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OPTIMALITY CONDITIONS IN SEMIDEFINITE PROGRAMMING

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□ *Semidefinite positiveness of operators on Euclidean spaces is characterized. Using this characterization, we compute in a direct way the first-order and second-order tangent sets to the cone of semidefinite positive operators on such a space. These characterizations are useful for optimality conditions in semidefinite programming.*

Keywords Cone programming; Manifold with corners; Optimality condition; Semidefinite programming; Tangent cone; Tangent set of order two.

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1. INTRODUCTION

The knowledge of the expression of the tangent cone and of the normal cone to the set defining the constraints of a mathematical programming problem enables to formulate first-order optimality conditions in primal or dual form. For second-order optimality conditions, the knowledge of the second-order tangent sets is usually not enough [5, 20, 24, 32] and additional "curvature" terms must be introduced in second-order optimality conditions. However, such a knowledge cannot be neglected. In [3] and [10], the second-order tangent sets of the positive cones of some classical function spaces are identified. It is the purpose of this article to do the same for the cone of positive semidefinite operators on a Hilbert space or a finite dimensional Euclidean space. In exploring this question, we detect the difficulties linked to the fact that not all symmetric operators have a closed range. However, the method we use

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seems to be the only one that is available in the infinite dimensional case. A second objective of our article is to offer a clear analysis of the additional "curvature" terms appearing in second-order optimality conditions. A third aim is the application of recent second-order optimality conditions (see [33]) to the case of semidefinite programming.

Although the cone of positive semidefinite operators on some Euclidean space is not polyhedral, it enjoys remarkable properties [5, 36–41]. Among them are the fact that this set is parabolically derivable in the sense of [35, Definition 13.11] (see Proposition 9 below) and the fact that its second-order tangent set is convex, a property that is not always satisfied [5]. We give a more precise form to such a result, showing that this set is locally a translate of a closed convex cone. This property is a consequence of the fact that the cone of positive semidefinite operators has the nice geometric structure of manifolds with corners. This structure is slightly more precise than the one revealed in [5].

The present study confirms the special place semidefinite programming occupies among mathematical programming problems (see [2, 14, 15, 18, 22, 27, 36, 40, 41]).

The article is organized as follows. In the next section, we revisit a characterization of positivite semidefiniteness when the space is split into an orthogonal decomposition (Schur's decomposition). This characterization is the basis of our study in section 3 of first-order and second-order tangent sets to the cone of positive semidefinite operators. It is geometric, natural and intrinsic. In [5, 37] these sets are described through a study of the least eigenvalue of the operator. Other approaches use the principal minors of order $k \leq n$, $n \times n$ being the format of the matrix, or the principal invariants of the matrix. Such analytical approaches are not as direct and simple as the one here and they cannot be extended to the infinite dimensional case. The route we take to optimality conditions is closely linked to the geometry of the set of symmetric operators. Section 4 is devoted to a review of optimality conditions for mathematical programming problems when the constraints are defined by abstract cones. In section 5, we specialize these conditions to the case the cone is the cone of positive semidefinite operators.

The importance of semidefinite programming for algorithms and applications (see [1, 2, 12, 28–30] for instance) justifies a fresh look at its fundamental features.

2. A CHARACTERIZATION OF POSITIVE SEMIDEFINITENESS

Let Z be a Hilbert space or a finite dimensional Euclidean space; for $w, z \in Z$ we write w^*z for the scalar product $(w \mid z)$, w^* being the image of w by the Riesz isomorphism. If W and X are Euclidean or Hilbert spaces and A is an element of the space $L(W, X)$ of continuous linear operators from W into X, $A^* : X \to W$ denotes the *adjoint* of A characterized by $(A^*x \mid w) = (Aw \mid x)$ or $w^*A^*x = x^*Aw$ for all $w \in W$, $x \in X$. We recall that if X is a closed linear subspace of Z, then the adjoint of the orthogonal projector $P_X \in L(Z, X)$ from Z onto X is the canonical injection J_X of X into Z. Here P_X is not considered as an operator from Z into Z, so that the orthogonal projector of Z onto $X^\perp$ cannot be written as $I_Z - P_X$ (where I_Z is the identity map on Z) but as $I_Z - J_X \circ P_X$ (when considered as an element of $L(Z, Z)$).

An element M of $L(Z, Z)$ is said to be *symmetric*, and we write $M \in \mathscr{S}(Z)$, or $M \in \mathscr{S}$ when there is no risk of confusion, if $M^* = M$. If Z is a Hilbert space, we denote by $\mathscr{S}^c(Z)$ or $\mathscr{S}^c$ the space of symmetric continuous linear maps with closed ranges. We write $M \succeq 0$ (resp. $M \succ 0$) to mean that M is positive semidefinite (resp. positive definite) and $M \succeq N$ (resp. $M \succ N$) stands for $M - N \succeq 0$ (resp. $M - N \succ 0$). We denote by $\mathscr{S}_+$ the cone of positive semidefinite operators. Given $M \in \mathscr{S}$ (identified with a matrix when Z is finite dimensional), let us suppose that the space Z splits into an orthogonal sum $Z = X \oplus Y$ of two closed linear subspaces, in such a way that we can write

$$M := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \tag{1}$$

with A nonsingular. The following characterization of positive semidefiniteness is akin an elementary fact about polynomial functions of degree two. It is well known as an application of the Schur Complement. For the sake of completeness we provide a proof valid in the infinite dimensional case.

Lemma 1. *Let M be a symmetric operator decomposed as above, with A invertible. Then a necessary and sufficient condition in order that M be positive semidefinite is that A be positive definite and $C - B^*A^{-1}B$ be positive semidefinite:*

$$(M \succeq 0) \Leftrightarrow \left(A \succ 0,\ C \succeq B^*A^{-1}B\right).$$

Proof. Suppose M is is positive semidefinite. Then, for each $x \in X$, we have $x^*Ax = x^*Mx \geq 0$, so that A is positive semidefinite and nonsingular, hence is positive definite (or elliptic), by a well known result. Moreover, for any $z := x + y \in X \oplus Y$, since $y^*B^*x = x^*By$, one has

$$\begin{aligned} z^*Mz &= x^*Ax + 2x^*By + y^*Cy \\ &= (x + A^{-1}By)^*A(x + A^{-1}By) - y^*B^*A^{-1}By + y^*Cy \\ &= (x + A^{-1}By)^*A(x + A^{-1}By) + y^*(C - B^*A^{-1}B)y. \end{aligned}$$

Taking $x = -A^{-1}By$ with $y \in Y$ arbitrary, we see that $C - B^*A^{-1}B \succeq 0$.

Conversely, if the two conditions $A \succ 0$, $C \succeq B^*A^{-1}B$ are satisfied, then the preceding relations show that $M \succeq 0$. □

Remark. When X is infinite dimensional and $B \in L(Y, X)$ is onto, for every $C \in \mathscr{S}(Y)$ and every $A \in \mathscr{S}(X)$ that is not invertible but satisfies $x^*Ax > 0$ for all $x \in X\backslash\{0\}$ one can find $z \in Z$ such that $z^*Mz < 0$.

A similar proof yields the following characterization.

Lemma 2. *Let M be a symmetric operator decomposed as above, with A nonsingular. Then a necessary and sufficient condition in order that M be positive definite is that A be positive definite and $C - B^*A^{-1}B$ be positive definite:* $(M \succ 0) \Leftrightarrow \big(A \succ 0, \quad C \succ B^*A^{-1}B\big)$.

Now we detect a property that is slightly more exacting than the notion of cone reduction introduced in [5]. It has the advantage of corresponding to a concept relevant to differential geometry.

Definition 3. A subset C of a normed vector space (n.v.s.) space Z is said to be a C^2-manifold with corners around $c \in C$ if there exists a C^2-diffeomorphism $\varphi : U \to V$ from a neighborhood U of c onto a neighborhood V of 0 in a n.v.s. W and a closed convex cone Q (with vertex 0) of W such that $\varphi(C \cap U) = Q \cap V$. It is a C^2-manifold with corners if it is a C^2-manifold with corners around each of its points.

When Q is a half-space, we recover the classical notion of manifold with boundary. The notion of manifold with corners is almost equivalent to the notion of cone reducible set, as the following example shows.

