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Optimal phasor measurement unit placement for numerical observability in the presence of conventional measurements using semi-definite programming

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Abstract: This study presents a new approach for optimal placement of synchronised phasor measurement units (PMUs) to ensure complete power system observability in the presence of non-synchronous conventional measurements and zero injections. Currently, financial or technical restrictions prohibit the deployment of PMUs on every bus, which in turn motivates their strategic placement across the power system. PMU allocation is optimised here based on measurement observability criteria for achieving solvability of the power system state estimation. Most of the previous work has proposed topological observability based methods for optimal PMU placement (OPP), which may not always ensure numerical observability required for successful execution of state estimation. The proposed OPP method finds out the minimum number and the optimal locations of PMUs required to make the power system numerically observable. The problem is formulated as a binary semi-definite programming (BSDP) model, with binary decision variables, minimising a linear objective function subject to linear matrix inequality observability constraints. The BSDP problem is solved using an outer approximation scheme based on binary integer linear programming. The developed method is conducted on IEEE standard test systems. A large-scale system with 3120 buses is also analysed to exhibit the applicability of proposed model to practical power system cases.

1 Introduction

The state estimator (SE) is the key data processing tool in modern energy management systems (EMS), used to maintain the power system in a secure operation and detect faulty equipment by providing the EMS with an accurate system state. Supervisory control and data acquisition (SCADA) system collects measurement data in real time from remote terminal units (RTUs) installed in substations across the power system and feeds the SE. Typical RTU measurements include power flows, power injections, and voltage magnitudes.

With the development of the global positioning satellite (GPS) technology, the measurement set can be enlarged, incorporating additional measurements obtained from phasor measurement units (PMUs) installed at selected substations in the system. These are high accuracy measuring devices providing synchronised positive sequence bus voltage and branch current phasor measurements [1]. Nowadays, the number of PMUs deployed in the power systems is drastically increased, especially after the major blackouts occurred in Italy and USA [2, 3], improving the performance of different essential functions concerning the monitoring, protection, and control of power systems [4–6].

The highly demanding operational level of PMUs renders their optimal placement a challenging task for the researchers, increasing their interest in developing methodologies that satisfy it [7]. The objective of the optimal PMU placement (OPP) problem is the strategic choice of the minimum number and locations of PMUs ensuring the full network observability.

The pioneering heuristic proposal [8] uses a modified bisecting search and a simulated annealing based method to find the optimal placement set. Several heuristic optimisation methods were developed to solve the OPP problem [9–16]. Despite some advantages, such as the better execution time and the parallel simulation, the major drawback of these methods is their non-determinism, not ensuring the existence of a globally optimal solution. This disadvantage is completely circumvented adopting mathematical algorithms to determine the optimal solution.

Integer linear programming (ILP) is the dominant mathematical optimisation technique used for solving the OPP problem. Several aspects of the problem have been addressed in literature [17–30]. An OPP method that ensures the topological and numerical observability of power systems is introduced in [17]. In [18–22], the OPP problem is formulated considering the existence of conventional measurements and/or zero injections. Single or multiple PMU loss and branch outage is studied in [21–23]. Moreover, methods [21, 22] solve the OPP problem by allowing or prohibiting placement of PMUs at zero injection buses. To avoid wide-area blackouts following cascading failures, an effective scheme considering power system control islanding mode is proposed [24]. Procedures for multistage PMU placement in a given time horizon, using an ILP framework, are presented in [25, 26]. A novel method for the simultaneous optimal placement of PMUs and PDCs, ensuring the power system observability and providing the highest reliability for the communication network, is suggested in [27]. The above studies assume that PMUs have unlimited number of channels. The impact of limited channel capacity to OPP problem is examined in [28–30]. The OPP formulation [31] incorporates the effect of dc lines on network observability and considers the PMU installation costs as a function of the number and type of measurement channels. In [32], the branch PMUs are considered taking into account PMU failures and network contingencies. A linear programming (LP) method that recovers the overall system observability using the network connectivity and the measurement Jacobian matrix as equality constraints and identifies the necessary branches or nodes needed for pseudo measurement placement, is studied in [33]. Exhaustive search [34, 35], integer quadratic programming [36], weighted least squares (WLS) [37] and sequential quadratic programming [38], are some other mathematical methods used to tackle the OPP problem. An efficient technique in [39] optimally allocates a predefined number of PMUs throughout an observable system to maximise the estimated state accuracy, utilising the concept of participation factors to account for the contribution of the off-diagonal elements of the eigenvalue decomposition of the relative state error covariance matrix. North American Synchrophasor Initiative has developed guidelines for the placement of PMUs in practical power systems, to augment the conventional SE and improve its fidelity, by forming a phasor-assisted SE [40].

The incorporation of phasor measurements into existing SCADA-based estimators presents some implementation challenges, due to the significant difference between the refresh rates of SCADA measurements (1 time every 2 to 6 s) versus PMU measurements (30 times per second). Therefore, there is a need to process these two different categories of measurements in the best possible way [41]. Since present-day power systems are generally not observable if only PMUs are considered, the system will likely be unobservable at instants when only PMU data are received [42]. In [42], a weighted least absolute value (WLAV)-based hybrid SE is proposed to handle a mixed set of PMU and SCADA measurements, received at different refresh rates.

A literature survey reveals that most of the existing literature on PMU placement strategies targets topological network observability, which may not always ensure total system observability required for successful execution of the SE. It is worth noting, that the majority of existing topological OPP methods use simple heuristic observability rules for the inclusion of pre-existing power flows and zero injections, which may not be valid for all possible measurement set configurations.

In this paper an alternative OPP problem formulation is suggested using a 0/1 semi-definite programming (SDP) method. The optimal solution is derived minimising a linear objective function subject to linear matrix inequality (LMI) observability constraints with binary decision variables. The performance of the proposed model is tested on IEEE standard systems and a large scale system.

The main contributions of the proposed PMU placement method are:

- It can consider any number, type, and position, of pre-existing conventional and synchronised measurements as well as any available zero injections.
- It develops a systematic and efficient procedure to obtain optimum PMU locations.
- It can employ either AC or DC state estimation models.
- It finds the optimal solution in a reasonable execution time, even for very large systems, using a robust mixed integer SDP optimisation tool.
- It delivers optimal solutions, as opposed to other sub-optimal SDP formulations using convex relaxation of 0–1 constraints to bypass the combinatorial search involved.
- It provides less number of PMUs in the presence of conventional measurements and zero injections, compared with existing techniques.

