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FULL LENGTH PAPER

A condition number theorem in convex programming

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Abstract A finite-dimensional mathematical programming problem with convex data and inequality constraints is considered. A suitable definition of condition number is obtained via canonical perturbations of the given problem, assuming uniqueness of the optimal solutions. The distance among mathematical programming problems is defined as the Lipschitz constant of the difference of the corresponding Kojima functions. It is shown that the distance to ill-conditioning is bounded above and below by suitable multiples of the reciprocal of the condition number, thereby generalizing the classical Eckart–Young theorem. A partial extension to the infinite-dimensional setting is also obtained.

Keywords Convex programming · Condition number · Condition number theorem

Mathematics Subject Classification 90C31 · 90C25

1 Introduction

Consider the finite-dimensional convex programming problem Q, to minimize the objective function $f(x)$ subject to the inequality constraints

$$g_1(x) \leq 0, \ldots, g_q(x) \leq 0.$$

The purpose of this paper is twofold. First, to find an appropriately defined condition number of Q. Second, to relate such condition number to the suitably defined distance of Q from the set of ill-conditioned problems of the same form. In this way we

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generalize to the convex programming setting the classical Eckart–Young theorem in its optimization version, see [16]. According to such result and to its generalizations, see [17] and [18], the distance to ill-conditioning is bounded from below and above by constant multiples of the reciprocal of the condition number.

Such results are of interest for the analysis of the computational complexity (see [4]), for the sensitivity and stability properties of the problem under perturbations, and for evaluating the performance of numerical methods of solution. Quoting [4, p. 232], "a very general theme in numerical analysis is a relationship between the condition number of a problem and the reciprocal of the distance to the set of ill-conditioned problems". However, as shown in [18], there exist classes of minimum norm least squares problems for which a condition number theorem as before cannot hold. This adds interest in finding classes of optimization problems for which a condition number theorem is available. To the knowledge of this author, no such result is known for convex programming problems. Moreover, the definition we shall employ of distance among mathematical programming problems, based on their Kojima functions (see below), is new. The results of Renegar [12,13] deal with the distance to instability (a different notion than ill-conditioning) within linear programming problems (following a different approach).

In this paper we consider a condition number as a measure of the sensitivity of the optimal solution with respect to a given class of data perturbations, according to standard notions of conditioning in numerical analysis. Hence the notion of conditioning crucially depends on the choice of the perturbations. We consider here the canonical perturbations $Q(p)$ of problem Q, defined by the parameter $p = (a, b)$, with the objective function $f(x) - a^T x$ subject to the constraints

$$g_1(x) \leq b_1, \dots, g_q(x) \leq b_q,$$

where $(b_1, \dots, b_q)^T = b$. Such canonical perturbations play a significant role in the analysis of stability and sensitivity of mathematical programming problems, see [9, chapter 8]. We assume existence, uniqueness and boundedness of the optimal solutions $s(p) = [m(p), u(p)]$ of $Q(p)$ for p sufficiently small, where $m(p)$ is the corresponding minimizer and $u(p)$ is the corresponding multiplier. Then the condition number of Q is defined as the Lipschitz modulus of s at $p = 0$. Problem Q is then declared well-conditioned if its condition number is finite. A solvable program is called here ill-conditioned iff either the unique optimal solution of the canonically perturbed problem is not locally Lipschitz around 0 with respect to the perturbation parameter, or the perturbed problems have many solutions.

We note that a sufficient condition for uniqueness and Lipschitz continuity of locally optimal solutions to (nonconvex) mathematical programming problems under (not necessarily canonical) perturbations can be obtained from [15, Th. 4.1]. See also [6], and [5, Cor. 3] for a necessary and sufficient condition of well-conditioning in the smooth convex case under more general perturbations.