Example. Suppose Y, Z are Banach spaces and there exists a map $\psi : U \to Y$ of class C^2 such that $\psi'(c)(Z) = Y$, Z is the topological sum of $\ker\psi'(c)$ and of a closed subspace X of Z and $C \cap U = \psi^{-1}(Q)$ for some neighborhood U of c in Z and some closed convex cone Q of Y. Then, the submersion theorem ensures that C is a C^2-manifold with corners around c. Shrinking U if necessary, one can take $W := X \times Y$, $\varphi(z) := (p_X(z), \psi(z))$, where p_X is the projection of Z onto Y associated with the direct sum $Z = X \oplus \mathit{ker}\, \psi'(c)$. In particular, when Z is finite dimensional, any cone reducible subset C of Z is a manifold with corner (the converse is always true).

Proposition 4. *The set $\mathscr{S}_+$ of positive semidefinite operators of Z is a C^2-manifold with corners: for all $M_0 \in \mathscr{S}_+$ there exist a closed convex cone $\mathbb{Q}$ of $\mathscr{S}$, open neighborhoods U of M_0, V of 0 and a diffeomorphism $\varphi : U \to V$ of class C^∞ such that $\varphi(U \cap \mathscr{S}_+) = V \cap \mathbb{Q}$, $\varphi(M_0) = 0$.*

Proof. Given $M_0 \in \mathcal{S}_+$, let U be the set of $M \in \mathcal{S}$ such that PMP^* is invertible and positive definite, P being the orthogonal projector of Z onto $X := Y^\perp$ with $Y := \ker M_0$. Such a set is an open neighborhood of M_0 since $M \mapsto PMP^*$ is continuous and the set $\mathcal{S}_{++}(X)$ of positive definite elements of $L(X,X)$ is an open neighborhood of $A_0 := PM_0P^*$ in $L(X,X)$. Now, using the decomposition (1) of $M \in U$, let us define $\varphi : U \to \mathcal{S}(X) \times L(Y,X) \times \mathcal{S}(Y)$ by

$$\varphi(M) := (A - A_0, B, C - B^*A^{-1}B).$$

Clearly, φ is a diffeomorphism of class C^∞ from U onto its image $V := \varphi(U)$ with inverse

$$\varphi^{-1}(M') := (A' + A_0, B', C' + B'^*(A' + A_0)^{-1}B') \quad \text{for } M' := (A', B', C')$$

Then $\varphi(U \cap \mathcal{S}_+) = (\mathcal{S}_{++}(X) - A_0) \times L(Y,X) \times \mathcal{S}_+(Y)$, $\varphi(M_0) = 0$, and $\mathcal{Q} := \mathcal{S}(X) \times L(Y,X) \times \mathcal{S}_+(Y)$ is a closed convex cone. □

Such a structure has important consequences for optimality conditions in semidefinite programming, as shown later.

Remark. The cone of positive semidefinite matrices and the Lorentz cone (or second-order cone) have been studied simultaneously by means of Jordan algebras (see [11, 25]). However, the Lorentz cone has a more classical structure. Defining it as the set

$$K := \{(x,t) \in X \times \mathbb{R} : \ \|x\|^2 - t^2 \geq 0\},$$

where $X := \mathbb{R}^n$ (or a Hilbert space), one can show that for every $(a,b) \neq (0,0)$ in the boundary of K (i.e., such that $\|a\|^2 - b^2 = 0$), the set K is a submanifold with boundary of $X \times \mathbb{R}$, a special case of manifold with corners.

3. TANGENT SETS TO THE CONE $\mathcal{S}_+$

Let us recall that the *tangent cone* (also called contingent cone or Bouligand tangent cone) to a subset S of a normed vector space X at some $x \in S$ is the set $T(S,x) := \limsup_{t \to 0_+} t^{-1}(S - x)$, that is, the set of $v \in X$ such that there exist sequences $(t_n) \to 0_+, (v_n) \to v$ with $x + t_n v_n \in S$ for each n.

The *incident cone* (or intermediate, or adjacent cone or derivate cone) is the set $T^i(S,x) := \liminf_{t \to 0_+} t^{-1}(S - x)$, that is, the set of $v \in X$ such that, for any sequence $(t_n) \to 0_+$, there exists a sequence $(v_n) \to v$ with $x + t_n v_n \in S$ for each n. When S is convex, the two cones coincide (it is also the case in many situations of practical interest); in such a case the tangent cone is convex but in general it differs from the radial tangent cone $T^r(S,x) := \mathbb{R}_+(S-x)$. In fact, when S is convex, $T(S,x)$ is the closure of $T^r(S,x)$ and quite often a convenient way of computing $T(S,x)$ consists in characterizing first $T^r(S,x)$.

The *second-order tangent set* to S at (x,v), where $x \in S$ and $v \in X$ is the set

$$T^2(S,x,v) := \limsup_{t\to 0_+} 2t^{-2}(S - tv - x),$$

that is, the set of $w \in X$ such that there exist sequences $(t_n) \to 0_+$, $(w_n) \to w$ with $x + t_n v + \frac{1}{2}t_n^2 w_n \in S$ for all n. The *second-order incident set* to S at (x,v), where $x \in S$ and $v \in X$ is the set

$$T^{ii}(S,x,v) := \liminf_{t\to 0_+} 2t^{-2}(S - tv - x),$$

that is, the set of $w \in X$ such that, for any sequence $(t_n) \to 0_+$, there exists a sequence $(w_n) \to w$ with $x + t_n v + \frac{1}{2}t_n^2 w_n \in S$ for all n. Even when S is convex these two second-order sets may be different and $T^2(S,x,v)$ may be nonconvex (see [5]). Of course, if $T^2(S,x,v)$ is nonempty, one has $v \in T(S,x)$. It is easy to see that $v \in T^i(S,x)$ iff there exists some arc $c : [0,1] \to S$ such that $c(0) = x$, c has v as a right derivative at 0; similarly, $w \in T^{ii}(S,x,v)$ iff there exists some arc $c : [0,1] \to S$ such that $c(0) = x$, c has v as a right derivative at 0 and can be expanded as $c(t) = x + tv + \frac{1}{2}t^2 w_t$ with $w_t \to w$ as $t \to 0_+$.

The following characterization is well known in the finite-dimensional case ([5, p. 472], [13], e.g.). The direct proof we provide has some interest even in the finite-dimensional case. It uses the preceding characterization of positive semidefinite operators.

Proposition 5. *The tangent cone to the set $\mathcal{S}_+$ at some $M_0 \in \mathcal{S}_+^c$ is the set of $V \in \mathcal{S}$ such that the "restriction" of V to the kernel Y of M_0 is positive semidefinite: if $P := P_Y \in L(Z,Y)$ is the orthogonal projector onto Y, then $V \in T(\mathcal{S}_+, M_0)$ iff $C := PVP^* \in \mathcal{S}_+(Y)$.*

Moreover, for all $V \in T(\mathcal{S}_+, M_0)$, one has $d(M_0 + tV, \mathcal{S}_+) = O(t^2)$.

Thus, denoting by $P_X \in L(Z,X)$ (resp. $P_Y \in L(Z,Y)$) the orthogonal projector of Z onto the complement X of Y (resp. onto Y) and decomposing M_0 and V as

$$M_0 := \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad V := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix},$$

where $A_0 := P_X M_0 P_X^*$ is the restriction (and corestriction) of M_0 to the subspace X orthogonal to the kernel Y of M_0 and $C := P_Y V P_Y^*$, one has $V \in T(\mathcal{S}_+, M_0)$ iff $C \succeq 0$.

Proof. Since $M_0 \in \mathcal{S}_+^c$ and $X := (Ker\, M_0)^\perp$, the operator A_0 is positive definite. Let us suppose that $V \in T(\mathcal{S}_+, M_0)$. Then V is the limit of a sequence (V_n) such that $M_0 + t_n V_n \in \mathcal{S}_+$ for all $n \in \mathbb{N}$ and some sequence $(t_n) \to 0_+$. Decomposing V_n as we did above for V, we get operators A_n, B_n, C_n with limits A, B, C respectively. For n large enough, the operator $A_0 + t_n A_n$ is nonsingular and the characterization of the preceding section yields

$$t_n C_n \succeq t_n B_n^*(A_0 + t_n A_n)^{-1} t_n B_n.$$

Simplifying by t_n and taking limits, we get $C \succeq 0$.