The remaining of the paper is organised as follows. Section 2 provides the measurement model and observability definitions. Section 3 presents a basic mathematical background for the semi-definite programming method. In Section 4 the OPP problem is formulated as an SDP optimisation method. Section 5 illustrates the proposed method using the IEEE 14-bus system. Section 6 presents and discusses numerical simulations with various test systems, and Section 7 concludes the paper.

Notation: $\mathbb{R}^n$ is the set of real $n-$ dimensional vectors; $\mathbb{R}^{n\times n}$ is the set of real $n\times n$ matrices; S^n is the set of symmetric matrices in $\mathbb{R}^{n\times n}$; S^n_+ is the set of symmetric positive semi-definite $n\times n$ matrices; S^n_{++} is the subset of S^n_+ consisting of positive definite matrices; $(.)^{\mathrm{T}}$ denotes transposition.

2 Measurement model and observability

The AC measurement model of the state estimation, for a $N-$bus power system, is described by an over-determined system of non-linear equations relating the measurements and the unknown states [43]

$$z = h(x) + e \tag{1}$$

where $z\in\mathbb{R}^m$ consists of conventional measurements (bus voltage magnitudes, active and reactive branch power flows, and active and reactive bus injections) provided by SCADA and synchronised phasor measurements (bus voltage phasors and branch current phasors) provided by PMUs, $x\in\mathbb{R}^{2n}$ comprises n bus voltage phase angles δ_i with respect to the time reference dictated by the GPS system and n bus voltage magnitudes V_i, $h(x)\in\mathbb{R}^{2n}\to\mathbb{R}^m$ is the non-linear vector function relating measurements to states, $e\in\mathbb{R}^m$ is independent identically distributed Gaussian measurement error vector with zero mean $E(e)=0$ and diagonal covariance matrix $R=\mathrm{cov}(e)=E(ee^{\mathrm{T}})$, m is the number of measurements, and n is the number of network buses. Matrix R is diagonal and each diagonal entry R_{ii} equals σ_i^2, where σ_i is the standard deviation of the ith measurement.

The WLS method minimises the following objective function to compute the optimal estimate of x

$$J(x) = \sum_{i=1}^{m}\left[z-h(x)\right]^{\mathrm{T}}R^{-1}\left[z-h(x)\right] \tag{2}$$

Using the Gauss–Newton method, the estimated state vector $\hat{\boldsymbol{x}}$ is computed by the following iterative solution scheme

$$G(x^k)(x^{k+1}-x^k) = H^{\mathrm{T}}(x^k)R^{-1}(z-h(x^k)) \tag{3}$$

where k is the iteration index

$$H(x^k) = \left.\frac{\partial h(x)}{\partial x}\right|_{x=x^k}$$

is the $m\times 2n$ Jacobian matrix of $h(x)$, and $G(x^k)=H^{\mathrm{T}}(x^k)R^{-1}H(x^k)$ is the $2n\times 2n$ gain matrix.

The elements (partial derivatives) of the Jacobian matrix $H(x)$ for the conventional and the phasor measurements can be found in [43, 44].

When the available measurements are adequate so that (2) can be uniquely solved for a state estimate, the system is said to be *observable*, otherwise it is said to be *unobservable*. The power system is said to be *numerically observable* if the Jacobian matrix H has full rank at each iteration [45]. The power system is said to be *topologically observable* if a spanning tree of full rank can be formed [45]. It is to be noted that topological observability does not necessarily guarantee numerical observability required for successful solution of (2). As can be seen by the example of Fig. 9 found in [46], although this network is observable in the topological sense, the gain matrix becomes singular and the system unsolvable. In the case of the linearised DC measurement model with 1.0 p.u. voltage magnitudes at all buses and $j1.0$ branch impedances, the numerical and topological observability problems become equivalent [45]. An efficient method for numerical observability analysis for systems including both SCADA and PMU measurements, can be found in [47].

3 Semi-definite programming background

Semi-definite programming is one of the recent main developments in mathematical programming, with many applications in applied mathematics and engineering [48, 49]. In this section, we compile some essential definitions which will be used in developing a mathematical formulation of SDP.

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A symmetric matrix $M \in S^n$ is called positive semi-definite (PSD) if $u^T M u \geq 0, \forall u \in \mathbb{R}^n$, which implies that all its eigenvalues and principal minors are non-negative. A symmetric matrix $M \in S^n$ is called positive definite (PD) if $u^T M u \geq 0, \forall u \neq 0 \in \mathbb{R}^n$, which implies that all its eigenvalues and leading principal minors are positive. In the sequel, when $M \in S^n_{++}$ ($M \in S^n_+$) we denote it by $M \succ 0 (\succeq 0)$. LMI is a constraint of the form

$$M(y) = M_0 + \sum_{i=1}^{n} y_i M_i \succ 0 (\succeq 0) \tag{4}$$

where the matrix valued function $M \in \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is an affine function of $y \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$ is a vector of scalar decision variables, and $M_0, M_1, \ldots, M_n \in S^n$ are given symmetric semi-definite matrices. The LMI $M(y) \succ 0 (M(y) \succeq 0)$ forms a strict (non-strict) convex constraint on $\boldsymbol{y}$.

SDP is a class of convex conic optimisation problems which amounts to minimising a linear objective function of vector variable $y \in \mathbb{R}^n$ that satisfies linear matrix inequality (4)

$$\begin{array}{ll} \min\limits_{y} & c^T y \\ \text{s.t.} & M(y) \succeq 0 \end{array} \tag{5}$$

where $c \in \mathbb{R}^n$. The linear matrix inequality $M(y) \succeq 0$ is feasible if the set $\{y | M(y) \succeq 0\}$ is non-empty. If there is no feasible solution, we say that the problem (5) is infeasible.