In the present setting of mathematical programming problems with inequality constraints, a significant modification of the Lagrangean function of the problem is given by the Kojima function (see [10]). The (pseudo-)distance between two programs is then defined as the Lipschitz constant of the difference of their Kojima functions. Then we characterize well-conditioning by properties of the Kojima function, and show that a lower bound of the distance to ill-conditioning is given by a constant multiple of the reciprocal of the condition number. This result holds true in the Banach space setting (with the same proof), so generalizing results of [17] obtained for unconstrained problems. Finally we prove an upper bound of the same form for the distance, exploiting the implicit function theorem of [8]. In both cases, the constants are explicitly obtained in terms of problem's data.

The results here generalize (for the finite-dimensional setting) the main results of [17] about unconstrained optimization problems. Results dealing with more general perturbations of unconstrained optimization problems are obtained in [2], and extensions to multiobjective optimization can be found in [3].

Section 2 describes the problem and the assumptions. Section 3 contains three examples of well- and ill-conditioned problems. Section 4 is devoted to the main results. Section 5 deals with an extension to the Banach space setting. Section 6 collects the proofs and some auxiliary results.

2 Problem statement and assumptions

We consider an open ball $B \subset R^n$, functions

$$f, g_1, \dots, g_q : B \to R$$

and points $a \in R^n$, $b \in R^q$. For simpler notation we write $p = (a, b)$ with $a \in R^n$, $b \in R^q$ instead of $p = (a^T, b^T)^T$. Then we consider the mathematical programming problem $Q(p)$ to minimize $f(x) - a^T x$ subject to the constraints

$$g_1(x) \le b_1, \dots, g_q(x) \le b_q \tag{1}$$

where $(b_1, \dots, b_q)^T = b$. We denote by g the vector with components $g_1, \dots g_q$ and write $Q = (f, g)$. Under suitable assumptions (specified later) we denote by $m(p)$ the minimizer of $Q(p)$, by $u(p)$ the corresponding multiplier, and by

$$s(p) = [m(p), u(p)] \tag{2}$$

the optimal solution of $Q(p)$. We consider the Kojima function K associated to problem $Q = Q(0)$, namely

$$K : B \times R^q \to R^{n+q}$$

given by

$$K(x, y) = \left(\nabla f(x) + \sum_{j=1}^{q} y_j^+ \nabla g_j(x), g_1(x) - y_1^-, \dots, g_q(x) - y_q^- \right). \tag{3}$$

Here $y^+ = \max\{y, 0\}$, $y^- = \min\{y, 0\}$ and (3) means that the first n components of $K(x, y)$ are those of $\nabla f(x) + \sum_{j=1}^{q} y_j^+ \nabla g_j(x)$ and the remaining are given by $g_i(x) -$

y_i^-, $i = 1, \ldots, q$. We denote by $|\cdot|$ the l_1 norm of R^m, namely $|(x_1, \ldots, x_m)^T| = \sum_{j=1}^m |x_j|$.

For every problem Q as before we are given a nonempty set $\Omega \subset R^{n+q}$ containing 0, such that 0 is a cluster point of Ω. Then we consider the canonically perturbed problems $Q(p)$ as $p \in \Omega$. Problem Q will be called *well-conditioned* if the following properties hold. First, $Q(p)$ has a unique global optimal solution $s(p)$ as in (2) for each sufficiently small $p \in \Omega$. Second, the *condition number* of Q, namely

$$\text{cond } Q = \text{limsup}_{p', p'' \to 0} \frac{|s(p'') - s(p')|}{|p'' - p'|} \tag{4}$$

is finite. Of course, in (4) we consider $p', p'' \in \Omega$. We posit the following conditions.

(A1) $f, g_1, \ldots, g_q$ are convex, their gradients exist and are Lipschitz continuous on B;
(A2) there exist $x_0 \in B$, $\varepsilon > 0$ such that $g_j(x_0) \le -\varepsilon$, $j = 1, \ldots, q$;
(A3) there exists $\delta \in (0, \varepsilon)$ such that for each $p \in \Omega$ with $|p| < \delta$ there exists a unique optimal solution $s(p)$ of $Q(p)$.