Conversely, let us suppose V decomposed as above is such that $C \succeq 0$. Then, setting for $t > 0$ small enough (so that $A_0 + tA$ is invertible)

$$C_t := C + tB^*(A_0 + tA)^{-1} B,$$

we see that $C_t \to C$ as $t \to 0$ and that $tC_t - t^2 B^*(A_0 + tA)^{-1} B = tC \succeq 0$, so that if

$$V_t := \begin{pmatrix} A & B \\ B^* & C_t \end{pmatrix}.$$

one has $M_t := M_0 + tV_t \in \mathcal{S}_+$ for $t > 0$ small enough. Since $V_t \to V$ as $t \to 0_+$, that shows that V is tangent to $\mathcal{S}_+$ at M_0 (and even that $V \in T^i(\mathcal{S}_+, M_0)$, but that is the same as $T(\mathcal{S}_+, M_0) = T^i(\mathcal{S}_+, M_0)$, as recalled above). Moreover, in view of the decomposition of $M_0 + tV - M_t$, by an obvious property of the norm on $L(Z, Z)$, one has

$$\|M_0 + tV - M_t\| = \|tC - tC_t\| = t^2 \left\|B^*(A_0 + tA)^{-1} B\right\| = O(t^2),$$

so that $d(M_0 + tV, \mathcal{S}_+) = O(t^2)$.

The characterization in terms of the orthogonal projector P onto the kernel Y of M_0 stems from the fact that $C = PVP^*$. □

We just considered the case of a point in the set $\mathcal{S}_+^c$ of operators in $\mathcal{S}_+$ with closed range. The following counterexample explains that choice: if $M_0 \in \mathcal{S}_+ \backslash \mathcal{S}_+^c$ the preceding characterization may not hold.

Example. Let $Z = X \oplus Y$ with X infinite-dimensional and let $M_0 : (x, y) \mapsto (A_0 x, 0)$ with $(A_0 x \mid x) > 0$ for all $x \in X \backslash \{0\}$ but $A_0 \notin \mathcal{S}_+^c(X)$, so that for all $c > 0$ there exists some $u \in X$ satisfying $(A_0 u \mid u) < c \|u\|^2$. Let

$V : (x, y) \mapsto (-2x, 0)$. Then $C := P_Y V P_Y^* = 0 \succeq 0$ but V is not tangent to $\mathcal{S}_+$ at M_0 since for any $V' \in B(V, 1)$ and any sequence $(t_n) \to 0_+$ there exists a sequence (x_n) of unit vectors in X such that $(A_0 x_n \mid x_n) < t_n$, hence, $((M_0 + t_n V')x_n \mid x_n) < ((A_0 - 2t_n I_X)x_n \mid x_n) + t_n \|V' - V\| < 0$.

As expected for a nonpolyhedral set, the tangent cone $T(\mathcal{S}_+, M_0)$ is much larger than the radial tangent cone $T^r(\mathcal{S}_+, M_0) := \mathbb{R}_+(\mathcal{S}_+ - M_0)$. A precise analysis is given in the following proposition; it completes the assertions of [5, p. 473]. It can be extended to the infinite-dimensional case when considering operators with closed ranges and when the kernel Y of M_0 is finite dimensional.

Proposition 6. *Let $M_0 \in \mathcal{S}_+^c$ be such that $Y := \mathrm{Ker}\, M_0$ is finite dimensional. The radial tangent cone to the set $\mathcal{S}_+$ at M_0 is the set of $V \in \mathcal{S}$ such that $PVP^* \in \mathcal{S}_+(Y)$ and there exists some linear operator $K \in L(Y, Z)$ from Y into Z with values in $X := Y^\perp$ such that $(I - P^*P)VP^* = KPVP^*$.*

Thus, in terms of the decomposition of M_0 and V along the subspaces X and Y of Z, identifying $I - P^*P$ with $P_X^* \circ P_X$ and considering K as an element of $L(Y, X)$, one has

$$V \in T^r(\mathcal{S}_+, M_0) \Leftrightarrow C \succeq 0, \exists K \in L(Y, X) : B = KC.$$

Proof. Decomposing V as above, one has $V \in T^r(\mathcal{S}_+, M_0)$ iff for $t > 0$ small enough one has $M_0 + tV \succeq 0$ or

$$C \succeq tB^*(A_0 + tA)^{-1}B. \tag{2}$$

When $C \succeq 0$ and $B = KC$ for some $K \in L(Y, X)$ this condition is satisfied for $t > 0$ small enough since the right-hand side of this relation is a small operator induced by an operator from $(\mathit{Ker}\, C)^\perp$ into itself (again, one applies Lemma 1 in the decomposition of Y into $(\mathit{Ker}\, C)^\perp \oplus (\mathit{Ker}\, C)$.

Conversely, if relation (2) holds for $t > 0$ small enough, then, for all $x \in \mathit{Ker}\, C$ one has $0 \geq x^* tB^*(A_0 + tA)^{-1}Bx$, hence $Bx = 0$, the operator $(A_0 + tA)^{-1}$ being positive definite. The existence of some $K \in L(Y, X)$ satisfying $B = KC$ follows. □

Using the bijection $V \mapsto (A, B, C)$ from $\mathcal{S}$ onto $\mathcal{S}(X) \times L(Y, X) \times \mathcal{S}(Y)$, Proposition 5 ensures that $T(\mathcal{S}_+, M_0)$ can be identified with $\mathcal{S}(X) \times L(Y, X) \times \mathcal{S}_+(Y)$. Thus, introducing the orthogonal projector $Q \in L(Y, Y \cap \ker V)$ from Y onto $Y \cap \ker V$, we have the following characterization obtained by replacing Z with Y.

Corollary 7. *Given $M_0 \in \mathcal{S}_+^c$ and $V \in T(\mathcal{S}_+, M_0)$, one has $W \in T(T(\mathcal{S}_+, M_0), V)$ if, and only if, $QPWP^*Q^* \succeq 0$.*

The following consequence of Proposition 5 will be improved soon.

Corollary 8. *If Z is finite dimensional, for any $M_0 \in \mathcal{S}_+$ and any $V \in T(\mathcal{S}_+, M_0)$ the set $T^2(\mathcal{S}_+, M_0, V)$ is nonempty.*

Proof. For any $M_0 \in \mathcal{S}_+$ and any $V \in T(\mathcal{S}_+, M_0)$, we have seen that we can find $c > 0$ and a curve $t \mapsto M_t$ in $\mathcal{S}_+$ such that $M_t = M_0$ for $t = 0$, $\left(\frac{d}{dt}\right)_{t=0} M_t = V$ and $\|M_0 + tV - M_t\| \leq ct^2$. Then, as $\mathcal{S}$ is finite dimensional, one can find a limit point W of $(t^{-2}(M_t - M_0 - tV))$ as $t \to 0_+$; such a W belongs to $T^2(\mathcal{S}_+, M_0, V)$. □

In order to characterize the second-order tangent set to the set $\mathcal{S}_+$ at some $M_0 \in \mathcal{S}_+^c$ in the direction $V \in T(\mathcal{S}_+, M_0)$, let us decompose the space $Y := Ker\, M_0$ into $Ker\, M_0 \cap Ker\, V$ and its orthogonal subspace, so that C, M_0, V, and $W \in T^2(\mathcal{S}_+, M_0, V)$ take the forms

$$C := \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

$$M_0 := \begin{pmatrix} A_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad V := \begin{pmatrix} A & E & F \\ E^* & D & 0 \\ F^* & 0 & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} A'' & G & H \\ G^* & D'' & J \\ H^* & J^* & K \end{pmatrix}.$$

Let us suppose C has a closed range, so that D is invertible. Note that the relation $V \in T(\mathcal{S}_+, M_0)$ amounts to $D \succeq 0$, whereas the relation $V \in T^r(\mathcal{S}_+, M_0)$ amounts to $D \succeq 0$, $F = 0$ and $E = LD$ for some $L \in L((Ker\, M_0 \cap Ker\, V)^\perp, Ker\, M_0^\perp)$. Again, let us denote by P (resp. P_X) the orthogonal projector from Z onto $Y := Ker\, M_0$ (resp. $X := (Ker\, M_0)^\perp$) and let us denote by Q the orthogonal projector from Y onto $Y \cap Ker\, V$, so that QP $(:= Q \circ P)$ is the orthogonal projector from Z onto $Y \cap Ker\, V$. Let $R := I_Y - Q^*Q$ considered as the orthogonal projector from Y onto the orthogonal complement of $Y \cap Ker\, V$ in Y. Then $A_0 = P_X M_0 P_X^*$, $D = RCR^* = RPVP^*R^*$, $D'' = RPWP^*R^*$, $E = BR^* = P_X VP^*R^*$, $F = P_X VP^*Q^*$, $G = P_X WP^*R^*$, $H = P_X WP^*Q^*$, $J = RQWP^*R^*$, $K = QPWP^*Q^*$.