If the decision variables are binary, that is $y_i \in \{0, 1\}$, the optimisation problem (5) becomes a binary SDP model, which is not convex. To solve a binary constrained SDP, we work with the intersection of two geometries. The first is the rather complex curved geometry created by the semi-definite (SD) constraint and the second the binary lattice points. Note that the convex hull of the intersection trivially is a polytope and we are thus effectively optimising over a polytope. The solver for the integrality constrained model relaxes the geometry induced by the SD constraint by replacing it with some simpler outer approximation (normally a polytope), and then solves an integrality constrained program over this outer approximation [50]. If the binary feasible solution satisfies the original SD constraint, the solution has been found. If not, additional constraints have to be added to the outer approximation to improve its strength.

The integer programs that have to be solved are thus binary ILP (BILP) problems. The outer approximation is generated by computing cuts based on eigenvectors associated with violated SD constraints. If an optimal solution y^* to a BILP fails to satisfy $M(y^*) \succeq 0$, there exists at least one negative eigenvalue λ, with associated normalised eigenvector υ satisfying $M(y^*)\upsilon = \lambda\upsilon$, that is, $\upsilon^T M(y^*)\upsilon = \lambda$. Hence, a scalar linear cut to add to the polytope approximation is thus $\upsilon^T M(y)\upsilon \geq 0$ as this will cut away the current BILP solution. As realistic problems solved in this paper are large, rendering already the eigenvalue computations challenging, great care is taken to exploit structure that might arise during solution process. As an example, the intermediate infeasible solutions $M(y^*)$ are often permuted block-diagonal matrices, which can be exploited to reduce the complexity of eigenvalue computations. Structure of the problem is also exploited in generation of the initial outer approximation. Apart from an initial set of constraints based on non-negativity of the diagonal, all 2×2 minors are investigated (this is possible as the model is very sparse). If the constant term of a 2×2 minor is not PSD, some of the variables in the 2×2 minor have to be non-zero. This is a simple constraint to add to a BILP model.

It is worth noting that the outer approximation is performed with respect to the semi-definite cone. The non-convexity which occurs due to binary variables is treated globally as the outer approximations of the semi-definite cone, intersected with the binary lattice, is solved using a mixed ILP (MILP) solver. Multiple optimal solutions of the BILP is not an issue. If an optimal solution to the BILP is feasible in the original binary SDP, it is optimal also, that is, any optimal solution to the BILP, which is feasible in the binary SDP, is optimal.

4 SDP formulation of the OPP problem

The objective of the PMU placement problem is to make the entire system observable by placing the minimum number of PMUs at strategic buses in the system. It is assumed that each PMU has enough channels to measure the voltage phasor at its own bus and the current phasors on all incident branches.

Assume that $y = (y_1, \ldots, y_n)^T$ is the binary decision variable vector, whose entries are defined as

$$y_i = \begin{cases} 1 & \text{if a PMU is installed at bus } i \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Based on decision variables y_i, the Jacobian matrix will have the following form

$$H(y) = \begin{array}{c} \begin{pmatrix} H_0 \\ y_1 H_1 \\ \vdots \\ y_i H_i \\ \vdots \\ y_n H_n \end{pmatrix} \end{array} \begin{array}{c} \text{SCADA} \\ \text{PMU}_1 \\ \vdots \\ \text{PMU}_i \\ \vdots \\ \text{PMU}_n \end{array} \tag{7}$$

where $H_0 \in \mathbb{R}^{m_0 \times 2n}$ is the Jacobian matrix associated with the existing conventional (SCADA) measurements and $H_i \in \mathbb{R}^{m_i \times 2n}$ is the Jacoban matrix associated with the phasor measurements acquired by a PMU located at bus i.

Jacobians H_0 and H_i are evaluated at $x = x^0$, where x^0 is an initial guess for the state vector x, which is usually assumed to the flat voltage profile. Note that other typical voltage profiles can be alternatively used, too. The dependence of the Jacobians on $\boldsymbol{x}$ has been suppressed for notational convenience. The corresponding gain matrix will be

$$G(y) = H^T(y)H(y) = \begin{pmatrix} H_0 \\ y_1 H_1 \\ \vdots \\ y_i H_i \\ \vdots \\ y_n H_n \end{pmatrix}^T \begin{pmatrix} H_0 \\ y_1 H_1 \\ \vdots \\ y_i H_i \\ \vdots \\ y_n H_n \end{pmatrix} = H_0^T H_0 + \sum_{i=1}^{n} y_i^2 H_i^T H_i \tag{8}$$

Since $y_i^2 = y_i$ for $y_i \in \{0, 1\}$, (8) yields:

$$G(y) = G_0 + \sum_{i=1}^{n} y_i G_i \tag{9}$$

where $G_0 = H_0^T H_0 \in S_+^{2n}$ and $G_i = H_i^T H_i \in S_+^{2n}$, $i = 1, \ldots, n$ are the corresponding gain matrices.

Matrix G_0 will be very sparse if power flow measurements are more than power and zero injection measurements. Matrices G_i are extremely sparse and can be directly formed and stored, without explicitly forming the Jacobians H_i. Matrix G_0 being positive definite ($G_0 \succ 0$) means that the system is observable with the existing conventional measurements and no additional phasor measurements are required to make the system completely observable. Matrix G_0 being positive semi-definite ($G_0 \succeq 0$) means that the system is unobservable. In this case, the system will

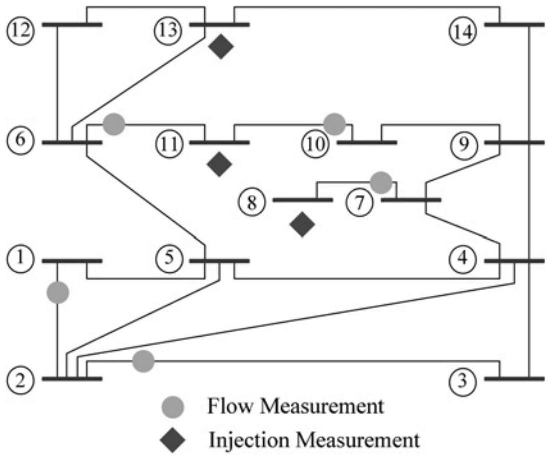

Fig. 1 *IEEE 14-bus system and existing conventional measurement configuration*

become observable if and only if $G(y)$ has full rank ($\text{rank}(G(y)) = 2n$), which is equivalent to $G(y) \succ 0$.