We denote by *IC* the set of all mathematical programming problems Q fulfilling (A1) and (A2), assuming that $Q(p)$ has optimal solutions for each p sufficiently small, such that Q is *ill-conditioned*. So $Q \in IC$ if either $Q(p_m)$ has infinitely many solutions for some sequence $p_m \to 0$, $p_m \in \Omega$, or $Q(p)$ has a unique optimal solution $s(p)$ for each $p \in \Omega$ sufficiently small and its Lipschitz modulus cond $Q = +\infty$ as defined by (4).

Remarks (1) The central property of ill-conditioning is the non Lipschitz behavior under perturbations. The lack of uniqueness is in some sense a secondary reason of ill-conditioning. An extension of some results of this paper to mathematical programming problems with many solutions will be presented elsewhere.

(2) Let J be a class of problems $Q = (f, g)$ such that the following properties hold. $f, g_1, \ldots, g_q : R^n \to R$, the admissible region is unbounded, the restriction to it of $f(x)$ goes to $+\infty$ faster that $|x|$ as $x \to \infty$ uniformly with respect to f, (A2) holds and there exists a constant N such that $|x_0| \le N$, $f(x_0) \le N$ for every $Q \in J$. Then, as easily checked, the minimizers $m(p)$ are uniformly bounded within J if p is, hence the existence of the ball B we assume from the beginning is guaranteed.

The Slater condition (A2), which plays no role in the definition of conditioning, will be however required in defining the distance between two mathematical programming problems. Indeed, if Q fulfills (A1) and (A2) and $Q(p)$ has optimal solutions for every $p \in \Omega$ sufficiently small, then the set of the optimal solutions (not necessarily unique) to $Q(p)$ is bounded, and there exists an open ball in R^q, centered at 0, which contains all points of the form

$$u(p) + g[m(p)] - b$$

where $[m(p), u(p)]$ is an optimal solution of $Q(p)$, as $p \in \Omega$ is sufficiently small, see Proposition 1 of Sect. 6. Moreover its radius can be estimated by using problem's data, as we show in Proposition 1. Thus, given Q fulfilling (A1) and (A2), such that $Q(p)$ has optimal solutions if $p \in \Omega$ and $|p| < \delta$ for some sufficiently small $\delta > 0$, we consider the infimum r^* of all $r > 0$ such that $|u(p) + g[m(p)] - b| \le r$ as $|p| < \delta$, and denote by $D = D(Q)$ the open ball in R^q of radius r^* if $r^* > 0$; let $D = \{0\}$ if $r^* = 0$. Given two mathematical programming problems Q_1, Q_2 such that $Q_1(p)$ and $Q_2(p)$ have optimal solutions if p is sufficiently small, fulfilling (A1) and (A2), let K_1, K_2 denote their corresponding Kojima functions and write $K = K_1 - K_2$. Let D_1, D_2 be the corresponding balls in R^q as defined above, and denote by D the largest between them. Then the pseudo-distance is defined by

$$\text{dist}\,(Q_1, Q_2) = \sup \left\{ \frac{|K(x'', y'') - K(x', y')|}{|x'' - x'| + |y'' - y'|} : x', x'' \in B; y', y'' \in D \right\}, \quad (5)$$

namely the Lipschitz constant of $K_1 - K_2$ over $B \times D$.

We compare the previous definition with the setting of [17], dealing with (infinite-dimensional) unconstrained optimization problems. The definition here of well-conditioned problems contains as a special case the one adopted in [17] (in the finite-dimensional case of course). Moreover, definition (4) of the condition number reduces to the one used there, namely the Lipschitz modulus c_2 of the unique minimizer with respect to the canonical (tilt) perturbations of Q. The definition (5) contains the one used in [17] for the unconstrained case (of course the Lipschitz constant of the difference of the Kojima functions there reduces to that of the gradients of the objective functions).