Proposition 9. *The second-order tangent set to the set $\mathcal{S}_+$ at some $M_0 \in \mathcal{S}_+^c$ in a direction $V \in \mathcal{S}$ such that $C = PVP^*$ has a closed range coincides with the second-order incident set at $M_0 \in T(\mathcal{S}_+, M_0)$ in the direction V. It is the set of $W \in \mathcal{S}$ such that $K \succeq 2F^*A_0^{-1}F$ for the preceding decompositions, that is,*

$$QPWP^*Q^* \succeq 2QPV^*P_X^* \left(P_X^* M_0 P_X\right)^{-1} P_X VP^*Q^*. \tag{3}$$

Moreover, for any $W \in T^2(\mathcal{S}_+, M_0, V)$, one has $d(M_0 + tV + \frac{1}{2}t^2 W, \mathcal{S}_+) = O(t^3)$.

Proof. Suppose $W \in T^2(\mathcal{S}_+, M_0, V)$. Let $(t_n) \to 0_+$ and $(W_n) \to W$ be such that $M_n = M_0 + t_n V + \frac{1}{2}t_n^2 W_n \in \mathcal{S}_+$ for all n. Let us decompose W_n as we did for W, introducing operators $A_n'', \dots, K_n$ with limits $A'', \dots, K$ respectively. For n large enough, the operator $A_n := A_0 + t_n A + \frac{1}{2}t_n^2 A_n''$ is invertible, hence positive definite, and we can apply Lemma 1 to the decomposition of $M_n = M_0 + t_n V + \frac{1}{2}t_n^2 W_n$ along $X \oplus Y$ as

$$M_n = M_0 + t_n V + \frac{1}{2}t_n^2 W_n = \begin{pmatrix} A_n & B_n \\ B_n^* & C_n \end{pmatrix}$$

The condition $C_n - B_n^* A_n^{-1} B_n \succeq 0$ of this lemma can be written

$$\begin{pmatrix} D_n & \frac{1}{2}t_n^2 J_n \\ \frac{1}{2}t_n^2 J_n^* & \frac{1}{2}t_n^2 K_n \end{pmatrix} - \begin{pmatrix} E_n & F_n \end{pmatrix}^* A_n^{-1} \begin{pmatrix} E_n & F_n \end{pmatrix} \succeq 0,$$

where $D_n = t_n D + \frac{1}{2}t_n^2 D_n''$, $E_n = t_n E + \frac{1}{2}t_n^2 G_n$, $F_n = t_n F + \frac{1}{2}t_n^2 H_n$, or

$$\begin{pmatrix} D_n & \frac{1}{2}t_n^2 J_n \\ \frac{1}{2}t_n^2 J_n^* & \frac{1}{2}t_n^2 K_n \end{pmatrix} - \begin{pmatrix} E_n^* A_n^{-1} E_n & E_n^* A_n^{-1} F_n \\ F_n^* A_n^{-1} E_n & F_n^* A_n^{-1} F_n \end{pmatrix} \succeq 0.$$

Since

$$L_n := t_n^{-1}\left(D_n - E_n^* A_n^{-1} E_n\right) = D + \frac{1}{2}t_n D_n'' - t_n(E^* + \frac{1}{2}t_n G_n^*)A_n^{-1}(E + \frac{1}{2}t_n G_n)$$

is positive definite for n large enough, applying Lemma 1 again, the preceding condition is equivalent to

$$\frac{1}{2}t_n^2 K_n - F_n^* A_n^{-1} F_n \succeq \left(\frac{1}{2}t_n^2 J_n^* - F_n^* A_n^{-1} E_n\right) t_n^{-1} L_n^{-1} \left(\frac{1}{2}t_n^2 J_n - E_n^* A_n^{-1} F_n\right),$$

or, after simplification by $\frac{1}{2}t_n^2$, setting $\widehat{J}_n := J_n - 2t_n^{-2} E_n^* A_n^{-1} F_n = J_n - 2\left(E + \frac{1}{2}t_n G_n\right)^* A_n^{-1}\left(F + \frac{1}{2}t_n H_n\right)$,

$$K_n - 2(F^* + \frac{1}{2}t_n H_n^*)A_n^{-1}(F + \frac{1}{2}t_n H_n) \succeq \frac{1}{2}t_n \widehat{J}_n^* L_n^{-1} \widehat{J}_n.$$

Taking limits, and using the fact that $\left(A_n^{-1}\right) \to A_0^{-1}$, $\left(\widehat{J}_n\right) \to J - 2E^* A_0^{-1} F$ and $\left(L_n^{-1}\right) \to D^{-1}$, we get $K - 2F^* A_0^{-1} F \succeq 0$, a rewriting of relation (3).

Conversely, let us suppose that W decomposed as above is such that $K \succeq 2F^* A_0^{-1} F$. Let us introduce for $t > 0$ small enough the operators

$$A_t := A_0 + tA + \frac{1}{2}t^2 A'', \qquad E_t := tE + \frac{1}{2}t^2 G \qquad F_t := tF + \frac{1}{2}t^2 H$$

$$D_t := tD + \frac{1}{2}t^2D'', \qquad J_t := \frac{1}{2}t^2J - E_t^*A_t^{-1}F_t,$$
$$K_t := (K - 2F^*A_0^{-1}F) + 2t^{-2}F_t^*A_t^{-1}F_t - 2t^{-2}J_t^*L_t^{-1}J_t,$$
$$L_t := D + \frac{1}{2}tD'' - t^{-1}E_t^*A_t^{-1}E_t$$

and note that L_t is invertible for t small enough. We rewrite the assumption $K \succeq 2F^*A_0^{-1}F$ as

$$\frac{1}{2}t^2K_t \succeq F_t^*A_t^{-1}F_t - J_t^*L_t^{-1}J_t,$$

hence, applying again Lemma 1,

$$\begin{pmatrix} D_t & \frac{1}{2}t^2J \\ \frac{1}{2}t^2J^* & \frac{1}{2}t^2K_t \end{pmatrix} - \begin{pmatrix} E_t & F_t \end{pmatrix}^* A_t \begin{pmatrix} E_t & F_t \end{pmatrix} \succeq 0. \tag{4}$$

Setting $M_t := M_0 + tV + \frac{1}{2}t^2W_t$ with

$$W_t := \begin{pmatrix} A'' & G & H \\ G^* & D'' & J \\ H^* & J^* & K_t \end{pmatrix}, \qquad M_t := \begin{pmatrix} A_t & B_t \\ B_t^* & C_t \end{pmatrix},$$

and twice applying Lemma 1, Condition (4) means that $C_t - B_t^*A_t^{-1}B_t \succeq 0$, or $M_t := M_0 + tV + \frac{1}{2}t^2W_t \in \mathscr{S}_+$ for $t > 0$ small enough. Moreover, since $t^{-2}F_t^*A_t^{-1}F_t \to F^*A_0^{-1}F$ and $t^{-2}J_t^*L_t^{-1}J_t \to 0$ as $t \to 0$, we have $K_t \to K$ as $t \to 0_+$. Thus, $W_t \to W$ and $W \in T^{ii}(\mathscr{S}_+, M_0, V)$. Moreover, as $\|K - K_t\| = O(t)$, we note that $\left\|M_0 + tV + \frac{1}{2}t^2W - M_t\right\| = O(t^3)$. □

One can notice that, as predicted by [35, Proposition 13.12], for any $V \in T(\mathscr{S}_+, M_0)$ one has

$$T^2(\mathscr{S}_+, M_0, V) + T(T(\mathscr{S}_+, M_0), V) = T^2(\mathscr{S}_+, M_0, V).$$