The proposed OPP problem is formulated as a 0–1 SDP model

$$\begin{aligned} \min \quad & w^{\mathrm{T}} y \\ \text{s.t.} \quad & G(y) = G_0 + \sum_{i=1}^{n} y_i G_i \succ 0 \\ & y \in \{0, 1\}^n \end{aligned} \tag{10}$$

where $w = (w_1, \ldots, w_n)^{\mathrm{T}}$ and w_i is the weight (reflecting the investment cost) of the PMU installed at bus i. Usually we set $w_i = 1, \forall i$, meaning that all PMUs have the same priority of placement.

4.1 Discussion

To *enforce* (*prohibit*) the allocation of a PMU at a specific bus i, we use two different modelling approaches:

- Consider bus i as a candidate for PMU placement, form the associated gain matrix G_i, and introduce the equality constraint $y_i = 1$ ($y_i = 0$), respectively.
- Eliminate PMU bus i from problem formulation, either replacing G_0 by $G_0 + G_i$ or keeping G_0 unchanged, respectively.

To *allow* placement of a PMU at a specific bus i, we introduce the candidate PMU-bus i in problem formulation, by imposing no constraint for y_i (unbounded decision variable).

Sometimes, it may be preferable not to place any PMU at zero injection buses. The reason is that the zero injection property is not used if a PMU is installed on that bus, since the PMU measures all incident branch currents [21]. However, preventing PMU allocation at zero injection buses may yield more PMUs at other buses. A proposed approach is to solve the OPP problem twice, by allowing and prohibiting it, respectively, to place PMUs at zero injection buses, and taking the solution which delivers the least number of installed PMUs.

If PMUs have already been allocated at some buses, the problem is formulated by enforcing their placement as explained before and new PMUs will be optimally placed at the remaining buses. Often it is not feasible to install a PMU at a specific bus to meet entire network observability, due to high infrastructure, installation and maintenance cost [40]. This installation can be prohibited as explained before. In practice, it is advantageous to site PMUs at buses that receive priority attention for equipment and communications network maintenance, to assure that these PMUs sustain high availability and performance [40]. These installations can be enforced as discussed before.

5 Illustrative example for SDP formulation

The IEEE 14-bus system, shown in Fig. 1, and the dc state estimation model [44] are used to illustrate the proposed SDP formulation for the OPP problem, assuming pre-existing active flow measurements on branches $1-2$, $2-3$, $6-11$, $7-8$ and $10-11$ and active injection measurements at buses 8, 11 and 13. The SDP formulation (10) for this case is

$$\begin{aligned} \min \quad & \sum_{i=1}^{14} w_i y_i \\ \text{s.t.} \quad & G_0 + \sum_{i=1}^{14} y_i G_i \succ 0 \\ & y_i \in \{0, 1\}, \quad i = 1, \ldots, 14 \end{aligned}$$

The dc Jacobian matrix H_0 for the existing conventional measurements is: (see equation at bottom of the page)

The dc Jacobian matrix H_1 associated with a PMU located at bus 1 is also given for illustration: (see equation at bottom of the page)

where $I_{i-j,r}$ designates the real part of the current phasor from bus i to bus j.

$$H_0 = \begin{array}{c} \begin{array}{cccccccccccccc} \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_6 & \delta_7 & \delta_8 & \delta_9 & \delta_{10} & \delta_{11} & \delta_{12} & \delta_{13} & \delta_{14} \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 \end{pmatrix} \end{array} \begin{array}{l} \\ P_{1-2} \\ P_{2-3} \\ P_{6-11} \\ P_{7-8} \\ P_{10-11} \\ P_8 \\ P_{11} \\ P_{13} \end{array}$$

$$H_1 = \begin{array}{c} \begin{array}{cccccccccccccc} \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_6 & \delta_7 & \delta_8 & \delta_9 & \delta_{10} & \delta_{11} & \delta_{12} & \delta_{13} & \delta_{14} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \delta_1 \\ I_{1-2,\mathrm{r}} \\ I_{1-5,\mathrm{r}} \end{array}$$

The corresponding gain matrices G_0 and G_1 are: (see equation at bottom of the page)

$$G_1 = H_1^{\mathrm{T}} H_1$$
$$= \begin{pmatrix} 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution of the SDP placement problem shows that for complete system observability two PMUs are required at buses 4 and 6. As can be verified, the gain matrix $G = G_0 + G_4 + G_6$, associated with the exisitng SCADA measurements and the chosen PMUs, is non-singular (positive definite).

6 Results and discussion

The proposed method is analysed and tested for the IEEE 14-, 30-, 57-, 118-, and 300-bus test systems, as well as for the Polish 3120-bus system. The proposed OPP algorithm has been developed in MATLAB [51] on a 3.4 GHz Intel(R) Core(TM) i7-2600 processor with 16 GB of RAM.

To solve the binary SDP problem (10), we use the optimisation modelling tool YALMIP [52], which includes a generic implementation of an outer approximation strategy. To solve the BILP models, we used the solver SCIP [53]. The power flow data are stored in ASCII file with the PTI PSS/E 'raw' format [54] and the conventional measurement data (types and locations) are given in ASCII file. Using the MATLAB-based software tool MATPOWER [54], which is an open-source package of MATLAB m-files for solving steady-state power system analysis problems, appropriate code has been developed to write the problem data, required by YALMIP, in an ASCII input file with postfix '.dat-s', using the SDPA sparse format [55]. It is to be noted that only the diagonal elements and the non-zero elements in the upper triangular part of matrices G_0 and G_i, incorporated in the LMI observability constraint (10), are stored in the '.dat-s' input file.

All the conventional power flow measurements and power and zero injection measurements are in active and reactive pairs. The gain matrices, associated with existing SCADA and PMU measurements as well as candidate PMUs are evaluated at flat start. To avoid ill-conditioning problems when using flat voltage profile [43, 44], the current phasor measurements are expressed in rectangular form [44]. The standard deviation for voltage, flow, and injection measurements, is assumed to be 0.004, 0.008, and 0.01 respectively. Voltage and current phasor measurements are assumed to have a standard deviation of 0.0001. Zero injections are considered as perfect measurements with a standard deviation of 0.00001.

6.1 IEEE standard test systems

The efficacy of the proposed technique is elucidated by simulations on the IEEE 14-, 30-, 57-, 118-, and 300-bus benchmark systems. Detailed system information and one-line diagram for each of the above networks is available in [56].