Remark By (5), the mapping dist $(\cdot, \cdot)$ fulfills all properties of a metric except that dist $(Q_1, Q_2) = 0$ does not imply $Q_1 = Q_2$; it is instead equivalent to the following (as easily checked). Writing $Q_i = (f_i, g_i)$, then $f_1 - f_2$ is an affine function, while $g_1 - g_2$ is constant, and conversely. Thus Q_2 is a canonical perturbation of Q_1 (up to an additive constant on the objective function).

3 Examples

(1) Every linear programming problem (with inequality constraints as before) is well-conditioned provided there exists a unique optimal solution for each sufficiently small perturbation, and the constraints of the primal and the dual problem are regular in the sense of Robinson, see Theorem 1 of [14].

(2) Let $n = 2, q = 1$, denote by $\| \cdot \|$ the euclidean norm and let $\Omega = R^2$. Let $c, \bar{x} \in R^2$ be fixed, with $c \neq 0$. Consider a sufficiently large disk B, p sufficiently small and let

$$f(x) = \|x - \bar{x}\|^2, g(x) = c^T x.$$

Then for each small (a, b), $Q(p)$ denotes the problem of minimizing $\|x - \bar{x}\|^2 - a^T x$ subject to the constraint $c^T x \le b$. Conditions (A1), (A2) and (A3) are fulfilled. The optimal solution $s(p)$ is Lipschitz around 0 by Theorem 4.1 of [15], since $c \neq 0$ and the strong second order sufficient condition holds.

(3) Let $n = q = 1$ and consider

$$\Omega = \{(a, b) \in R^2 : a \leq 0, b \geq 0\},\ \ f(x) = \frac{x^4}{4},\ \ g_1(x) = e^x - 1,\ \ B = (-1, 0).$$

Then conditions (A1), (A2) and (A3) are fulfilled. For $(a, b) \in \Omega$ sufficiently small we have

$$m(a, b) = \sqrt[3]{a},\ \ u(a, b) = 0$$

hence this problem is ill-conditioned.

4 Main results

A characterization of well-conditioned mathematical programming problems in our setting can be obtained making use of properties of their Kojima functions. We exploit the known link between critical points, optimal solutions and the behavior of the Kojima functions (see [9], chapter 7).

Theorem 1 *Given problem Q fulfilling (A1) and (A2), let K denote its Kojima function. Then the following are equivalent:*

Q is well-conditioned;
K is one-to-one with K^{-1} Lipschitz continuous around 0.

In order to obtain a condition number theorem, we start by establishing a lower bound of the distance of a given problem to ill-conditioning by the reciprocal of its condition number. We obtain an estimate of the size of perturbations acting on a well-conditioned problem, which do not destroy well-conditioning. We consider a problem $Q = (f, g)$ and denote by L the Lipschitz constant of g on B.

Theorem 2 *Let Q, $\bar{Q}$ be given problems fulfilling (A1) and (A2). Let Q be well-conditioned such that*

$$(1 + L)\,\text{cond}\,Q\,\text{dist}\,(Q, \bar{Q}) < 1. \tag{6}$$

Then $\bar{Q}$ is well-conditioned.

Let now

$$\text{dist}\,(Q, IC) = \inf\,\{\text{dist}\,(Q, \bar{Q}) : \bar{Q} \in IC\}.$$

As an immediate corollary of Theorem 2 we get

Corollary 1 *Let Q be well-conditioned fulfilling (A1) and (A2), with cond $Q > 0$. Then*

$$\text{dist}\,(Q, IC) \geq \frac{1}{(1 + L)\,\text{cond}\,Q}. \tag{7}$$

Corollary 1 extends the finite-dimensional version of [17, Th. 3.1] dealing with unconstrained problems (where of course $L = 0$). In a sense, (7) can be considered one half of a condition number theorem. To obtain the second half, we need an upper bound of the distance to ill-conditioning in terms of the condition number. The estimate of the Lipschitz constant obtained in [8] will be used to this purpose.