In fact, given $W \in T^2(\mathscr{S}_+, M_0, V)$, $W' \in T(T(\mathscr{S}_+, M_0), V)$, Corollary 7 ensures that the element K' of the decomposition of W' satisfies $K' \succeq 0$. Thus, since $K \succeq 2F^*A_0^{-1}F$, we also have $K + K' \succeq 2F^*A_0^{-1}F$, hence $W + W' \in T^2(\mathscr{S}_+, M_0, V)$. Since $0 \in T(T(\mathscr{S}_+, M_0), V)$, the reverse inclusion $T^2(\mathscr{S}_+, M_0, V) \subset T^2(\mathscr{S}_+, M_0, V) + T(T(\mathscr{S}_+, M_0), V)$ is obvious and equality holds. One can even write

$$T^2(\mathscr{S}_+, M_0, V) = W_0 + T(T(\mathscr{S}_+, M_0), V),$$

where

$$W_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2F^* A_0^{-1} F \end{pmatrix}$$

Corollary 10. *For any $M_0 \in \mathcal{S}_+, V \in T(\mathcal{S}_+, M_0)$ the set $T^2(\mathcal{S}_+, M_0, V)$ is convex and stable by addition of elements of $\mathcal{S}_+$, $T(\mathcal{S}_+, M_0)$, $T(T(\mathcal{S}_+, M_0), V)$.*

Remark. One may wonder whether one can replace the assumption that C has a closed range with the assumption that V has a closed range. However, as shown by the following counterexample, the closed range property is not always inherited by restrictions. To see that, let $Z = X \oplus X$, where X is an infinite dimensional Hilbert space and let $T \in \mathcal{S}_+(X) \backslash \mathcal{S}_+^c(X)$. Let us define $S \in \mathcal{S}(Z)$ by $S(x, y) = (Tx + y, x)$ for $(x, y) \in Z$, so that $S(x, 0) = (Tx, 0)$ and T is the restriction of S to $X \times \{0\}$. Then S has a closed range: if $(S(x_n, y_n)) \to (u, v)$ one has $(x_n) \to v$ and $(Tx_n + y_n) \to u$, so that $(y_n) \to u - Tv$ and $(u, v) = (Tv + (u - Tv), v) \in S(Z)$.

Remark. It is of interest to make a comparison with semi-infinite programming. For that purpose, we consider the embedding $j : \mathcal{S} \to C(S_Z)$ of $\mathcal{S}$ into the space of continuous functions on the unit sphere S_Z of Z defined by $j(M) := q_M \mid S_Z$, the restriction to S_Z of the quadratic form $q_Z : z \mapsto (Mz \mid z)$. Clearly, one has $j(\mathcal{S}_+) = j(\mathcal{S}) \cap C(S_Z)_+$ where $C(S_Z)_+$ is the cone of nonnegative continuous functions on S_Z. The map j is a linear isomorphism onto its image and j maps $T(\mathcal{S}_+, M)$ onto $T(j(\mathcal{S}_+), j(M))$. It is known that $C(S_Z)_+$ is derivable in the sense of [35, p. 198] that $T(C(S_Z)_+, f)$ coincides with the incident cone $T^i(C(S)_+, f)$ for all $f \in C(S_Z)$; however, in general the cone $C(S_Z)_+$ is not parabolically derivable, on the contrary of $\mathcal{S}_+$.

Let us turn now to the so-called augmented second-order tangent cone introduced in [33, Proposition 3.1, 3.4]. It is defined as follows: the *augmented second-order tangent cone* $\widehat{T}^2(\mathcal{S}, M_0, V)$ to the set $\mathcal{S}_+$ at some $M_0 \in \mathcal{S}_+$ in the direction $V \in \mathcal{S}$ is the set of $(W, r) \in \mathcal{S} \times \mathbb{R}_+$ such that there exist sequences $(W_n) \to W$ in $\mathcal{S}$, (t_n), (r_n) in $\mathbb{P}$ with limits 0 and r, respectively, satisfying $\left(r_n^{-1} t_n\right) \to 0$ and

$$M_n := M_0 + t_n V + \frac{t_n^2}{2} \frac{W_n}{r_n} \in \mathcal{S}_+.$$

Replacing W_n with W_n / r_n in the proof of Proposition 9, we get the following characterization.

Proposition 11. *The augmented second-order tangent cone* $\widehat{T}^2(\mathcal{S}, M_0, V)$ *to the set* $\mathcal{S}_+$ *at some* $M_0 \in \mathcal{S}_+$ *in the direction* V *is the set of* $(W, r) \in \mathcal{S} \times \mathbb{R}_+$ *such that* $K \succeq 2rF^*A_0^{-1}F$ *for the preceding decompositions, that is,*

$$QPWP^*Q^* \succeq 2rQPV^*P_X^* \left(P_X^*M_0P_X\right)^{-1} P_XVP^*Q^*. \tag{5}$$

The preceding characterization enables to apply the optimality conditions of [33, Proposition 3.1, 3.4]. Given a function $f : \mathcal{S} \to \mathbb{R}$ that is twice differentiable at a local minimizer $M_0 \in \mathcal{S}_+$ of f on $\mathcal{S}_+$, [33, Proposition 3.1] asserts that for all $V \in T(\mathcal{S}, M_0)$ one has $f'(M_0)V \geq 0$, and if $f'(M_0)V = 0$ one has

$$\forall (W, r) \in \widehat{T}^2(\mathcal{S}, M_0, V) \qquad f'(M_0)W + rf''(M_0)VV \geq 0.$$

This condition can be decomposed into the case $r = 1$ yielding $f'(M_0)W + f''(M_0)VV \geq 0$ for all $W \in T^2(\mathcal{S}, M_0, V)$ and the case $r = 0$ yielding $f'(M_0)W \geq 0$ for all $W \in \mathcal{S}$ such that $QPWP^*Q^* \succeq 0$, that is, for all $W \in T(T(\mathcal{S}, M_0), V)$ by Corollary 7. A similar analysis can be conducted for the related sufficient condition.

Before applying the preceding results to semidefinite programming, we review the general case of cone programming.

4. OPTIMALITY CONDITIONS IN CONE PROGRAMMING

Let us consider the conic programming problem

$$\text{(P)} \qquad \text{minimize } f(x) \quad \text{subject to } g(x) \in C,$$

where X, Z are Banach spaces, $f : X \to \mathbb{R}$, $g : X \to Z$ are twice differentiable maps and C is a closed convex cone in Z. A well-known first-order necessary local optimality condition is that the set

$$M(a) := \{y \in N(C, g(a)) : f'(a) + y \circ g'(a) = 0\}$$

of *multipliers* at the local minimizer a be nonempty provided a constraint qualification condition is satisfied; here $N(C, g(a))$ is the normal cone to C at $g(a)$, that is, the set of $y \in Y := Z^*$ such that $\langle y, w\rangle \leq 0$ for all $w \in C$ and $\langle y, g(a)\rangle = 0$. Second-order conditions are not as simple. In order to state the general condition of [32, Theorem 3.5], let us introduce a convenient formalism.

Let V, W be n.v.s. and let $(h_t)_{t>0}$ be a family of maps from V into W. Suppose W is ordered by a closed convex cone W_+. Let *epi* $h_t := \{(v, w) \in V \times W : w \geq h_t(v)\}$. Then the set $H := \limsup_{t \to 0_+}$ *epi* h_t is a pseudo-epigraph in the sense that it is closed and for any $(v, w) \in H$ and

$w' \geq w$ one has $(v, w') \in H$. Then, we write symbolically $w \geq h(v)$ with $h := e - \liminf_{t \to 0_+} h_t$ although the existence of such an epi-limit can be guaranteed just when W is a complete lattice for the order induced by W_+ assumed to be pointed.