We consider three different measurement sets of pre-existing conventional measurements for each test system, as shown in Table 1. OPP results are given for four different cases: (1) no conventional measurements and zero injections, (2) flow measurements, (3) zero injections (4) flow measurements as well as zero and power injections, as shown in Table 2. For comparison purposes, the zero injection bus locations found in [19] are used for the simulations with the IEEE 14-, 30-, 57-, 118-bus systems. For the IEEE 300-bus system all the available zero injections are considered.

All numerical simulations were conducted by allowing PMU allocation at all zero injection buses. The minimum number of PMUs and the corresponding bus locations are given in Table 3. As can be observed in Table 3 for case 4, some zero injection buses have been included in locating PMUs: 30-bus system {6}, 118-bus system {37}, 300-bus system {60, 132, 134, 160}. It is to noted that, for case 3 in the IEEE 14-, 30-, 57-, 118-bus systems, the same number of PMUs as in [19] were found, but in different bus locations. Results of Table 3 indicate that the number of PMUs required to make the system fully observable, is decreased when considering the existence of widely dispersed conventional measurements. Table 4 shows the computational time requirements to find the optimal PMU locations for the different IEEE systems. As can be observed, if no conventional measurements are considered, the CPU time is very low and increases gradually as the number of already installed SCADA measurements increases.

The proposed formulation can be also applied to measurement sets consisted of pre-installed PMUs along with conventional measurements. To meet the PMU availability requirements at specific buses, we can prohibit or enforce PMU installation at these buses. The PMU availability at bus i can be described by an additional equality constraint: $y_i = 0$ (not install PMU at bus i) or $y_i = 1$ (install PMU at bus i). Moreover, numerical results are provided in presence of pre-installed PMUs, for the measurement sets shown in Tables 5 and 6. Fifteen cases, namely case 5 to case 19, are considered. For all cases, the existing measurement sets include conventional power flow and injection measurements as well as phasor measurements. Cases 6, 9, 12, 15 and 18 consider the effect of prohibiting PMU installation at specific buses,

$$G_0 = H_0^{\mathrm{T}} H_0 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 & -3 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & -3 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & -3 & 9 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 1 \end{pmatrix}$$

Table 1 Measurement sets for the IEEE standard test systems

Test system	Measurement set 1	Measurement set 2	Measurement set 3
	Power flows	Zero injections	Power injections
IEEE 14-bus	1-2, 2-3, 6-11, 7-8, 10-11	7	8, 11, 13
IEEE 30-bus	1-2, 1-3, 2-4, 2-5, 2-6, 6-8, 9-10, 10-17, 12-4, 12-13, 12-14, 12-15, 12-16, 18-19, 20-19, 21-22, 23-24, 24-25, 27-29, 29-30	6, 9, 11, 25, 28	1, 2, 19
IEEE 57-bus	1-2, 1-15, 1-16, 1-17, 3-15, 4-5, 4-6, 4-18, 7-29, 8-9, 9-10, 10-12, 10-51, 11-41, 11-43, 12-13, 14-46, 19-20, 20-21, 22-38, 23-24, 24-25, 24-26, 27-26, 28-27, 29-52, 30-31, 32-34, 34-35, 36-35, 38-37, 38-44, 38-48, 40-36, 41-42, 42-56, 47-46, 49-38, 51-50, 53-54	4, 7, 11, 21, 22, 24, 26, 34, 36, 37, 39, 40, 45, 46, 48	1, 15, 32, 38, 51, 57
IEEE 118-bus	1-2, 1-3, 2-12, 3-5, 3-12, 4-5, 4-11, 5-6, 5-11, 6-7, 7-12, 8-5, 8-9, 8-30, 9-10, 11-12, 11-13, 19-20, 20-21, 21-22, 22-23, 23-24, 23-25, 23-32, 24-70, 24-72, 26-25, 26-30, 27-25, 27-28, 27-32, 27-115, 28-29, 29-31, 33-37, 34-36, 34-37, 35-36, 35-37, 37-39, 37-40, 38-30, 38-37, 38-65, 39-40, 40-41, 40-42, 41-42, 42-49, 43-34, 43-44, 44-45, 45-46, 45-49, 50-57, 51-58, 54-55, 54-56, 4-59, 55-56, 55-59, 56-57, 56-58, 56-59, 59-60, 59-61, 60-61, 60-62, 61-62, 62-66, 62-67, 63-59, 63-64, 64-61, 68-81, 69-77, 75-77, 76-77, 76-118, 77-78, 77-80, 77-82, 78-79, 79-80, 80-96, 80-97, 80-98, 80-99, 81-80, 82-83, 82-96, 83-84, 83-85, 84-85, 85-86, 85-88, 85-89, 86-87, 88-89, 92-102, 100-101, 100-103, 100-104, 100-106, 101-102, 103-104, 103-105, 103-110, 104-105, 105-106, 105-107, 105-108, 106-107, 108-109, 109-110, 110-111, 110-112	5, 9, 30, 37, 38, 63, 64, 68, 71, 81	11, 12, 13, 14, 31, 32, 33, 50, 51, 52, 53, 54, 73, 74, 75, 76, 77, 78, 91, 92, 93, 94, 95, 100, 101, 102, 103
IEEE 300-bus	1-5, 2-6, 2-8, 3-2, 3-4, 3-7, 3-19, 3-150, 4-16, 5-9, 7-5, 7-6, 7-12, 7-131, 8-11, 8-14, 9-11, 10-11, 11-13, 12-10, 12-21, 13-20, 14-15, 15-17, 15-37, 15-89, 15-90, 16-15, 16-42, 19-21, 19-87, 20-22, 20-27, 21-20, 21-24, 22-23, 23-25, 24-23, 24-319, 25-26, 26-27, 26-320, 33-34, 33-38, 33-40, 33-41, 34-42, 35-72, 35-76, 35-77, 36-35, 36-88, 37-38, 37-40, 37-41, 37-49, 37-89, 37-90, 37-9001, 38-41, 38-43, 39-42, 40-48, 41-42, 41-49, 41-51, 42-46, 43-44, 43-48, 43-53, 44-47, 44-54, 45-44, 45-46, 45-60, 45-74, 46-81, 47-73, 47-113, 48-107, 49-51, 51-52, 52-55, 53-54, 54-55, 55-57, 57-58, 57-63, 58-59, 59-61, 60-62, 62-61, 62-64, 62-144, 63-64, 63-526, 69-79, 69-211, 70-71, 70-528, 71-72, 71-73, 72-77, 72-531, 73-74, 73-76, 73-79, 74-88, 74-562, 76-77, 77-78, 77-80, 77-552, 77-609, 78-79, 78-84, 79-211, 80-211, 81-88, 81-194, 81-195, 85-86, 85-99, 86-87, 86-102, 86-323, 87-94, 89-91, 90-92, 91-94, 91-97, 92-103, 92-105, 94-97, 97-100, 97-102, 97-103, 98-100, 98-102, 99-107, 99-108, 99-109, 99-110, 100-102, 102-104, 103-105, 104-108, 104-322, 105-107, 105-110, 108-324, 109-110, 109-113, 109-114, 110-112, 112-114, 114-207, 121-115, 115-122, 116-120, 116-124, 117-118, 118-119, 118-121, 118-1201, 119-120, 119-121, 120-1201, 122-123, 122-125, 122-157, 123-124, 123-125, 125-126, 126-127, 126-129, 126-132, 126-157, 126-158, 126-169, 127-128, 127-134, 127-168, 128-130, 128-133, 129-130, 129-133, 130-131, 159-117, 160-124, 201-69, 7001-1, 7002-2, 7003-3, 7011-11, 7012-12, 7017-17, 7023-23, 7024-24, 7039-39, 7044-44, 7049-49, 7055-55, 7057-57, 7061-61, 7062-62, 7071-71	4, 7, 12, 16, 19, 24, 34, 35, 36, 39, 42, 45, 46, 60, 62, 64, 69, 74, 78, 81, 85, 86, 87, 88, 100, 115, 116, 117, 128, 129, 130, 131, 132, 133, 134, 144, 150, 151, 158, 160, 164, 165, 166, 168, 169, 174, 193, 194, 195, 210, 212, 219, 226, 237, 240, 244, 1201, 2040, 7049, 9001, 9005, 9006, 9007, 9012, 9023, 9044	13, 14, 15, 17, 39, 40, 41, 47, 48, 49, 79, 80, 84, 104, 105, 107, 108, 109, 110, 136, 137, 138, 139, 140, 141, 142, 143, 156, 157, 159, 161, 196, 197, 199, 200, 201, 202, 205, 206, 207, 7001, 7002, 7003, 7011, 7012, 7017, 7023, 7024, 7039, 7044, 7055, 9026, 9031, 9032, 9033, 9034, 9035, 9036, 9037