Let $Q = (f, g)$ be well-conditioned fulfilling (A1), (A2) and (A3), let K be its Kojima function. Let $c(0)$ be the critical point of problem Q, see [9, p. 150]. Suppose that Clarke's generalized Jacobian $\partial K[c(0)]$ of K at $c(0)$ is nonsingular. Then consider the $(n+q) \times (n+q)$ matrices W, the vectors $v \in R^{n+q}$, and positive numbers

$$\omega > \sup\{|A^{-1}| : A \in \partial K[c(0)]\},$$
$$\alpha = \min\left\{\left|\left(A + \frac{W}{\omega}\right) v\right| : A \in \partial K[c(0)], |W| = 1, |v| = 1\right\}. \tag{8}$$

Denote by F the Lipschitz constant of $\bigtriangledown f$ on B, and let L (as before) be the Lipschitz constant of g on B.

Theorem 3 *Let Q fulfill (A1),(A2),(A3). Let $\partial K[c(0)]$ be nonsingular. Then Q is well-conditioned, and if cond $Q > 0$ we have*

$$\text{dist}\,(Q, IC) \leq \frac{F(2+L)}{\alpha \text{ cond } Q}. \tag{9}$$

Summarizing, by Corollary1 and Theorem 3 we obtain the following final result.

Condition number theorem *For every problem Q which is well-conditioned with cond $Q > 0$, and fulfills (A1), (A2) and (A3), if $\partial K[c(0)]$ is non singular, we have*

$$\frac{1}{(1+L)\text{ cond } Q} \leq \text{dist}\,(Q, IC) \leq \frac{F(2+L)}{\alpha \text{ cond } Q}.$$

Remarks (1) An explicit characterization on nonsingularity of ∂K at a given point is obtained in [8, Th. 3.1] (under suitable regularity conditions).

(2) The second order condition we require in Theorem 3, namely nonsingularity of $\partial K[c(0)]$, is a natural assumption here, since Theorem 3 generalizes the result of [11] about tilt stability of minimizers, which is intimately related to the second order condition for unconstrained problems.

5 Extension to the infinite dimensional setting

Theorem 2 and Corollary 1 hold true in the Banach space setting, as follows. Let E be a real Banach space, let B be an open ball of E with positive radius, let $f, g_1, \ldots, g_q : B \to R$ be convex functions which are Frèchet differentiable at each point of B, such that their gradients can be extended to the closure of B so to be Lipschitz continuous there. Given the mathematical programming problem $Q = (f, g)$ let $p = (a, b) \in E^* \times R^q$, where E^* denotes the dual space of E, and consider the problem $Q(p)$, of minimizing $f(x) - < a, x >$ subject to the constraints (1). Here $< \cdot, \cdot >$ denotes the duality pairing between E^* and E. We denote by $\| \cdot \|$ the norm in E and in E^*, and (abusing notation) we write here

$$\|(a, b)\| = \|a\| + |b|.$$

The definition of the Kojima function

$$K : B \times R^q \to E^* \times R^q$$

of problem Q is exactly (3) as before. Let Q_1, Q_2 be two programs with data $f, g_1, \dots, g_q$ as before, fulfilling (A2), such that $Q_1(p)$ and $Q_2(p)$ have optimal solutions if p is sufficiently small. Then the definition of the distance of the programs Q_1, Q_2 is the same as (5) (up to obvious changes in notation), namely

$$\text{dist}\,(Q_1, Q_2) = \sup \left\{ \frac{\|K(x'', y'') - K(x', y')\|}{\|x'' - x'\| + \|y'' - y'\|} : x', x'' \in B; y', y'' \in D \right\},$$

owing to the uniform boundedness of the multipliers of $Q(p)$ for $\|p\|$ sufficiently small, which is true also in the infinite dimensional setting. With the same notations of Sect. 4 we obtain

Theorem 4 *Let Q, $\bar{Q}$ be given problems as before, fulfilling (A2). Let Q be well-conditioned such that (6) holds. Then $\bar{Q}$ is well-conditioned. Moreover, if cond $Q > 0$, then (7) holds.*

6 Auxiliary results and proofs

The following lemma will be used in the sequel.

