integer polyhedral set P_B as follows:

$$P_B = \{(x, y) \in \mathbb{Z}^n \times \mathbb{R} : (x, y) \in B'\}.$$

Let ihull $(\cdot)$ stand for the integer hull of its argument; then B' and ihull $\left(B'\right)$ have dimension $n + 1$, and so does conv (P_B). All mixed-integer solutions of P_B satisfy $y \leq 0$; this is in fact a facet-defining inequality for conv (P_B) as conv $(P_B) \cap \{(x, y) : y = 0\}$ equals ihull $(B) \times \{0\}$ which has dimension n.

Let $S_1, \ldots, S_t$ be t arbitrary split sets in $\mathbb{R}^{n+1}$ defined on the x variables, i.e., they are of the form $\hat{S}_i \times \mathbb{R}$, where $\hat{S}_i$ are split sets in $\mathbb{R}^n$. Recall that $y \leq 0$ is a t-branch split cut for P_B derived from the disjunction associated with the split sets $S_1, \ldots, S_t$ if and only if it is valid for $B' \setminus \cup_{i=1}^t S_i$. The split sets $\hat{S}_i$ do not cover the interior of B. Let $\hat{x} \in \text{int}(B) \setminus \cup_{i=1}^t \hat{S}_i$. Then $B' \setminus \cup_{i=1}^t S_i$ contains a point of the form $(\hat{x}, \varepsilon')$ for some $\varepsilon' > 0$. This point violates the inequality $y \leq 0$, and thus $y \leq 0$ cannot be expressed as a t-branch split cut. □

This result when combined with Theorem 4.11 implies the existence of mixed-integer polyhedral sets with n integer variables with the property that their convex hull has a facet-defining inequality which cannot be expressed as a $(3 \times 2^{n-2} - 1)$-branch split cut.

Corollary 4.14 *For any $n \geq 3$ there exists a nonempty rational mixed-integer polyhedral set in $\mathbb{Z}^n \times \mathbb{R}$ with a facet-defining inequality that cannot be expressed as a $(3 \times 2^{n-2} - 1)$-branch split cut.*

5 Concluding remarks

As mentioned in the introduction, every cut based on a maximal lattice-free convex set in $\mathbb{R}^2$ is implied by a crooked cross cut and therefore by a 3-branch split cut [11]. This result is derived using the classification of maximal lattice-free sets in $\mathbb{R}^2$ by Dey and Wolsey [16]. An analogous classification result is not yet known in $\mathbb{R}^3$, and seems unattainable in $\mathbb{R}^n$ for larger n using current tools.

Combining the fact that any lattice-free convex set in $\mathbb{R}^2$ is contained in a 3-branch split set with Theorem 3.2 that bounds the lattice width of lattice-free, convex sets in $\mathbb{R}^3$, it is possible to show that cuts based on lattice-free convex sets in $\mathbb{R}^3$ are implied by 21-branch split cuts.

Theorem 5.1 *Any strictly lattice-free, convex set B in $\mathbb{R}^3$ is contained in the union of 21 split sets. Further, there is a disjunction not intersecting B that can be constructed using at most 22 atoms.*

Proof Let a stand for the direction of minimum lattice width and remember that $w(B) < 4.25$. Therefore, B is strictly contained in a set of the form $\{x \in \mathbb{R}^3 : q < a^T x < q + 4.25\} \subset \{x \in \mathbb{R}^3 : \lfloor q \rfloor < a^T x < \lceil q + 4.25 \rceil$ for some number q. Consider split sets of the form $\{x \in \mathbb{R}^3 : b < a^T x < b + 1\}$ for up to 6 consecutive values of $b = \lfloor q \rfloor, \ldots, \lceil q + 4.25 \rceil - 1$. Then B minus the union of these sets consists of two-dimensional, strictly lattice-free sets of the form $\{x \in B : a^T x = b\}$ for at most 5 consecutive values of b. Each such lattice-free set needs 3 split sets for a total of 21 split sets.

It is possible to construct a disjunction which does not intersect B using at most 22 atoms as follows: the first two atoms are $\{x \in \mathbb{R}^3 : a^T x \leq \lfloor q \rfloor\}$ and $\{x \in \mathbb{R}^3 : a^T x \geq \lceil q + 4.25 \rceil\}$. In addition, there are at most 4 atoms for each nonempty set of the form $\{x \in B : a^T x = b\}$ where $b = \lfloor q \rfloor + 1, \ldots, \lceil q + 4.25 \rceil - 1$. □

The above upper bound of 21 split sets is quite a bit higher than the lower bound of 7 split sets we obtained earlier. It would be interesting to obtain the smallest number of split sets needed to cover every lattice-free set in $\mathbb{R}^3$.

Acknowledgments We would like to thank the two anonymous referees for their careful and detailed reports.

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