Table 2 Cases of pre-existing conventional measurements for the IEEE standard test systems

Test system	Case 1		Case 2		Case 3		Case 4	
	Meas. Set	No. of meas.	Meas. set	No. of meas.	Meas. set	No. of meas.	Meas. sets	No. of meas.
IEEE 14-bus	—	0	1	5	2	1	1, 2, 3	9
IEEE 30-bus	—	0	1	20	2	5	1, 2, 3	28
IEEE 57-bus	—	0	1	40	2	15	1, 2, 3	61
IEEE 118-bus	—	0	1	117	2	10	1, 2, 3	154
IEEE 300-bus	—	0	1	207	2	66	1, 2, 3	332

Table 3 Simulation results for the IEEE standard test systems

Test system	Case							
	1		2		3		4	
	No. of PMUs	Bus #	No. of PMUs	Bus #	No. of PMUs	Bus #	No. of PMUs	Bus #
IEEE 14-bus	4	2, 7, 11, 13	2	4, 13	3	2, 6, 9	2	4, 6
IEEE 30-bus	10	1, 2, 6, 9, 10, 12, 15, 18, 25, 27	4	6, 9, 10, 25	7	1, 2, 10, 12, 19, 24, 30	2	6, 19
IEEE 57-bus	17	1, 4, 7, 13, 19, 22, 25, 26, 29, 32, 36, 39, 41, 44, 47, 51, 54	6	8, 15, 22, 32, 54, 57	11	1, 6, 13, 19, 25, 29, 32, 38, 41, 51, 54	2	23, 35
IEEE 118-bus	32	1, 5, 9, 11, 12, 17, 21, 25, 28, 34, 37, 40, 45, 49, 52, 56, 62, 63, 68, 70, 71, 76, 78, 85, 86, 90, 92, 96, 100, 105, 110, 114	10	12, 17, 32, 46, 52, 68, 70, 71, 90, 94	28	1, 10, 11, 12, 17, 21, 25, 28, 34, 35, 40, 45, 49, 52, 56, 62, 72, 75, 77, 80, 85, 86, 90, 94, 101, 105, 110, 114	3	11, 37, 66
IEEE 300-bus	87	1, 2, 3, 11, 12, 13, 15, 17, 23, 24, 26, 33, 35, 39, 43, 44, 47, 49, 55, 57, 61, 62, 63, 70, 71, 72, 74, 77, 78, 81, 86, 97, 100, 104, 105, 108, 114, 119, 120, 122, 124, 130, 132, 133, 134, 137, 139, 140, 143, 153, 156, 159, 164, 166, 173, 178, 184, 188, 189, 194, 204, 205, 210, 211, 214, 217, 221, 225, 229, 231, 232, 234, 237, 238, 240, 244, 247, 249, 9002, 9003, 9004, 9005, 9007, 9012, 9021, 9023, 9053	47	119, 120, 130, 132, 133, 136, 137, 139, 140, 143, 153, 154, 164, 166, 173, 178, 184, 188, 194, 198, 204, 208, 210, 211, 214, 217, 221, 225, 229, 231, 232, 234, 237, 238, 240, 245, 246, 249, 9002, 9003, 9004, 9005, 9007, 9012, 9021, 9023, 9053	70	1, 2, 3, 11, 15, 17, 20, 23, 26, 41, 43, 44, 48, 55, 57, 61, 63, 70, 71, 72, 77, 97, 104, 105, 108, 109, 114, 119, 120, 122, 126, 137, 139, 140, 143, 153, 154, 162, 175, 178, 181, 184, 189, 190, 191, 199, 205, 211, 214, 217, 221, 229, 231, 232, 234, 238, 241, 245, 249, 7024, 9002, 9003, 9004, 9021, 9025, 9051, 9052, 9053, 9054, 9071	21	26, 27, 33, 43, 60, 61, 70, 86, 105, 120, 122, 124, 127, 132, 134, 137, 139, 142, 145, 160, 176