Lemma 1 *Let $Q = (f, g)$ and assume (A1), (A2) and (A3). Let $p = (a, b)$ with $a \in R^n, b \in R^q$. For $x \in B$, $y \in R^q$ the following are equivalent:*

$$x = m(p), y = u(p); \tag{10}$$

$$\nabla f(x) + \sum_{i=1}^{q} (y_i + g_i(x) - b_i)^+ \nabla g_i(x) = a, \text{ and } g_i(x) - (y_i + g_i(x) - b_i)^- = b_i, i = 1, \dots, q. \tag{11}$$

Proof We need to prove that (11) is equivalent to the optimality system

$$\nabla f(x) + \sum_{i=1}^{q} y_i \nabla g_i(x) = a, g_i(x) \le b_i, y_i(g_i(x) - b_i) = 0, y_i \ge 0, i = 1, \dots, q. \tag{12}$$

Assume (12). Let l be such that (without restriction)

$$y_i + g_i(x) \ge b_i, i = 1, \dots, l; \;\; y_i + g_i(x) < b_i, i = l + 1, \dots, q \tag{13}$$

with the obvious modifications if $l = q$ or $l = 0$. Then remembering the first equality of (12) we get

$$\bigtriangledown f(x) + \sum_{i=1}^{q} (y_i + g_i(x) - b_i)^+ \bigtriangledown g_i(x) = a$$
$$- \sum_{i=l+1}^{q} y_i \bigtriangledown g_i(x) + \sum_{i=1}^{l} (g_i(x) - b_i) \bigtriangledown g_i(x). \qquad (14)$$

Since $y_i \geq -g_i(x) + b_i$, $i = 1, \ldots, l$, we have

$$0 = y_i[b_i - g_i(x)] \geq (g_i(x) - b_i)^2$$

hence $g_i(x) = b_i$ and the last sum of (14) is 0. Moreover $y_i < b_i - g_i(x), i = l + 1, \ldots, q$ hence

$$0 \leq y_i^2 \leq y_i[b_i - g_i(x)] = 0$$

whence $y_i = 0$, and by (14) the first equality of (11) is proved. The second equality follows by the previous calculations. Then (12) implies (11). Conversely, assume (11). With l as in (13), we have

$$\bigtriangledown f(x) + \sum_{i=1}^{q} y_i \bigtriangledown g_i(x) = a - \sum_{i=1}^{l} (g_i(x) - b_i) \bigtriangledown g_i(x) + \sum_{i=l+1}^{q} y_i \bigtriangledown g_i(x). \qquad (15)$$

If $1 \leq i \leq l$ then $g_i(x) = b_i$ by (11) hence the first sum in the right-hand side of (15) is 0. If $l + 1 \leq i \leq q$ then $y_i = 0$ by (11) and also the second sum is 0, so that the first equality of (12) is proved. Moreover $y_i = 0$ if $i = l + 1, \ldots, q$, while $y_i \geq b_i - g_i(x) = 0$ if $i = 1, \ldots, l$, hence $y_i \geq 0$ for all i. It follows that all conditions in (12) are fulfilled, and this completes the proof of the lemma. □

The following proposition extends to the parameterized problems $Q(p)$ known results about bounds of the multipliers (see Sect. 2.3 of chapter 7 in [7]).

Proposition 1 *Let Q fulfill (A1) and (A2). Suppose that $Q(p)$, $p \in \Omega$, has optimal solutions if $|p| < \delta$ for some $\delta > 0$. Then if $|p| < \delta < \varepsilon$, for every multiplier u_k of $Q(p)$ we have*

$$u_k \leq \frac{(M + |p|)\,\mathrm{diamB}}{\varepsilon - \delta}, k = 1, \ldots, q$$

where

$$M = \sup\{|\bigtriangledown f(x)| : x \in B\}.$$

Proof Given p let (x, u) be any optimal solution of $Q(p)$. If $g_j(x) = b_j$ we have by convexity and the Slater condition (A2)

$$\nabla g_j(x)^T (x_0 - x) \leq g_j(x_0) - b_j \leq \delta - \varepsilon. \tag{16}$$