Returning to problem (P), given $a \in F := g^{-1}(C)$, we consider the maps $h_t : X \to W := Z \times \mathbb{R}$ $(t \in \mathbb{P} := (0, +\infty))$ given by

$$h_t(x) := 2t^{-2} \left[(g(a), 0) + t(g'(a)x, f'(a)x) \right].$$

In such a case, we have $(-z, r) \geq h(v)$ if, and only if, there exist sequences $(t_n) \to 0_+$, $((v_n, z_n, r_n)) \to (v, z, r)$ such that $(-z_n, r_n) \in h_{t_n}(v_n) + Z_+ \times \mathbb{R}_+$ for all $n \in \mathbb{N}$ with $Z_+ := -C$, $W_+ := Z_+ \times \mathbb{R}_+$. That means that

$$\begin{aligned}(z, -r) &\in S_v \\ &:= \limsup_{(t,u) \to (0_+, v)} \frac{2}{t^2} \left[C \times (f(a) - \mathbb{R}_+) - (g(a), f(a)) - t(g'(a)u, f'(a)u) \right],\end{aligned}$$

the set introduced in [32, Theorem 3.5]. Let us note that a third formalism using a true epi-limit can be introduced. It involves the functions $k_t : X \times Z \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ given for $t \in \mathbb{P} := (0, +\infty)$ by

$$k_t(x, z) := \frac{2}{t} f'(a)x + \iota_C(g(a) + tg'(a)x + \frac{t^2}{2} z)$$

where ι_C is the *indicator function* of C given by $\iota_C(z) := 0$ for $z \in C$, $\iota_C(z) := +\infty$ for $z \in Z \backslash C$. In fact, one easily checks that $r \geq k(v, z) := e - \liminf_{t \to 0_+} k_t(v, z)$ if, and only if $(-z, r) \geq h(v)$, that is, $(v, -z, r) \in H$ or $(z, -r) \in S_v$. Thus, $k(v, z) = \inf\{r : (v, -z, r) \in H\}$. We call k *the critical function* of (P) at a. We notice that for $v \in g'(a)^{-1}(T(C, g(a)))$ we have $k(v, z) = +\infty$ whenever $z \in Z \backslash T_G^2(C, g(a), g'(a)v)$, where, for $G := g'(a)$, $T_G^2(C, g(a), g'(a)v)$ is the (second-order) *compound tangent set of* C at $g(a)$ in the direction $g'(a)v$ defined by

$$T_G^2(C, g(a), g'(a)v) := \limsup_{(t,u) \to (0_+, v)} \frac{2}{t^2} \left[C - g(a) - tg'(a)u \right].$$

We notice that S_v is the compound tangent set of $C \times (f(a) - \mathbb{R}_+)$ at $(g(a), f(a))$ in the direction $(g'(a)v, f'(a)v)$ for the map $x \mapsto (g'(a)x, f'(a)x)$.

We also introduce the *critical cone* $C(a)$ of (P) at $a \in F := g^{-1}(C)$,

$$C(a) = \{v \in X : f'(a)v = 0,\ g'(a)v \in T(C, g(a))\}$$

and the qualification condition

$$Z_a := g'(a)(X) - T(C, g(a)) = -Z_a = cl(Z_a). \tag{6}$$

This condition means that Z_a is a closed linear subspace. It is obviously more general than Robinson's qualification condition:

$$g'(a)(X) - \mathbb{R}_+(C - g(a)) = Z. \tag{7}$$

We say that g is *metrically regular in the direction* v (with respect to C) if there exist some $c > 0$, $\rho > 0$ such that

$$\forall u \in B(v, \rho),\ t \in (0, \rho) \qquad d(a + tu, g^{-1}(C)) \le cd(g(a + tu), C).$$

Such a condition is satisfied when (7) holds. On the other hand, it ensures that $v \in T(F, a)$, with $F := g^{-1}(C)$, whenever $g'(a)v \in T(C, g(a))$ and then that $w \in T^2(F, a, v)$ if $g'(a)w + g''(a)vv \in T^2(C, g(a), g'(a)v)$, as easily checked.

Theorem 12. *Let a be a local minimizer of (P) for which the qualification condition (6) is satisfied. If g is metrically regular in the direction $v \in C(a)$, in particular if condition (7) holds, then for all $z \in Z$ there exists some multiplier $y \in M(a)$ such that*

$$f''(a)vv + \langle y, g''(a)vv\rangle \ge \langle y, z\rangle - k(v, z).$$

The proof being almost the same as the proof of [32, Theorem 3.5], we omit it. It suffices to use the preceding analysis and to observe that the metric regularity of g in the direction v is enough to obtain [32, Lemma 3.2] in which we take $B = X$, $S_{v,B} = -\{(z, r) : (v, z, r) \in H\}$.

In [32, Theorem 3.7] it has been shown that the necessary condition of Theorem 12 corresponds as precisely as possible to a sufficient condition. We present a proof since here we show that a is not just a strict local minimizer of f on F but is an *essential local minimizer of second-order* of (P) in the sense of [33]: There exist $\alpha, \beta, \gamma > 0$ such that

$$x \in B(a, \beta) \cap \{u : d(g(u), C) \le \gamma \|u - a\|^2\} \Longrightarrow f(x) \ge f(a) + \alpha \|x - a\|^2.$$

This notion is even stronger than the notion of quadratic growth condition of [5, Definition 3.1] since in the preceding implication one can take x outside the feasible set F.

Theorem 13. *Suppose X is finite dimensional, $a \in F := g^{-1}(C)$ and that for all $v \in C(a)$ and all $z \in Z$ there exists some multiplier $y \in M(a)$ such that*

$$f''(a)vv + \langle y, g''(a)vv\rangle > \langle y, z\rangle - k(v, z). \tag{8}$$

Then a is an essential local minimizer of second-order of (P).

Moreover, the proof below shows that it suffices to have (8) satisfied for all $v \in C(a)$ and $z = g''(a)vv$.

Proof. Suppose on the contrary that there exist sequences $(x_n) \to a$ in X, $(\varepsilon_n) \to 0_+$ such that $d(g(x_n), C) \leq \varepsilon_n \|x_n - a\|^2$, $f(x_n) < f(a) + \varepsilon_n \|x_n - a\|^2$. Setting $t_n := \|x_n - a\| > 0$, we may assume that $v_n := t_n^{-1}(x_n - a)$ converges to some v. Then $v \in C(a)$ and, for $z := g''(a)vv$, one has $-f''(a)vv \geq k(v, z)$ since, by Taylor expansions, there exist $(z_n) \to z$ and $(r_n) \to -f''(a)vv$ such that $g(a) + t_n g'(a)v_n + (1/2)t_n^2 z_n \in C$ and $f(a) + t_n f'(a)v_n - (1/2)t_n^2 r_n \leq f(a)$, that is, $r_n \geq 2t_n^{-1} f'(a)v_n := k_{t_n}(v_n, z_n)$. Thus, we get a contradiction with (8). □

As in [33, Proposition 5.1], one can formulate the preceding result in terms of a John's multiplier $(y, t) \in N(C, g(a)) \times \mathbb{R}_+$ by replacing f by tf in the preceding proof. Such a result differs from [5, Theorem 3.63] by the presence of the auxiliary terms $\langle y, z\rangle - tk(v, z)$ in (9).

Proposition 14. *Suppose X is finite dimensional, $a \in F := g^{-1}(C)$ and that for all $v \in C(a)$ and all $z \in Z$ there exists some $y \in N(C, g(a))$, $t \in \mathbb{R}_+$ such that $tf'(a) + y \circ g'(a) = 0$ and*

$$tf''(a)vv + \langle y, g''(a)vv\rangle > \langle y, z\rangle - tk(v, z). \tag{9}$$

Then a is an essential local minimizer of second-order of (P).

One may wonder whether one can drop the assumption that X is finite dimensional by replacing $C(a)$ by some approximate critical set and k by some approximate critical function. Such a direction is taken in [5, Theorem 3.63 (i)], but at the expense of replacing the right-hand side of (9) by $\beta \|v\|^2$ for some $\beta > 0$, whereas the right-hand side of (9) is nonpositive by the definition of $k(v, z)$.