Table 4 CPU time for the IEEE standard test systems

Test system	CPU time, s			
	Case 1	Case 2	Case 3	Case 4
IEEE 14-bus	0.159	0.204	0.213	0.355
IEEE 30-bus	0.193	0.219	0.313	0.361
IEEE 57-bus	0.234	0.402	0.647	0.648
IEEE 118-bus	0.261	0.417	0.984	0.995
IEEE 300-bus	0.370	0.426	2.361	2.366

whereas cases 7, 10, 13, 16 and 19 simulate the effect of enforcing PMU allocation at specific buses. Table 7 presents the corresponding simulation results concerning the minimum number and locations of PMUs. In most cases, prohibiting or enforcing installation of PMUs at specific buses, increases the minimum number of PMUs.

A major advantage of the proposed method is that it provides lesser PMUs, in the presence of conventional measurements, compared with existing methods. Table 8 confirms that, by using the conventional measurement sets from [19, 36]. In Table 8, the minimum number of PMUs is boldfaced and PMU sites are enclosed in braces. The SDP method delivers optimal solutions with 1 and 2 PMUs less than those obtained by [19, 36], respectively.

6.2 Large power system

To investigate the performance of the proposed approach in large-scale power systems, the 3120-bus Polish power system [54] is studied. The effectiveness and flexibility of the proposed algorithm is assessed with several cases. Table 9 shows different measurement sets of pre-installed conventional measurements and zero injections as well as the associated minimum number of optimal PMU locations required to ensure complete system

Table 5 Conventional and PMU measurement sets for cases 5–16

Case	Test system	Measurement set			Additional constraints	
		Power flows	Power injections	Pre-installed PMU at bus #	Prohibited PMU at bus #	Enforced PMU at bus #
5	IEEE 14-bus	2-1, 3-4, 5-2, 11-10	9	1	—	—
6					6	—
7					—	13
8	IEEE 30-bus	1-2, 2-5, 7-6, 8-28, 19-18, 21-10	2, 4, 22, 23, 27	3, 15	—	—
9					10, 21	—
10					—	10, 25
11	IEEE 57-bus	1-16, 3-15, 4-5, 18-19, 20-21, 29-52, 30-31, 42-41, 50-51, 57-39	6, 12, 20, 28, 32, 49	9, 37, 38, 56	—	—
12					22, 25, 29, 47	—
13					—	1, 5, 18, 23
14	IEEE 118-bus	1-2, 2-12, 4-11, 13-15, 24-70, 48-49, 50-57, 56-55, 60-62, 62-66, 78-79, 86-87, 89-92, 108-109	3, 11, 19, 54, 70, 79, 85, 92, 104	16, 23, 32, 51, 68, 72, 88, 90, 106, 110	—	—
15					17, 34, 60, 74, 85, 108	—
16					—	11, 75, 80, 94, 100, 105

Table 6 Conventional and PMU measurement sets for cases 17 to 19

Case	Test system	Measurement set			Additional constraints	
		Power flows	Power injections	Pre-installed PMU at bus #	Prohibited PMU at bus #	Enforced PMU at bus #
17	IEEE 300-bus	1-5, 2-6, 2-8, 3-1, 3-2, 3-4, 3-7, 3-19, 3-150, 4-16, 5-9, 7-5, 7-6, 7-12, 7-131, 8-11, 8-14, 9-11, 10-11, 11-13, 12-10, 12-21, 15-37, 21-20, 24-23, 21-24, 22-23, 23-25, 24-319, 25-26, 26-27, 26-320, 33-34, 33-38, 33-40, 33-41, 34-42, 35-72, 35-76, 35-77, 36-35, 36-88, 37-38, 37-40, 37-41, 37-49, 37-89, 37-90, 37-9001, 38-41, 38-43, 45-60, 57-58, 58-59, 60-62, 62-61, 62-64, 62-144, 63-64, 63-526, 69-211, 69-79, 70-71, 70-528, 71-72, 71-73, 72-77, 72-531, 73-62, 201-69, 7001-1, 7002-2, 7003-3, 7011-11, 7023-23, 7024-24, 7061-61, 7071-71	36, 38, 46, 47, 64,71, 84, 90, 98, 109, 112, 135, 140, 147, 170, 171, 199, 201, 205, 324, 7049, 9004, 9051	3, 15, 26, 97, 121, 133, 140, 141, 157, 179, 189, 198, 202, 206, 220, 224, 228, 9005	—	—
18					115, 121, 126, 133, 220, 224	—
19					—	40, 92, 130, 163, 225, 7002

observability. The locations of the conventional measurements, zero injections, and installed PMUs, are not provided due to space limitations. The method proved very efficient in finding the globally optimal PMU locations.

As can be seen, the CPU time spent by the optimisation algorithm is reasonable, and that makes the proposed method computationally very attractive to effectively solve the OPP problem for large scale power systems. From Table 9 is obvious that, without considering conventional and zero injections, the solution time is only 5.753 s, and starts gradually increasing, when mixed sets of conventional and zero injections are incorporated.

The last column of Table 9 shows the ratio, λ, of the number of installed PMUs to the total number of buses. From simulation results, performed on various test systems, it has been observed [8] that ratio λ ranges from 1/4 to 1/3. As can be seen in Table 6, there are cases where the ratio λ significantly exceeds the lower limit 1/4, but is always less than the upper limit 1/3. In general, this rule does not hold true for every studied system, since the minimum number of PMUs is highly depended on its topology. This can be easily verified, using the 8-bus system shown in Fig. 2, where three PMUs, at buses 2, 6, and 7, are required for complete observability. It is obvious that ratio $\lambda = 0.375 > 1/3$ violates this rule.

From the numerical simulations, it was realised that for well-conditioned gain matrices the algorithm converged successfully in all cases. Whenever the gain matrices are ill-conditioned, the state estimation solution process may diverge and the convergence of the 0–1 SDP algorithm may also fail. Concluding, the convergence behaviour of the proposed algorithm depends on the conditioning of the involved gain matrices.