Let $d = x - x_0$, then by the Kuhn–Tucker conditions and (16)

$$\sum_{j=1}^{q} u_j \nabla g_j(x)^T d = [a - \nabla f(x)]^T d \leq (\delta - \varepsilon) \sum_{j=1}^{q} u_j$$

hence for every $k = 1, \dots, q$

$$0 \leq u_k \leq \sum_{j=1}^{q} u_j \leq \frac{[\nabla f(x) - a]^T d}{\varepsilon - \delta}$$

whence the conclusion. □

Proof of Theorem 1 Let Q be well-conditioned, hence (A3) holds. Let $x_1, x_2 \in B, z_1, z_2 \in R^q$ be such that

$$K(x_1, z_1) = K(x_2, z_2) = p = (a, b).$$

Consider

$$y_i = z_i - g(x_i) + b, i = 1, 2.$$

Then

$$K(x_1, y_1 + g(x_1) - b) = K(x_2, y_2 + g(x_2) - b) = p$$

hence

$$x_1 = x_2 = m(p), y_1 = y_2 = u(p)$$

by Lemma 1, thus $z_1 = z_2$ too and K is one-to-one. Moreover

$$K^{-1}(p) = (m(p), u(p) + g[m(p)] - b). \tag{17}$$

Then

$$|K^{-1}(p'') - K^{-1}(p')| \leq |s(p'') - s(p')| + |g[m(p'')] - g[m(p')]|$$

if $s(p) = [m(p), u(p)]$ is the unique optimal solution of $Q(p)$. Then Lipschitz continuity of K^{-1} comes from well-conditioning of Q and the Lipschitz continuity of g. The first half of Theorem 1 is thereby proved. Conversely, let Q be ill-conditioned.

If $Q(p)$ has a unique optimal solution for each sufficiently small $p \in \Omega$, then K is one-to-one as checked above, and we prove that

$$K^* = \limsup_{p',p''\to 0} \frac{|K^{-1}(p'') - K^{-1}(p')|}{|p'' - p'|} = +\infty, \tag{18}$$

ending the proof for this case. By ill-conditioning, at least one between

$$m^* = \limsup_{p',p''\to 0} \frac{|m(p'') - m(p')|}{|p'' - p'|}, \quad u^* = \limsup_{p',p''\to 0} \frac{|u(p'') - u(p')|}{|p'' - p'|},$$

is $+\infty$. If $m^* = +\infty$ then $K^* = +\infty$ since

$$|K^{-1}(p'') - K^{-1}(p')| \geq |m(p'') - m(p')|.$$

Finally if

$$u^* = +\infty \quad \text{and} \quad m^* < +\infty, \tag{19}$$

let

$$y(p) = u(p) + g[m(p)] - b,\ y^* = \limsup_{p',p''\to 0} \frac{|y(p'') - y(p')|}{|p'' - p'|}.$$

Suffices to prove that $y^* = +\infty$ since by (17)

$$|K^{-1}(p'') - K^{-1}(p')| \geq |y(p'') - y(p')|.$$

Indeed

$$|u(p'') - u(p')| \leq |y(p'') - y(p')| + |g[m(p'')] - g[m(p')]|$$

and the conclusion follows by (19), thus (18) is proved. If finally $Q(p)$ has many optimal solutions, then K cannot be one-to-one, and this ends the proof. □

Proof of Theorem 2 Let K be the Kojima function of Q. By well-conditioning and Theorem 1, K^{-1} is Lipschitz continuous around 0. Let $\bar{K}$ be the Kojima function of $\bar{Q}$. We prove that $\bar{K}$ is one-to-one and $\bar{K}^{-1}$ is Lipschitz continuous around 0. By Theorem 1, this will prove that $\bar{Q}$ is well-conditioned. Write $\bar{K} = \bar{K} - K + K$, and consider the balls $D_1 = D(Q)$, $\bar{D} = D(\bar{Q})$. Let $D = D_1 \cup \bar{D}$ and