Let us observe that for any $v \in C(a)$ and $z \in T^2(C, g(a), g'(a)v)$ we have $k(v, z) \leq 0$ since there exists a sequence $((t_n, z_n)) \to (0_+, z)$ such that $g(a) + t_n g'(a)v + (1/2)t_n^2 z_n \in C$ and $t_n^{-1} f'(a)v \leq 0$, or $k_{t_n}(v, z_n) \leq 0$. Thus, the conclusion of the following statement is a consequence of the conclusion of Theorem 12. However, it relies on the assumption

$$Z_{a,v} := g'(a)(X) - T(T(C, g(a)), g'(a)v) = -Z_{a,v} = cl(Z_{a,v}) \tag{10}$$

meaning that $Z_{a,v}$ is a closed linear subspace. Let us note that this assumption is weaker than (6). In fact, since C is convex we have $T(C, b) \subset T(T(C, b), w)$ for $b := g(a)$, $w := g'(a)(v) \in T(C, b)$, hence $Z_a \subset Z_{a,v}$. Conversely, given $z \in T(T(C, b), w)$, one can find sequences $(t_n) \to 0_+$, (w_n) in $T(C, b)$ such that $z = \lim_n t_n^{-1}(w_n - w) \in cl(-Z_a) = Z_a$ when (6) holds and since $0 \in T(C, b)$ we also have $g'(a)(X) \subset Z_a$ so that $Z_{a,v} = Z_a$. Since the assumptions of Theorem 12 and those of [5, 3.45], [9, Theorem 4.2] are weakened, a proof is required.

Proposition 15. *Let a be a local minimizer of (P) and let $v \in C(a)$ be such that g is metrically regular in the direction v and such that the qualification condition (10) is satisfied. Then, for all $z \in T^2(C, g(a), g'(a)v)$ there exists some $y \in M(a)$ such that*

$$f''(a)vv + \langle y, g''(a)vv\rangle \geq \langle y, z\rangle. \tag{11}$$

Moreover, for any convex subset T of $T^2(C, g(a), g'(a)v)$ there exists some $y \in M(a)$ such that

$$f''(a)vv + \langle y, g''(a)vv\rangle \geq \sup_{z\in T}\langle y, z\rangle. \tag{12}$$

Proof. Given $z \in T^2(C, g(a), g'(a)v) \subset T(T(C, g(a)), g'(a)v))$ (by [5, Relation (3.63)]), let $x \in X$ and $z'' \in T(T(C, g(a)), g'(a)v)$ be such that $g'(a)x - z'' = z - g''(a)vv$. Then, by [9, Proposition 3.1],

$$\begin{aligned} z' := z + z'' &\in T^2(C, g(a), g'(a)v) + T(T(C, g(a)), g'(a)v) \\ &\subset T^2(C, g(a), g'(a)v), \end{aligned}$$

so that there exist sequences $(t_n) \to 0_+$, $(z'_n) \to z'$ such that $g(a) + t_n g'(a)v + (1/2)t_n^2 z'_n \in C$. Since there exists $c > 0$ such that

$$d(a + tv + (1/2)t^2x, g^{-1}(C)) \leq cd(g(a + tv + (1/2)t^2x), C)$$

for $t > 0$ small enough, and $g(a + tv + (1/2)t^2x) = g(a) + tg'(a)v + (1/2)t^2z' + o(t^2)$, we can find a sequence $(x_n) \to x$ such that $a + t_nv + (1/2)t_n^2x_n \in g^{-1}(C)$ for all n. Since a is a local minimizer and $f'(a)v = 0$, using again a Taylor expansion, we get

$$f'(a)x + f''(a)vv \geq 0,$$

or $f'(a)x \geq -f''(a)vv$ for any $x \in X$ satisfying $g'(a)x + g''(a)vv - z \in T(T(C, g(a)), g'(a)v)$. Since (10) holds, the Lagrange multiplier rule of [32, Corollary 3.4] yields some y in the polar cone of $T(T(C, g(a)), g'(a)v)$ such that

$$f'(a)x + \langle y, g'(a)x + g''(a)vv - z\rangle \geq -f''(a)vv$$

for every $x \in X$. Thus, $y \in N(C, g(a))$, $\langle y, g(a)\rangle = 0$, $f'(a) + y \circ g'(a) = 0$ and $\langle y, g''(a)vv - z\rangle \geq -f''(a)vv$. Thus, $y \in M(a)$ and (11) holds.

Now let T be a convex subset of $T^2(C, g(a), g'(a)v)$. Using [31, Theorem 3.6] with $Y := Z_{a,v}$, $u := g'(a)$, $F := \{f'(a)\}$, $M := X$, $P := T(T(C, g(a)), g'(a)v)$, we get that

$$M'(a) := \{y' \in Z^*_{a,v} : y' \in T(T(C, g(a)), g'(a)v)^0,\ f'(a) + y' \circ g'(a) = 0\}$$

is bounded, hence, weak* compact. Then, setting $\ell''_{xx}(x, y) := f''(x) + \langle y, g''(x)\rangle$, using the Moreau's minimax theorem, we get

$$\max_{y' \in M'(a)} \inf_{z \in T} (\ell''_{xx}(a, y')vv - \langle y, z\rangle) = \inf_{z \in T} \max_{y' \in M'(a)} (\ell''_{xx}(a, y')vv - \langle y, z\rangle) \geq 0.$$

Thus there exists $y' \in M'(a))$ such that $\ell''_{xx}(a, y')vv \geq \langle y, z\rangle$ for all $z \in T$. Taking $y \in Z^*$ such that $y|_{Z_{a,v}} = y'$, we get $y \in M(a)$ and $\ell''_{xx}(a, y)vv = \ell''_{xx}(a, y')vv \geq \langle y, z\rangle$ □

Let us end this section with another second-order necessary condition avoiding constraint qualifications. For such a purpose, let us introduce the *performance function* (or value function) $p : Z \to \overline{\mathbb{R}}$ given by

$$p(z) := \inf\{f(x) : g(x) + z \in C\}.$$

Its *directional subdifferential* (or Dini-Hadamard subdifferential) at 0 is the set

$$\partial_D p(0) := \{y \in Y : \forall z \in Z\ \langle y, z\rangle \leq \liminf_{(t,w)\to(0_+,z)} \frac{p(tw) - p(0)}{t}\}.$$

Clearly, when p is directionally differentiable at 0 in the sense that the above liminf is a limit and is a continuous linear form in z, one has $\partial_D p(0) = \{p'(0)\}$.

Proposition 16. *Let a be a solution to (P). Then, $\partial_D p(0) \subset M(a)$ and for any $y \in \partial_D p(0)$, any critical direction $v \in C(a)$ and any $z \in Z$ the following second-order necessary condition is satisfied:*

$$f''(a)vv + \langle y, g''(a)vv\rangle \geq \langle y, z\rangle - k(v, z).$$

Denoting by k^*_v the Fenchel conjugate of $k_v := k(v, \cdot)$ this condition can be written

$$\forall v \in C(a) \qquad f''(a)vv \geq \sup\{k^*_v(y) - \langle y, g''(a)vv\rangle : y \in \partial_D p(0)\}.$$

Proof. The inclusion $\partial_D p(0) \subset M(a)$ is well known (see [34, Proposition 4.125]).

Let $y \in \partial_D p(0)$, $v \in C(a)$ and $z \in Z$ be given. If $k(v,z) = +\infty$ the inequality is obvious. Thus, we may suppose $k(v,z) < +\infty$. Let $(v_n) \to v$, $(z_n) \to z$, $(t_n) \to 0_+$ be such that $k(v,z) = \lim_n k_{t_n}(v_n, z_n)$. Thus $(2t_n^{-1} f'(a)v_n) \to k(v,z)$ and $c_n := g(a) + t_n g'(a)v_n + (t_n^2/2)z_n \in C$ for all $n \in \mathbb{N}$. Then, for some sequence $(w_n) \to 0$, we have

$$g(a + t_n v_n) - c_n = (t_n^2/2)(g''(a)v_n v_n - z_n + w_n)$$

Since $g(a + t_n v_n) + (t_n^2/2)(z_n - w_n - g''(a)v_n v_n) = c_n \in C$, we have

$$p((1/2)t_n^2(z_n - w_n - g''(a)v_n v_n) \leq f(a + t_n v_n),$$

hence, since $p(0) = f(a)$,

$$\begin{aligned}\langle y, z - g''(a)vv\rangle &\leq \liminf_n \frac{2}{t_n^2}(f(a + t_n v_n) - f(a)) \\ &\leq \liminf_n \frac{2}{t_n^2}\left(t_n f'(a)v_n + \frac{t_n^2}{2} f''(a)v_n v_n\right) = k(v,z) + f''(a)vv.\end{aligned}$$

That is the announced inequality. □

5. OPTIMALITY CONDITIONS IN SEMIDEFINITE PROGRAMMING

In this section, we take advantage of the special structure of the second-order tangent set to the set $\mathcal{S}_+$ of semidefinite matrices. That enables to give simplified optimality conditions. We first present such a condition in that special mathematical programming case. Then we show that the set $\mathcal{S}_+$ enjoys a special geometric property that entails such a special structure of the second-order tangent set.