Table 7 Simulation results for case 5–19

Test system	Case	No. of PMUs	Bus #
IEEE 14-bus	5	2	6, 7
	6	3	7, 11, 12
	7	3	6, 7, 13
IEEE 30-bus	8	5	7, 8, 10, 21, 24
	9	6	7, 8, 11, 14, 22, 24
	10	5	7, 8, 10, 21, 25
IEEE 57-bus	11	10	3, 10, 13, 19, 22, 24, 25, 29, 41, 47
	12	10	1, 4, 10, 21, 24, 28, 30, 39, 41, 48
	13	11	1, 5, 18, 21, 23, 28, 30, 38, 40, 41, 47
IEEE 118-bus	14	21	4, 8, 10, 13, 17, 24, 25, 27, 33, 37, 40, 43, 52, 58, 59, 66, 74, 77, 86, 93, 109
	15	22	4, 8, 10, 13, 18, 24, 25, 27, 33, 37, 40, 43, 52, 58, 59, 66, 70, 72, 77, 88, 92, 94
	16	25	4, 8, 10, 11, 13, 17, 25, 27, 32, 37, 40, 43, 52, 60, 62, 66, 74, 75, 76, 80, 86, 94, 100, 105, 109
IEEE 300-bus	17	54	12, 17, 37, 39, 41, 42, 51, 52, 64, 70, 77, 87, 91, 98, 104, 107, 108, 113, 115, 119, 125, 127, 132, 139, 142, 148, 152, 160, 165, 168, 175, 178, 183, 193, 195, 196, 210, 212, 213, 215, 218, 219, 221, 230, 247, 2040, 7001, 7011, 7012, 7017, 7024, 9001, 9003, 9006
	18	57	12, 17, 37, 39, 41, 42, 51, 52, 64, 70, 77, 87, 91, 94, 107, 108, 112, 113, 119, 123, 127, 132, 139, 140, 145, 150, 152, 160, 165, 168, 172, 175, 178, 183, 193, 195, 196, 210, 212, 213, 215, 218, 219, 221, 225, 226, 230, 247, 2040, 7001, 7011, 7012, 7017, 7024, 9001, 9003, 9006
	19	60	12, 17, 37, 39, 40, 41, 42, 51, 52, 64, 70, 74, 77, 87, 91, 92, 104, 107, 108, 113, 115, 119, 125, 127, 130, 132, 139, 142, 148, 152, 160, 163, 165, 168, 175, 178, 183, 195, 196, 210, 212, 213, 215, 218, 219, 221, 225, 227, 230, 247, 2040, 7001, 7002, 7011, 7012, 7017, 7024, 9002, 9003, 9006

Table 8 Comparison of the proposed SDP with other OPP methods

Test system	Measurements		No. and locations of PMUs		
	Power flows	Power injections	IP [19]	IQP [33]	Proposed SDP
IEEE 14- bus	1-2, 7-4, 7-8, 7-9, 9-7, 9-4	1, 2, 3, 4, 6, 9, 10, 12, 13	—	**2** {6, 9}	**1** {5}
IEEE 118-bus	1-3, 3-5, 3-12, 5-6, 6-7, 8-5, 8-9, 8-30, 12-16, 12-117, 15-19, 19-20, 19-34, 23-32, 25-27, 26-25, 27-32, 28-29, 29-31, 32-114, 34-37, 35-36, 35-37, 38-37, 43-44, 46-47, 49-50, 47-69, 51-52, 51-58, 54-59, 55-59, 59-60, 65-68, 68-69, 68-116, 70-74, 74-75, 82-83, 83-84, 92-94, 93-94, 94-100, 98-100, 99-100, 105-107, 106-107, 110-111, 110-112	5, 9, 12, 19, 21, 27, 28, 30, 32, 37, 38, 41, 44, 47, 50, 53, 59, 62, 63, 64, 68, 71, 81, 83, 86, 94, 96, 108, 110	**19** {2, 11, 17, 21, 24, 40, 49, 56, 62, 71, 77, 80, 86, 89, 91, 100, 102, 108, 118}	—	**17** {6, 10, 11, 14,18, 29, 35, 43, 46, 51, 55, 58, 61, 66, 75, 76, 89}

Table 9 Optimal number of installed PMUs for the 3120-bus Polish system

Case	No. of power flows	No. of zero injections	No. of PMUs	CPU time, s	$\lambda = \dfrac{\text{No. of PMUs}}{\text{No. of buses}}$
1	0	0	992	5.753	0.318
2	160	0	963	11.362	0.310
3	220	0	947	11.635	0.304
4	320	0	926	22.193	0.297
5	400	0	909	33.686	0.291
6	500	0	888	35.132	0.285
7	0	34	982	10.355	0.315
8	0	70	975	21.396	0.313
9	0	100	960	48.866	0.308
10	0	200	921	89.323	0.295
11	50	50	971	55.581	0.311
12	280	100	900	73.264	0.289
13	400	100	881	97.719	0.287
14	400	200	841	132.453	0.270

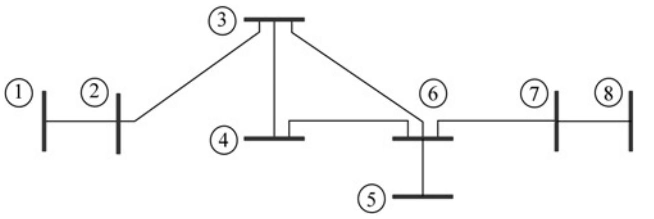

Fig. 2 *8-bus system*

7 Conclusion

This paper presented a simple, flexible, and easy-to-implement SDP-based method, which minimises the number of installed PMUs, ensuring full numerical network observability with consideration of existing conventional measurements. The optimal solution is obtained by minimising a linear objective function with binary decision variables, subject to linear matrix inequality observability constraints. The associated binary SDP problem is solved using an outer approximation scheme based on BILP. The proposed model is successfully tested on the IEEE standard test systems in addition to a large-scale system. Unlike existing counterpart, the proposed approach is more flexible in that (i) it can consider any number and type of pre-existing SCADA and PMU measurements (ii) it can employ either AC or DC measurement models (iii) it guarantees globally optimal solutions (iv) it provides lesser PMUs compared with existing techniques. Results from five IEEE test systems as well as the Polish 3120-bus power system prove the efficiency of the proposed method in terms of accuracy and speed.

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