$$d = \text{dist}\,(Q, \bar{Q}),$$

then $\bar{K} - K$ is Lipschitz continuous on $B \times D$ with Lipschitz constant d. Let K^* denote the Lipschitz modulus of K^{-1} at 0, and consider for $\alpha > 0$ sufficiently small

$$H(\alpha) = \sup \left\{ \frac{|K^{-1}(p'') - K^{-1}(p')|}{|p'' - p'|} : p', p'' \in \Omega, |p'| < \alpha, |p''| < \alpha \right\},$$

then $H(\alpha) \to K^*$ as $\alpha \to 0$. By (17) we have

$$K^* \leq (1 + L) \operatorname{cond} Q$$

hence by (6) $K^* d < 1$, whence $H(\alpha)d < 1$ if $\alpha > 0$ is sufficiently small. By Lemma 3.1 of [17] it follows that $\bar{K}^{-1}$ is Lipschitz continuous around 0, thus $\bar{Q}$ is well-conditioned. □

Remarks (1) From the proof of Theorem 2 we see that the condition number of $\bar{Q}$ is $\leq K^*/(1 - dK^*)$.

(2) The solution map $s(p)$ of $Q(p)$ is implicitly obtained by solving the equation $K(x, y) = p$, as shown by (17). This suggests that some connection could be established with metric regularity theory.

Proof of Theorem 3 We show that Q is well-conditioned. Let K denote the Kojima function of Q. The implicit function theorem of [8] can be applied to the equation $K(x, y) = 0$. It follows that the equation $K(x, y) = p$ has a unique solution $c(p)$ for each sufficiently small $p \in \Omega$. Thus $c(p)$ is the unique Kojima critical point of $Q(p)$. Moreover by the implicit function theorem, there exists a constant $T > 0$ such that

$$|c(p'') - c(p')| \leq T|p'' - p'| \tag{20}$$

provided p', p'' are sufficiently small. By (17)

$$c(p) = (m(p), u(p) + g[m(p)] - b).$$

Write

$$y(p) = u(p) + g[m(p)] - b,$$

then by (20) we have

$$\begin{aligned}
&|m(p'') - m(p')| \leq T|p'' - p'|, \\
&|u(p'') - u(p')| \leq |y(p'') - y(p')| + L|m(p'') - m(p')| \leq T(L + 1)|p'' - p'|
\end{aligned}$$

where L is the Lipschitz constant of g on B. Hence

$$|s(p'') - s(p')| \leq T(L + 2)|p'' - p'| \tag{21}$$

proving well-conditioning of Q. Now we estimate cond Q from above. By assumption, all matrices $A \in \partial K[c(0)]$ are nonsingular. Given A, by the Eckart–Young theorem [4, Theorem 1 p. 203], if $W \in R^{n+q,n+q}$ and $|W| \leq 1$, we have that $A + W/\omega$ is nonsingular. Thus, remembering (8), by the proof of Theorem 2.1 of [8] we see (p. 130 there) that $T = 1/\alpha$. Hence by (21)

$$\operatorname{cond} Q \leq \frac{2 + L}{\alpha}. \tag{22}$$

Now we estimate dist (Q, IC). Let $\bar{Q} = (\bar{f}, g)$ where $\bar{f}(x) = 0$ for every x. Of course $\bar{Q} \in IC$ by (A2), and

$$\text{dist}\,(Q, IC) \leq \text{dist}\,(Q, \bar{Q}) = F,$$

hence by (22)

$$\text{dist}\,(Q, IC)\,\text{cond}\,Q \leq \frac{F(2+L)}{\alpha},$$

proving (9). □

Proof of Theorem 4 As well known, the multiplier theorem holds in the Banach space setting (see Corollary 1.1 p. 177 of [1]). An inspection of the proofs shows that, in the Banach space setting, Lemma 1 and Proposition 1 hold true (with obvious changes in the proof of Proposition 1). Then also the proofs of Theorem 1 and 2 are unchanged, thus proving Theorem 3. □

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