Let us say that a subset T of a n.v.s. Z is an *affine cone* if there exists some $w \in T$ and some cone P of Z containing 0 such that $T = P + w$. Then w is called a *vertex* of T. When P is pointed (i.e., $P \cap (-P) = \{0\}$), w is unique.

Lemma 17. *Let $T = P + w$ be an affine cone with vertex w in a n.v.s. Z. If Q is a subset of Z such that $T + Q \subset T$, then one has $P + Q \subset P$, $Q \subset P$.*

If Q is a closed cone of Z such that $T \subset Q$, then one has $P \subset Q$.

Thus, if Q is a closed cone such that $T + Q \subset T \subset Q$ one has $P = Q$.

Proof. The inclusion $T + Q \subset T$ can be written $w + P + Q \subset w + P$, so that one has $P + Q \subset P$. Taking $0 \in P$, we get $Q \subset P$. Suppose now that $T \subset Q$, where Q is a closed cone of Z. Then, for any $p \in P$ and any $r > 0$ one has $rp + w \in Q$ or $p + r^{-1}w \in r^{-1}Q \subset Q$. Taking the limit as $r \to +\infty$, we get $p \in Q$ since Q is closed. □

Given a convex cone C of a Banach space Z, $z \in C$, $z' \in T(C,z)$, we deduce from the preceding lemma and the inclusions

$$T^2(C,z,z') + T(T(C,z),z') \subset T^2(C,z,z') \subset T(T(C,z),z') \qquad (13)$$

of [9, Proposition 3.1] that when $T^2(C,z,z')$ is an affine cone with vertex w, then $T^2(C,z,z') = w + T(T(C,z),z')$. In such a case, the necessary condition of Proposition 15 can be restricted to taking the multiplier y corresponding to $z := w$ in (11).

Proposition 18. *Suppose that $a \in F := g^{-1}(C)$ is a local minimizer of f on F such that for some $v \in C(a)$ the set $T^2(C, g(a), g'(a)v)$ is an affine convex cone with vertex $w(v)$ and (10) holds. Suppose that g is metrically regular in the direction v. Then, there exists some multiplier $y \in M(a)$ such that*

$$f''(a)vv + \langle y, g''(a)vv\rangle \geq \langle y, w(v)\rangle. \qquad (14)$$

Moreover the term $\langle y, w(v)\rangle$ is equal to $\sup\{\langle y,z\rangle : z \in T^2(C, g(a), g'(a)v)\}$, *hence does not depend on the choice of the vertex $w(v)$ in $T^2(C, g(a), g'(a)v)$.*

Proof. The existence of a multiplier $y \in M(a)$ satisfying (14) is obtained in taking $z := w(v)$ in Proposition 15. It remains to prove the last assertion. Let $w' \in T^2(C, g(a), g'(a)v)$. Since $\langle y, g'(a)v\rangle = -f'(a)v = 0$, y is in the polar cone of $T(T(C,g(a)), g'(a)v)$, and since $w' - w(v) \in T(T(C,g(a)), g'(a)v)$, we get $\langle y, w' - w(v)\rangle \leq 0$, so that $\langle y, w(v)\rangle = \max\{\langle y,z\rangle : z \in T^2(C, g(a), g'(a)v)\}$. □

Question. It would be interesting to know whether the condition

$$\forall v \in C(a) \qquad f''(a)vv + \langle y, g''(a)vv\rangle > \sup\{\langle y,z\rangle : z \in T^2(C, g(a), g'(a)v)\}$$

suffices to ensure that a is a local solution to (P). We give a positive answer in Proposition 20 in the case C is a manifold with corners.

When C is a manifold with corners, let us show that $T^2(C,z,z')$ is an affine cone.

Proposition 19. *If C is a C^2-manifold with corners around $c \in C$, then, for all $v \in T(C,c)$, the second-order tangent set $T^2(C,c,v)$ is an affine cone.*

If C is a C^2-manifold with boundary around $c \in C$, then, for all $v \in T(C,c)$, the second-order tangent set $T^2(C,c,v)$ is an affine half-space or the whole space.

Proof. Given a diffeomorphism φ as in Definition 3, we have $z \in T^2(C,c,v)$ if, and only if $\varphi'(c)z + \varphi''(c)vv \in T^2(Q,0,\varphi'(c)v) = T(Q,\varphi'(c)v)$, as easily seen, if, and only if

$$z \in \varphi'(c)^{-1}(T(Q,\varphi'(c)v)) - \varphi'(c)^{-1}\left(\varphi''(c)vv\right).$$

Thus, $w(v) := -\varphi'(c)^{-1}(\varphi''(c)vv)$ is a vertex of $T^2(C,c,v)$.

If C is a C^2-manifold with boundary around $c \in C$, then Q is a half-space, so that $T(Q, \varphi'(c)v)$ is either Q (when $\varphi'(c)v \in Q \backslash intQ$) or the whole space (when $\varphi'(c)v \in intQ$) □

In the following proposition, we deduce the sufficient condition of [5, Theorem 3.137] from Theorem 13.

Proposition 20. *Suppose X is finite dimensional, $a \in F := g^{-1}(C)$, C is a C^2-manifold with corners around $c := g(a)$ and that for all $v \in C(a)$ there exists some $y \in M(a)$ such that*

$$f''(a)vv + \langle y, g''(a)vv \rangle > \sup\{\langle y, w \rangle : w \in T^2(C, c, g'(a)v)\}. \tag{15}$$

Then a is a local minimizer of f on F.

Proof. Using a chart φ as in the preceding definition, we may suppose C is a cone and $g(a) = 0$. Note that substituting $\varphi'(0)^{-1} \circ \varphi$ to φ, we may suppose that $\varphi'(0)$ is the identity mapping, so that multipliers for the original problem are multipliers for the reduced problem and vice versa. Using Theorem 13, it suffices to show that for every $z \in Z$ one has

$$\sup\{\langle y, w \rangle : w \in T^2(C, 0, g'(a)v)\} \geq \langle y, z \rangle - k(v, z). \tag{16}$$

The left-hand side is 0 since $T^2(C, 0, g'(a)v)$ is the cone $T(C, g'(a)v) = cl(C + \mathbb{R}g'(a)v)$ and $y \in M(a)$, $v \in C(a)$. The right-hand side is $-\infty$ when $k(v, z) = +\infty$. When $k(v, z) < +\infty$, there exist sequences $(t_n) \to 0_+$, $(v_n) \to v$, $(z_n) \to z$, $(r_n) \to k(v, z)$, with $c_n := g(a) + t_n g'(a)v_n + (1/2)t_n^2 z_n \in C$,

$$\begin{aligned} r_n &\geq 2t_n^{-1} f'(a)v_n = -2t_n^{-2}\langle y, t_n g'(a)v_n \rangle \\ &\geq -2t_n^{-2}\langle y, c_n - (1/2)t_n^2 z_n \rangle \geq \langle y, z_n \rangle. \end{aligned}$$

Passing to the limit, we get $k(v, z) \geq \langle y, z \rangle$ and inequality (16) holds. □

Conclusion. The structure of $\mathcal{S}_+$ as a manifold with corners revealed in Proposition 4 allows to give simplified optimality conditions in semidefinite programming. In particular, in the right-hand side of Relation (15) the single term $\langle y, w(v) \rangle$ involving the vertex $w(v)$ corresponding to the critical direction v can be substituted to the supremum of y over $T^2(C, g(a), g'(a)v)$. Setting $\overline{w} := g'(a)v$ and using the notation of Proposition 9, the vertex $w(v)$ can be identified with the element $\overline{w}$ of $\mathcal{S}$ all elements $A'', \ldots, J$ of which are null but the term $K = 2F^* A_0^{-1} F = 2QPV^* P_X^* \left(P_X^* M_0 P_X\right)^{-1} P_X V P^* Q^*$. Thus the additional "sigma term" in the terminology of [5] can be easily computed.

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